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MODULI SPACES OF PARABOLIC HIGGS BUNDLES AND PARABOLIC K(D) PAIRS OVER SMOOTH CURVES

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1. INTRODUCTION

A deep result of Narasimhan and Seshadri states that, over a compact curve C, there is a one-to-one correspondence between irreducible *unitary* representations of $\pi_1(C)$ and stable bundles with $c_1(E) = 0$ [19]. This theorem was later extended to cover arbitrary compact Kähler manifolds [8, 9, 27]. Hitchin then introduced *Higgs* bundles, which are pairs (E, Φ) consisting of a holomorphic bundle E and holomorphic map $\Phi : E \to E \otimes K$ called the Higgs field. The nonabelian Hodge theorem establishes a correspondence between irreducible representations of $\pi_1(X)$ and stable Higgs bundles with $c_i(E) = 0$ for X a compact, Kähler manifold [13, 10, 7, 23, 25].

For noncompact curves C_0 with compactification $C = C_0 \cup \{p_1, \dots, p_n\}$, Mehta and Seshadri proved a correspondence between stable *parabolic* bundles E_* over C with $\operatorname{pardeg} E_* = 0$ and unitary representations of $\pi_1(C_0)$ with fixed holonomy around each p_i, the so-called parabolic points [18]. For regular bundles, the space of Higgs fields $H^0(\operatorname{End}(E) \otimes K)$ is naturally dual to $H^1(\operatorname{End}(E))$, but because parabolic endomorphisms satisfy a vanishing condition at the parabolic points, duality implies that Higgs fields of parabolic bundles can have poles of order one at those points. Allowing the Higgs field to have either *parabolic* or *nilpotent* residues at the p_i, we obtain the two moduli spaces $\mathcal{P}_\alpha$ of parabolic $K(D)$ pairs and $\mathcal{N}_\alpha$ of parabolic Higgs bundles. The subscript α refers to a particular choice of weights. In [28], $\mathcal{P}_\alpha$ is constructed using Geometric Invariant Theory and is proved to be a normal, quasi-projective variety. In [24], stable parabolic $K(D)$ pairs are called filtered regular Higgs bundles and are shown to correspond to filtered regular $\mathcal{D}_X$ modules. In [15], Konno constructs $\mathcal{N}_\alpha$ using gauge theory and shows that stable parabolic Higgs bundles correspond to irreducible parabolic Hermitian-Einstein Higgs bundles. In [22], the nonabelian Hodge theorem for elliptic surfaces is used to show that parabolic Higgs bundles with rational weights correspond to irreducible representations of $\pi_1(C_0)$ with holonomy around each p_i equal to some root of unity.

In this article, we study the topological properties of the moduli spaces of stable, rank two parabolic Higgs bundles and parabolic $K(D)$ pairs, using the approach of Hitchin [13]. There is a circle action on $\mathcal{N}_\alpha$ which preserves its complex and symplectic structure, and the associated moment map is a Morse function. Identifying the critical submanifolds and their indices, we prove that $\mathcal{N}_\alpha$ is a noncompact, connected, simply connected manifold and compute its Betti numbers, which turn out to be independent of the weights α. This behavior is in marked contrast to that exhibited by the moduli space $\mathcal{M}_\alpha$ of parabolic bundles, where the Betti numbers do depend on α [4]. It follows that the Euler characteristic of this moduli space vanishes for $g \geq 2$. All of the main results in this article was done in a joint work [6] with Hans U. Boden.

2. Definitions and Preliminary Results

2.1. Three moduli spaces. Let X be a smooth curve of genus g with n marked points in the reduced divisor $D = p_1 + \cdots + p_n$ and E a holomorphic bundle over X.

Definition 2.1. *A parabolic structure on E consists of weighted flags*

$$E_p = F_1(p) \supset \cdots \supset F_{s_p}(p) \supset 0$$
$$0 \leq \alpha_1(p) < \cdots < \alpha_{s_p}(p) < 1$$

over each $p \in D$. A holomorphic map $\phi : E^1 \longrightarrow E^2$ between parabolic bundles is called parabolic if $\alpha_i^1(p) > \alpha_j^2(p)$ implies $\phi(F_i^1(p)) \subset F_{j+1}^2(p)$ for all $p \in D$. We call ϕ strongly parabolic if $\alpha_i^1(p) \geq \alpha_j^2(p)$ implies $\phi(F_i^1(p)) \subset F_{j+1}^2(p)$ for all $p \in D$.

We use E_* to denote the bundle together with a parabolic structure. Also, we use $\mathrm{ParHom}(E_*^1, E_*^2)$ and $\mathrm{ParHom}(E_*^1, \widehat{E}_*^2)$ to denote the sets of parabolic and strongly parabolic morphisms from E^1 to E^2, respectively. (The decorative notation will become clear in §2.2.) If $\alpha_i^1(p) \neq \alpha_j^2(p)$ for all i, j and $p \in D$, then a parabolic morphism is automatically strongly parabolic. On the other hand, using the notation $\mathrm{ParEnd}(E_*) = \mathrm{ParHom}(E_*, E_*)$ and $\mathrm{ParEnd}^\wedge(E_*) = \mathrm{ParHom}(E_*, \widehat{E}_*)$, then strongly parabolic endomorphisms are nilpotent with respect to the flag data at each $p \in D$.

Let K denote the canonical bundle of X and give $E \otimes K(D)$ the obvious parabolic structure.

Definition 2.2. *A parabolic $K(D)$ pair is a pair (E, Φ) consisting of a parabolic bundle E and a parabolic map $\Phi : E \to E \otimes K(D)$. Such a pair is called a parabolic Higgs bundle if, in addition, Φ is a strongly parabolic morphism.*

Viewing α as a vector-valued function on D, we use it as an index to indicate the parabolic structure on E_*. Let $m_i(p) = \dim(F_i(p)) - \dim(F_{i+1}(p))$, the multiplicity of $\alpha_i(p)$, and $f_p = \frac{1}{2}(r^2 - \sum_{i=1}^{s_p}(m_i(p))^2)$, the dimension of the associated flag variety. Define the parabolic degree and slope of E_* by

$$\begin{aligned} \operatorname{pardeg} E_* &= \deg E + \sum_{p \in D} \sum_{i=1}^{s_p} m_i(p)\alpha_i(p), \\ \mu(E_*) &= \frac{\operatorname{pardeg} E_*}{\operatorname{rank} E}. \end{aligned}$$

If L is a subbundle of E, then L inherits a parabolic structure from E by pullback. We call the bundle E_* *stable* (*semistable*) if, for every proper subbundle L of E, we have $\mu(L_*) < \mu(E_*)$ (respectively $\mu(L_*) \leq \mu(E_*)$). Likewise, we will call a parabolic $K(D)$ pair (E_*, Φ) *stable* (or *semistable*) if the same inequalities hold on those proper subbundles L of E which are, in addition, Φ-invariant.

Denote by $\mathcal{M}_\alpha$ the moduli space of α-semistable parabolic bundles, by $\mathcal{N}_\alpha$ the moduli space of α-semistable parabolic Higgs bundles, and by $\mathcal{P}_\alpha$ the moduli space of α-semistable parabolic $K(D)$ pairs. By [18], $\mathcal{M}_\alpha$ is a normal, projective variety of dimension

$$\dim \mathcal{M}_\alpha = (g-1)r^2 + 1 + \sum_{p\in D} f_p.$$

(If $g=0$, this holds only when $\mathcal{M}_\alpha \neq \emptyset$.) Further, in [28, 29], $\mathcal{P}_\alpha$ is shown to be a normal, quasi-projective variety of dimension

$$\dim \mathcal{P}_\alpha = (2g-2+n)r^2 + 1$$

which contains $\mathcal{N}_\alpha$ as a closed subvariety of $\mathcal{P}_\alpha$ of dimension

$$\dim \mathcal{N}_\alpha = 2(g-1)r^2 + 2 + 2\sum_{p\in D} f_p.$$

For generic α, a bundle (or pair) is α-semistable $\Leftrightarrow$ it is α-stable. In these cases, the moduli spaces $\mathcal{M}_\alpha, \mathcal{N}_\alpha$ and $\mathcal{P}_\alpha$ are smooth and can be described topologically as certain quotients of the gauge group $\mathcal{G}^{\mathbb{C}} = \mathrm{ParAut}(E_*)$. The same is true for $\mathcal{M}^0_\alpha, \mathcal{N}^0_\alpha$ and $\mathcal{P}^0_\alpha$, the moduli spaces with fixed determinant and trace-free Φ. In this way, it is shown in [15] that $\mathcal{N}^0_\alpha$ is, for generic α, a smooth, hyperkähler manifold of complex dimension

$$\dim \mathcal{N}^0_\alpha = 2(g-1)(r^2-1) + 2\sum_{p\in D} f_p.$$

2.2. Parabolic sheaves and Serre duality. Suppose now that E is a locally free sheaf on X and $D = p_1 + \cdots + p_n$ is a reduced divisor.

Definition 2.3. *A parabolic structure on E consists of a weighted filtration of the form*

$$\begin{gathered} E = E_0 = E_{\alpha_1} \supset \cdots \supset E_{\alpha_l} \supset E_{\alpha_{l+1}} = E(-D), \\ 0 = \alpha_0 \leq \alpha_1 < \cdots < \alpha_l < \alpha_{l+1} = 1. \end{gathered}$$

We can define E_x for $x \in [0,1]$ by setting $E_x = E_{\alpha_i}$ if $\alpha_{i-1} < x \leq \alpha_i$, and then extend to $x \in \mathbb{R}$ by setting $E_{x+1} = E_x(-D)$. We call the resulting filtered sheaf E_* a parabolic sheaf.

We define the coparabolic sheaf $\widehat{E}_*$, by

$$\widehat{E}_x = \begin{cases} E_x & \text{if } x \neq \alpha_i \\ E_{\alpha_{i+1}} & \text{if } x = \alpha_i. \end{cases}$$

A morphism of parabolic sheaves $\phi : E^1_* \to E^2_*$ is a called parabolic if $\phi(E^1_x) \subseteq E^2_x$ and strongly parabolic if $\phi(E^1_x) \subseteq \widehat{E}^2_x$ for all $x \in \mathbb{R}$.

We shall denote by $\mathfrak{ParHom}(E^1_*, E^2_*)$ and $\mathfrak{ParHom}(E^1_*, \widehat{E}^2_*)$ the sheaves of parabolic and strongly parabolic morphisms, and by $\mathrm{ParHom}(E^1_*, E^2_*)$ and $\mathrm{ParHom}(E^1_*, \widehat{E}^2_*)$ their global sections. We now show that there is an equivalence of the categories of parabolic bundles on X and parabolic sheaves on X.

Given a parabolic bundle E with flags and weights as in Definition 2.1, we define the filtered sheaf E_* following Simpson [26]. For $p \in D$ and $\alpha_{i-1}(p) < x \leq \alpha_i(p)$, set

$$\begin{aligned} E_x^p &= \ker(E \to E_p/F_i(p)), \\ E_x &= \bigcap_{p\in D} E_x^p. \end{aligned}$$

Now extend to all x by $E_{x+1} = E_x(-D)$.

Conversely, given a parabolic sheaf E_*, the quotient E/E_1 is a skyscraper sheaf with support on D and, for each $p \in D$, we get weighted flags in E_p by intersecting with the filtration at p. To be precise, let $\alpha_1(p), \ldots, \alpha_{s_p}(p)$ be the subset of weights such that

$$\alpha_{i-1}(p) < x \leq \alpha_i(p) \Leftrightarrow (E_x/E_1)_p = (E_{\alpha_i(p)}/E_1)_p. \tag{1}$$

Setting $F_i(p) = (E_{\alpha_i(p)}/E_1)_p$, we obtain a parabolic bundle in the sense of Definition 2.1.

Suppose now E^1 and E^2 are parabolic bundles and $\phi \in \mathrm{ParHom}(E^1, E^2)$. We want to show that ϕ induces a morphism of the parabolic sheaves. So, suppose $\alpha_{i-1}^1(p) < x \leq \alpha_i^1(p)$ and $\alpha_{j-1}^2(p) < x \leq \alpha_j^2(p)$. Since $\alpha_i^1(p) > \alpha_{j-1}^2(p)$, $\phi(F_i^1(p)) \subset F_j^2(p)$ and we see that ϕ maps $\ker(E^1 \to E_p^1/F_i^1(p))$ to $\ker(E^2 \to E_p^2/F_i^2(p))$ for all $p \in D$, from which it follows that ϕ induces a map $\phi : E_x^1 \to E_x^2$.

Suppose conversely that E_*^1 and E_*^2 are parabolic sheaves, $\phi \in \mathrm{ParHom}(E_*^1, E_*^2)$ and $\alpha_i^1(p) > \alpha_j^2(p)$. Set $x = \alpha_i^1(p)$ and $y = \alpha_{j+1}^2(p)$ for notational convenience. Then $\phi(E_x^1) \subset E_x^2$. Since $x > \alpha_j^2(p)$, it follows from (1) that $(E_x^2/E_1^2)_p \subset (E_y^2/E_1^2)_p$ and hence $\phi(F_i^1(p)) \subset F_{j+1}^2(p)$.

It is not hard to see the same correspondence for strongly parabolic morphisms. Thus, we have an equivalence of the categories of parabolic bundles and parabolic sheaves. We use the definitions interchangeably and denote by E_* a parabolic bundle or sheaf, reserving $E = E_0$ for the underlying holomorphic bundle.

For the convenience of readers, we briefly summarize the results in [29] dealing with exact sequences and tensor products of parabolic sheaves. This is necessary for the statement of Serre duality for parabolic bundles, which is a tool we use throughout the article.

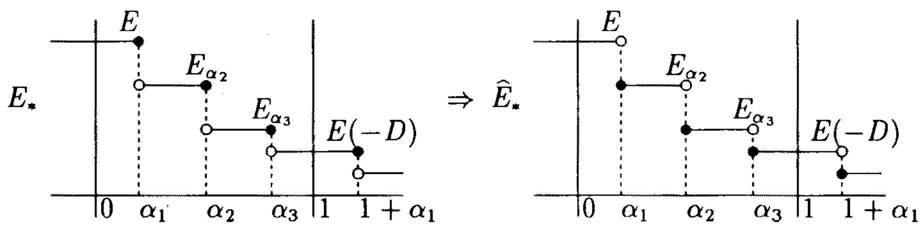

FIGURE 1. The simple relationship between E_* and $\widehat{E}_*$.

The category of parabolic sheaves $\mathcal{P}$ is not abelian, but is contained in an abelian category $\widetilde{\mathcal{P}}$ as a full subcategory. Objects in $\widetilde{\mathcal{P}}$ are also written by E_* and a morphism $f : E^1_* \to E^2_*$ is a family of morphisms $f_x : E^1_x \to E^2_x$. A coparabolic sheaf $\widehat{E}_*$ is realized in $\widetilde{\mathcal{P}}$. The set $\mathrm{ParHom}(E^1_*, \widehat{E}^2_*)$ is just the set of morphisms in $\widetilde{\mathcal{P}}$. In $\widetilde{\mathcal{P}}$, a sequence

$$0 \longrightarrow L_* \longrightarrow E_* \longrightarrow M_* \longrightarrow 0 \tag{2}$$

is exact if and only if the induced sequence at x is exact for all $x \in \mathbb{R}$.

Remark. If the sequence (2) is exact, then so is the sequence obtained by tensoring (2) with any parabolic bundle (cf. Proposition 3.3 of [29]) and

$$\mathrm{pardeg}\, E_* = \mathrm{pardeg}\, L_* + \mathrm{pardeg}\, M_*.$$

We can define dual parabolic sheaves $E^\vee_*$, parabolic tensor products $L_* \otimes M_*$, Hom-parabolic sheaves $\mathfrak{ParHom}(L_*, M_*)_*$, and cohomology groups $\mathrm{Ext}^i(L_*, M_*)$. Clearly,

$$\mathrm{pardeg}(L_* \otimes M_*) = \mathrm{rank}(M)\,\mathrm{pardeg}\, L_* + \mathrm{rank}(L)\,\mathrm{pardeg}\, M_*.$$

In addition, we have

$$\begin{aligned}
\mathrm{Ext}^0(L_*, M_*) &= H^0(L^\vee_* \otimes M_*) = H^0(\mathfrak{ParHom}(L_*, M_*)) = \mathrm{ParHom}(L_*, M_*),\\
\mathrm{Ext}^1(L_*, M_*) &= H^1(L^\vee_* \otimes M_*) = H^1(\mathfrak{ParHom}(L_*, M_*)).
\end{aligned}$$

We can identify $\mathrm{Ext}^1(M_*, L_*)$ with the set of equivalence classes of exact sequences of type (2).

The Serre duality theorem is generalized as follows (see Proposition 3.7 of [29]).

Proposition 2.4. *For parabolic sheaves L_* and M_*, there is a natural isomorphism*

$$\theta^i : H^i(L^\vee_* \otimes M_* \otimes K(D)) \xrightarrow{\simeq} H^{1-i}(M^\vee_* \otimes \widehat{L}_*)^\vee.$$

Given E_* and $\beta \in \mathbb{R}^n$, define $E_*[\beta]_*$, the parabolic sheaf E_* shifted by β, by

$$E_*[\beta]_x = \bigcap_i E^{p_i}_{x+\beta_i}.$$

Example. The Picard group of parabolic line bundles.

A holomorphic bundle E is regarded as a parabolic bundle with the trivial parabolic structure $E_p \supset 0, \alpha_1(p) = 0$ at each $p \in D$. We call this the special structure on E. Note that every parabolic line bundle L_* is gotten by shifting the special structure on the underlying bundle L, i.e., there is a unique $\beta \in [0,1)^n$ with $L_* = L[\beta]_*$ Viewing $\mathcal{O}_X$ as a parabolic bundle with the special structure, then it is not difficult to verify that

$$E_*[\beta]_* = E_* \otimes \mathcal{O}_X[\beta]_* \tag{3}$$

Let e_i denote the standard basis vector in $\mathbb{R}^n$. From (3) we have

$$\begin{aligned}
E^1_*[\beta^1]_* \otimes E^2_*[\beta^2]_* &= E^1_* \otimes E^2_*[\beta^1 + \beta^2]_*,\\
E_*[\beta]^\vee_* &= E^\vee_*[-\beta]_*,\\
E_*[e_i]_* &= E_* \otimes \mathcal{O}_X(-p_i).
\end{aligned}$$

These three formulas determine the Picard group of parabolic line bundles on X.

Remark. For any parabolic line bundle L_*, the stability (or semistability) of $E_* \otimes L_*$ is equivalent to that of E_*. Similarly, the stability (or semistability) of $(E_* \otimes L_*, \Phi \otimes 1)$ is equivalent to that of (E_*, Φ).

In particular, apply this to the case of a rank two parabolic bundle E_* with full flags at each p_i and weights $0 \leq \alpha_1(p_i) < \alpha_2(p_i) < 1$. Using equation (3) with $\beta_i = \frac{1}{2}(\alpha_1(p_i) + \alpha_2(p_i) - 1)$ notice that $E_*[\beta]_*$ has weights $0 < a_1(p_i) < 1 - a_1(p_i) < 1$ at p_i, where $a_1(p_i) = \frac{1}{2}(\alpha_1(p_i) - \alpha_2(p_i) + 1)$.

3. A Topological Description of $\mathcal{N}_\alpha^0$ in Rank Two

3.1. The function spaces of Biquard and construction of Konno. We begin with a brief overview of the gauge theoretical description of $\mathcal{N}_\alpha$ following [15].

It is convenient to think of the parabolic bundle separate from its holomorphic structure, so we use E_* to denote the underlying topological parabolic bundle (weights α) and $\overline{\partial}_E$ its holomorphic structure. By tensoring with an appropriate line bundle, we can always assume that $\mu(E_*) = 0$. We shall also restrict our attention to generic weights, i.e., weights α for which α-stability and α-semistability coincide. Let $\mathcal{C}$ denote the affine space of all holomorphic structures on E, and $\mathcal{G}_{\mathbb{C}}$ the group of smooth bundle automorphisms of E preserving the flag structure. Introduce a metric κ adapted to E (κ is unitary and smooth on $E|_{X \setminus D}$, but singular at $p \in D$ in a prescribed way, see Definition 2.3 [3]), and let $\mathcal{A}$ denote the affine space of κ-unitary connections. Define $\mathcal{G}$ to be the subgroup of $\mathcal{G}_{\mathbb{C}}$ consisting of κ-unitary gauge transformations. Letting $\mathcal{C}_{ss}$ and $\mathcal{A}_{flat}$ be the subspaces of α-semistable holomorphic structures and the flat connections, respectively, Biquard proved that

$$\mathcal{M}_\alpha \stackrel{\text{def}}{=} \mathcal{C}_{ss}/\mathcal{G}_{\mathbb{C}} \cong \mathcal{A}_{flat}/\mathcal{G}$$

by introducing the norms $\| \; \|_{D_k^p}$, defining the weighted Sobolev spaces $\mathcal{C}^p$ and $\mathcal{A}^p$ of D_1^p holomorphic structures and D_1^p κ-unitary connections, and taking quotients by the groups $\mathcal{G}_{\mathbb{C}}^p$ and $\mathcal{G}^p$ of D_2^p gauge transformations for a certain $p > 1$ [3].

The same approach works for parabolic Higgs moduli, at least for generic weights, as was shown by Konno. The arguments in [15] are given for moduli with fixed determinant, but remain equally valid without this condition. We set

$$\begin{aligned} \mathcal{H} \;=\; & \{(\overline{\partial}_E, \Phi) \in \mathcal{C} \times \Omega^{1,0}(\operatorname{End} E) \mid \overline{\partial}_E \Phi = 0 \text{ on } X \setminus D \text{ and at each } p \in D, \\ & \Phi \text{ has a simple pole with nilpotent residue with respect to the flag}\}. \end{aligned}$$

Note that $\mathcal{H}$ (this is denoted by $\mathcal{D}$ in [15]) is just the differential geometric definition of the space of parabolic Higgs bundle structures on E_*, for example, the nilpotency condition implies that Φ is strongly parabolic.

For $A \in \mathcal{A}$, we use d_A for its covariant derivative, F_A for its curvature, and d_A'' for the $(0,1)$ component of d_A, so $d_A'' \in \mathcal{C}$. Define $\mathcal{E} = \mathcal{A} \times \Omega^{0,1}(\operatorname{End} E)$ and $\mathcal{E}^p$ as its completion with respect to the norms $\| \; \|_{D_1^p}$, and set

$$\mathcal{E}_{flat} = \{(d_A, \Phi) \in \mathcal{E}^p \mid d_A'' \Phi = 0, F_A + [\Phi, \Phi^*] = 0\}.$$

(This last space is denoted $\mathcal{D}^p_{HE}$ by Konno.) Using the usual definition of stability on $\mathcal{H}$, Theorem 1.6 of [15] shows that for some $p > 1$,

$$\mathcal{N}_\alpha \stackrel{\text{def}}{=} \mathcal{H}_{ss}/\mathcal{G}_{\mathbb{C}} \cong \mathcal{E}_{flat}/\mathcal{G}^p.$$

The advantage of the second quotient is that it endows $\mathcal{N}_\alpha$ with a natural hyperkähler structure, namely by viewing it as a hyperkähler quotient of $\mathcal{E}^p$ (in the sense of [14]), whose hyperkähler structure is given by the metric

$$g((\xi,\phi),(\xi,\phi)) = 2i\int_X \mathrm{Tr}(\xi^*\xi + \phi\phi^*),$$

which is Kähler with respect to each of three complex structures

$$I(\xi,\phi) = (i\xi, i\phi), \quad J(\xi,\phi) = (i\phi^*, -i\xi^*), \quad K(\xi,\phi) = (-\phi^*, \xi^*).$$

3.2. The Morse function for the moduli space of parabolic Higgs bundles. Assume that E_* is a rank two parabolic bundle with generic weights α_i and $1-\alpha_i$ at p_i and that $\mu_\alpha(E_*) = 0$. Write $\alpha = (\alpha_1, \dots, \alpha_n)$. We will always assume $n \geq 1$. We consider the moduli with fixed determinant and trace-free Higgs fields, requiring the following minor modifications in the definitions of the previous section:

(i) the induced connection d_Λ or holomorphic structure $\overline{\partial}_\Lambda$ on $\Lambda^2 E$ be fixed;
(ii) the Higgs field be trace-free, i.e. $\Phi \in \Omega^{1,0}(\mathrm{End}_0 E)$.

We denote the corresponding spaces by $\mathcal{A}^0, \mathcal{C}^0, \mathcal{E}^0$, and $\mathcal{H}^0$.

As in [13], we consider the circle action defined on $\mathcal{E}^0$ by $e^{i\theta}\cdot(d_A,\Phi) = (d_A, e^{i\theta}\Phi)$. This action preserves the subspace $\mathcal{E}^0_{flat}$ and commutes with the action of the gauge group $\mathcal{G}^p$, thus it descends to give a circle action ρ on $\mathcal{N}^0_\alpha$. This action commutes with the complex structure defined by I and preserves the symplectic form $\omega_1(X,Y) = g(IX,Y)$, so the associated moment map $\mu_\rho(d_A,\Phi) = \frac{1}{4\pi}\|\Phi\|^2_{D^p_1}$, renormalized for convenience, is a Bott-Morse function and can be used to determine the Betti numbers of $\mathcal{N}^0_\alpha$.

We introduce some notation which will be used throughout the rest of this section. For any line subbundle L_* of E_*, let $e_i(L) = \dim L_{p_i} \cap F_2(p_i) \in \{0,1\}$. The weight inherited by L_* is then $\beta_i(L) = e_i + (-1)^{e_i}\alpha_i$. We will often suppress the dependence on L and simply write $e = (e_1, \dots, e_n)$ and $\beta = (\beta_1, \dots, \beta_n)$. We will also write $\beta(\alpha, e)$ when we want to emphasize the functional dependence of β on α and e. We also use $|e| = \sum_{i=1}^n e_i$.

Theorem 3.1. (a) *The map $\mu_\rho : \mathcal{N}^0_\alpha \longrightarrow \mathbb{R}$ is a proper Morse function.*

(b) *Whenever nonempty, $\mathcal{M}^0_\alpha$ is the unique critical submanifold corresponding to the minimum value $\mu_\rho = 0$. The other critical submanifolds are given by $\mathcal{M}_{d,e}$ for an integer d and $e \in \mathbb{Z}_2^n$ satisfying*

$$-\sum_{i=1}^n \beta_i(\alpha,e) < d \leq g - 1 - |e|/2. \tag{4}$$

Along $\mathcal{M}_{d,e}$, μ_ρ takes the value $d + \sum_{i=1}^n \beta_i$.

(c) *The critical submanifold* $\mathcal{M}_{d,e}$ *is* $\widetilde{S}^{h_{d,e}}X$, *the* 2^{2g} *cover of the symmetric product* $S^{h_{d,e}}X$ *under the map* $x \mapsto 2x$ *on* J_X. *Here,* $h_{d,e} = 2g - 2 - 2d - |e|$.

(d) *The Morse index of* $\mathcal{M}_{d,e}$ *is given by* $\lambda_{d,e} = 2(n + 2d + g - 1 + |e|)$.

Remark. If $g = 0$, there are always α with $\mathcal{M}^0_\alpha = \emptyset$ (but $\mathcal{N}^0_\alpha \neq \emptyset$). For these α, the minimum value is achieved along some $\mathcal{M}_{d,e}$, which we identify in the next section.

Proof. Properness of μ_ρ follows from the global compactness result for parabolic bundles of Biquard (Theorem 2.14 in [3]). This proves (a). All the other statements rely on the following correspondence between the circle action and the moment map given in [11].

(1) Critical submanifolds are connected components of the fixed point set of ρ.

(2) The Morse index of a critical submanifold equals the dimension of the negative weight space of the infinitesimal circle action on its normal bundle.

Suppose that (d_A, Φ) is a fixed point of the circle action upstairs in $\mathcal{E}_{flat}$. Then $\Phi = 0$ and this shows that one component of the fixed point set in $\mathcal{N}^0_\alpha$ consists of $\mathcal{M}^0_\alpha$, the moduli of stable parabolic bundles with fixed determinant.

The other fixed points arise from when $e^{i\theta} \cdot (d_A, \Phi)$ is gauge equivalent to (d_A, Φ), i.e., when there is a one parameter family $g_\theta \in \mathcal{G}^p$ such that

$$\begin{aligned} g_\theta^{-1} \Phi g_\theta &= e^{i\theta}\Phi, \\ g_\theta^{-1} d_A g_\theta &= d_A. \end{aligned}$$

By the first equation, g_θ is not central, and by the second, we see that d_A is reducible and consequently the holomorphic parabolic bundle splits according to the eigenvalues of g_θ. Write $E_* = L_* \oplus M_*$ as a direct sum of parabolic bundles. We assume (wlog) that $\mu_\alpha(L_*) > 0 > \mu_\alpha(M_*)$. Let $d = \deg L$ and $e = (e_1, \ldots, e_n)$ where $e_i = \dim L_{p_i} \cap F_2(p_i)$. Then L inherits the weight $\beta_i = e_i + (-1)^{e_i}\alpha_i$ at p_i as a parabolic subbundle of E_* and

$$0 < \mu_\alpha(L_*) = d + \sum_{i=1}^{n} \beta_i. \tag{5}$$

Since g_θ is diagonal with respect to this decomposition, Φ is either upper or lower diagonal, which means either L or M is Φ-invariant. But α-stability of the pair (E_*, Φ) implies that

$$\Phi = \begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix},$$

where $0 \neq \phi \in \mathrm{ParHom}(L_*, \widehat{M}_* \otimes K(D))$. Thus

$$0 \neq H^0(L_*^\vee \otimes \widehat{M}_* \otimes K(D)) = H^0(L^\vee \otimes M \otimes K(\textstyle\sum_{i=1}^n (1 - e_i)p_i)).$$

Let $|e| = \sum_{i=1}^n e_i$, then a necessary condition is that

$$0 \leq \deg(L^\vee \otimes M \otimes K(\textstyle\sum_{i=1}^n (1 - e_i)p_i)) = 2(g-1) - 2d - |e|. \tag{6}$$

Now (4) follows from (5) and (6).

We can use the defining equations for $\mathcal{E}^0_{flat}$ to determine the associated critical values. Take (E_*, Φ) as above, then

$$0 = F_A + [\Phi, \Phi^*] = \begin{pmatrix} F_L - \phi\phi^* & 0 \\ 0 & F_M + \phi^*\phi \end{pmatrix}.$$

Using the Chern-Weil formula for parabolic bundles (Proposition 2.9 of [3]), we get

$$\mu_\rho(d_A, \Phi) = \frac{1}{4\pi}\|\Phi\|^2 = \frac{i}{2\pi}\int_X \mathrm{Tr}(\Phi\Phi^*) = \frac{i}{2\pi}\int_X \phi\phi^* = \frac{i}{2\pi}\int_X F_L = \mathrm{pardeg}(L_*).$$

This completes the proof of (b).

Given $E_* = L_* \oplus M_*$ and Φ as above, then the zero set of ϕ is a nonnegative divisor of degree

$$h_{d,e} = \deg(L^\vee \otimes M \otimes K(\textstyle\sum_{i=1}^n (1 - e_i)p_i)) = 2g - 2 - 2d - |e|$$

on X, which is just an element of $S^{h_{d,e}}X$. Conversely, given a nonnegative divisor of degree $h_{d,e}$, then we obtain a line bundle U of degree $2d + n$ along with a section of $U^\vee \otimes K(\sum_{i=1}^n (1 - e_i)p_i))$ vanishing on that divisor. There are 2^{2g} choices of L so that $U = L^{\otimes 2} \otimes \Lambda^2 E$, and each choice gives a stable parabolic Higgs bundle (E_*, Φ). The line subbundle L_* is canonically determined from E_*, but Φ is only determined up to multiplication by a nonzero constant. However, it is easy to see that (E_*, Φ) is gauge equivalent to $(E_*, \lambda\Phi)$ for $\lambda \neq 0$, and (c) now follows.

We now calculate the index $\lambda_{d,e}$ of the critical submanifold $\mathcal{M}_{d,e}$, which is given by the negative weight space of the infinitesimal action of ρ, or equivalently, of the gauge transformation g_θ. Letting $H^0(\mathrm{ParEnd}_0(E)) \cdot \Phi$ be the subspace of Higgs fields of the form $[\Psi, \Phi]$ for $\Psi \in H^0(\mathrm{ParEnd}_0(E))$, then the subspace

$$W = H^0(\mathrm{ParEnd}_0^\wedge(E) \otimes K(D))/H^0(\mathrm{ParEnd}_0(E)) \cdot \Phi$$

is Lagrangian with respect to the complex symplectic form

$$\omega((\xi_1, \phi_1), (\xi_2, \phi_2)) = \int_X \mathrm{Tr}(\phi_2\xi_1 - \phi_1\xi_2).$$

So once we determine the weights on W, the weights on the dual space W^* are given by $1 - \nu$ for some weight ν on W (since $\rho(\theta)^*\omega = e^{i\theta}\omega$). With respect to the decomposition $E_* = L_* \oplus M_*$, we have

$$g_\theta = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

with weights $(0, 1, -1)$ on

$$\mathrm{ParEnd}_0^\wedge(E_*) = \mathrm{ParHom}(L_*, \hat{L}_*) \oplus \mathrm{ParHom}(L_*, \widehat{M}_*) \oplus \mathrm{ParHom}(M_*, \hat{L}_*).$$

Further, there are no negative weights on $H^0(\mathrm{ParEnd}_0(E)) \cdot \Phi$ and the weights on W^* are $(1, 0, 2)$, so we get

$$\lambda_{d,e} \;=\; 2h^0(M_*^\vee \otimes \hat{L}_* \otimes K(D)) = 2(n + 2d + g - 1 + |e|).$$

This completes the proof of (d). □

3.3. The topology of $\mathcal{N}_\alpha^0$. Using the results of the previous section, we deduce the following theorem.

Theorem 3.2. (a) *If $g > 0$ or $g = 0$ and $n > 3$, then $\mathcal{N}_\alpha^0$ is noncompact.*
(b) *The Betti numbers of $\mathcal{N}_\alpha^0$ depend only on the quasi-parabolic structure of E_*.*
(c) *If $g > 0$ or $g = 0$ and $n \geq 3$, then $\mathcal{N}_\alpha^0$ is connected and simply connected.*

Proof. Notice that, whenever $\dim \mathcal{N}_\alpha^0 > 0$, then for all (d, e), $\lambda_{d,e} < \dim \mathcal{N}_\alpha^0$. Thus, the Morse function μ_ρ has no maximum value and (a) follows. The only case where $\dim \mathcal{N}_\alpha^0 = 0$ is, of course, $g = 0$ and $n = 3$.

We first recall Theorem 3.1 of [4]. Let $W = \{\alpha \mid 0 < \alpha_i < \frac{1}{2}\}$ be the weight space and for any (d, e), define the hyperplane $H_{d,e} = \{\alpha \mid d + \beta(\alpha, e) = 0\}$. The set $W \setminus \cup_{d,e} H_{d,e}$ consists of the generic weights, i.e., those for which stability and semistability coincide. Suppose $\delta \in H_{d,e}$, then stratifying $\mathcal{M}_\delta^0$ by the Jordan-Hölder type of the underlying parabolic bundle, we see that

$$\mathcal{M}_\delta^0 = (\mathcal{M}_\delta^0 \setminus \Sigma_\delta) \cup \Sigma_\delta,$$

where Σ_δ consists of strictly semistable bundles, i.e., semistable bundles E_* with $\operatorname{gr} E_* = L_* \oplus M_*$ for two parabolic line bundles of parabolic degree zero. Suppose that α and α' are generic weights on either side of $H_{d,e}$ and that $\operatorname{pardeg}_\alpha(L_*) < 0$. If both $\mathcal{M}_\alpha^0$ and $\mathcal{M}_{\alpha'}^0$ are nonempty, then Theorem 3.1 of [4] states that there are canonical, projective maps

$$\begin{array}{ccc} \mathcal{M}_\alpha^0 & & \mathcal{M}_{\alpha'}^0 \\ & \phi \searrow \; \swarrow \phi' & \\ & \mathcal{M}_\delta^0 & \end{array}$$

which are isomorphisms on $\mathcal{M}_\delta^0 \setminus \Sigma_\delta$ and are $\mathbb{P}^a$ and $\mathbb{P}^{a'}$ bundles along Σ_δ, where $a = h^1(M_*^\vee \otimes L_*) - 1$ and $a' = h^1(L_*^\vee \otimes M_*) - 1$. In particular, since $\Sigma_\delta = J_X$, Corollary 3.2 of [4] gives

$$P_t(\mathcal{M}_\alpha^0) - P_t(\mathcal{M}_{\alpha'}^0) = (P_t(\mathbb{P}^a) - P_t(\mathbb{P}^{a'}))P_t(J_X).$$

To prove (b), we must show that $P_t(\mathcal{N}_\alpha^0) = P_t(\mathcal{N}_{\alpha'}^0)$ for weights on either side of a hyperplane $H_{d,e}$. Note that $d = \deg L$ and $e = e(L)$, and set $\hat{d} = -n - d$ and $\hat{e}_i = 1 - e_i$. Since

$$d + \beta(\alpha, e) = \operatorname{pardeg}_\alpha(L) < 0 < \operatorname{pardeg}_{\alpha'}(L) = d + \beta(\alpha', e),$$

and $\hat{d} + \beta(\alpha', \hat{e}) < 0 < \hat{d} + \beta(\alpha, \hat{e})$, it follows that the indexing sets of (d, e) satisfying (4) for $\mathcal{N}_\alpha^0$ and $\mathcal{N}_{\alpha'}^0$ are identical except for (d, e) and $(\hat{d}, \hat{e})$ listed above; the pair (d, e) satisfies (4) for α but not for α' and vice versa for $(\hat{d}, \hat{e})$. Thus, we claim

$$0 \; = \; P_t(\mathcal{M}_\alpha^0) - P_t(\mathcal{M}_{\alpha'}^0) + t^{\lambda_{d,e}} P_t(\mathcal{M}_{d,e}) - t^{\lambda_{\hat{d},\hat{e}}} P_t(\mathcal{M}_{\hat{d},\hat{e}}),$$

which, setting $\Delta = t^{\lambda_{\hat{d},\hat{e}}} P_t(\mathcal{M}_{\hat{d},\hat{e}}) - t^{\lambda_{d,e}} P_t(\mathcal{M}_{d,e})$ is equivalent to

$$\Delta = \frac{(t^{2a'+2} - t^{2a+2})(1 + t)^{2g}}{1 - t^2}. \tag{7}$$

First, we compute

$$h_{d,e} = 2g - 2 - 2d - |e|, \quad \lambda_{d,e} = 2(n + 2d + g - 1 + |e|),$$
$$h_{\hat{d},\hat{e}} = 2g - 2 + n + 2d + |e|, \quad \lambda_{\hat{d},\hat{e}} = 2(g - 1 - 2d - |e|).$$

Next, notice that if $h > 2g-2$, then $P_t(\tilde{S}^h(X)) = P_t(S^h(X))$ (see p. 98 of [13]). But both $h_{d,e}$ and $h_{\hat{d},\hat{e}}$ are greater than $2g-2$, which we see as follows. Since $\frac{e_i}{2} \leq \beta_i(\alpha, e) \leq \frac{1+e_i}{2}$, we have $\frac{|e|}{2} \leq \sum_{i=1}^n \beta_i(\alpha, e) \leq \frac{n+|e|}{2}$. It now follows that $2d+|e| < 2d+2\beta(\alpha, e) < 0$ and $2d + n + |e| > 2d + 2\sum_{i=1}^n \beta(\alpha', e) > 0$.

Now use the result of [16] to interpret $P_t(S^h X)$ as the coefficient of x^h in

$$\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)},$$

and compute in terms of residues to see

$$\begin{aligned}\Delta &= t^{\lambda_{\hat{d},\hat{e}}} P_t(S^{h_{\hat{d},\hat{e}}} X) - t^{\lambda_{d,e}} P_t(S^{h_{d,e}} X) \\ &= \operatorname*{Res}_{x=0} \left(\frac{t^{\lambda_{\hat{d},\hat{e}}}}{x^{h_{\hat{d},\hat{e}}+1}} - \frac{t^{\lambda_{d,e}}}{x^{h_{d,e}+1}} \right) \left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} \right).\end{aligned}$$

This last function is analytic at $x = \infty$ and has a removable singularity at $x = 1/t^2$, thus

$$\begin{aligned}\Delta &= -\operatorname*{Res}_{x=1} \left(\frac{t^{\lambda_{\hat{d},\hat{e}}}}{x^{h_{\hat{d},\hat{e}}+1}} - \frac{t^{\lambda_{d,e}}}{x^{h_{d,e}+1}} \right) \left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} \right) \\ &= \frac{(t^{\lambda_{\hat{d},\hat{e}}} - t^{\lambda_{d,e}})(1+t)^{2g}}{1-t^2}.\end{aligned}$$

But we can compute directly that $2a'+2 = \lambda_{\hat{d},\hat{e}}$ and that $2a+2 = \lambda_{d,e}$ and (7) follows. This proves (b) in case both $\mathcal{M}_\alpha^0$ and $\mathcal{M}_{\alpha'}^0$ are nonempty. In case one of the moduli is empty, we use the following lemma (see the remark).

To prove (c), we use the fact that $\mathcal{M}_\alpha^0$ is connected and simply-connected, which follows for $g = 0$ from [2] and for $g \geq 1$ from [5]. Since $\lambda_{d,e}$ is always even, (c) will follow if $\lambda_{d,e} > 0$ for all (d, e). This is true if $\mathcal{M}_\alpha^0 \neq \emptyset$. However, if $g = 0$ we must be careful since there are weights α with $\mathcal{M}_\alpha = \emptyset$. In that case, we must show that there is a unique pair (d, e) with $\lambda_{d,e} = 0$, and also that $\mathcal{M}_{d,e}$ is connected and simply connected. This is the content of the following lemma. □

Lemma 3.3. (i) *If $g \geq 1$, then $\lambda_{d,e} > 0$ for every (d, e) satisfying* (4).

(ii) *If $g = 0$ and $n \geq 3$, then there is at most one pair (d, e) satisfying* (4) *with $\lambda_{d,e} = 0$. Such a pair (d, e) exists if and only if $\mathcal{M}_\alpha = \emptyset$, and in that case, $\mathcal{M}_{d,e} = \mathbb{P}^{n-3}$. Here, $\mathcal{M} = \mathcal{M}^0$ since $g = 0$.*

Remark. We now explain why this lemma proves part (b) of the Proposition when one of the moduli is empty. Suppose $\mathcal{M}_\alpha = \emptyset$, then it follows that the moment map μ_ρ is positive with minimum value $d + \sum_{i=1}^n \beta(\alpha, e)$ for the pair (d, e) identified in part (ii) of the lemma. Since (d, e) does not satisfy (4) for α', $H_{d,e}$ is the relevant hyperplane. This identifies the birth and death strata as $\mathcal{M}_{\alpha'}$ and $\mathcal{M}_{d,e}$, and thus all the other strata for α and α' are identical. The rest follows from the fact that $\mathcal{M}_{\alpha'} = \mathbb{P}^{n-3}$, first proved by Bauer [2].

Proof. Suppose that $\lambda_{d,e} = 0$ for a pair (d, e) satisfying (4). We first show that $g = 0$. Recall that $\beta_i(\alpha, e) = e_i + (-1)^{e_i}\alpha_i$. Using the fact that $0 = \lambda_{d,e} = n + 2d + g + |e| - 1$, the condition (4) and the inequality $\beta_i(\alpha, e) < \frac{e_i+1}{2}$, we see that

$$\frac{n + |e| + g - 1}{2} < \sum_{i=1}^{n} \beta_i(\alpha, e) < \frac{n + |e|}{2}. \tag{8}$$

This is only possible if $g = 0$, which we now assume.

Setting $\gamma_i = 1 - \beta_i = (1 - e_i)(1 - \alpha_i) + e_i\alpha_i$, then equation (8) is equivalent to

$$\frac{n - |e|}{2} < \sum_{i=1}^{n} \gamma_i < \frac{n - |e| + 1}{2}.$$

Writing $\gamma_i = \frac{1-e_i}{2} + (1 - e_i)(\frac{1}{2} - \alpha_i) + e_i\alpha_i$, we get immediately

$$0 < \sum_{i=1}^{n}(1 - e_i)(\frac{1}{2} - \alpha_i) + e_i\alpha_i < \frac{1}{2}. \tag{9}$$

The advantage of the (9) is that each summand is positive.

We now prove uniqueness of the pair (d, e). If $\lambda_{d',e'} = 0$ for $(d', e') \neq (d, e)$, then it follows that $|e| - |e'| = 2(d' - d)$ is even, which implies that $e_i \neq e'_i$ for at least two i, which we assume (wlog) to include $i = 1, 2$. Now (α, e) and (α, e') both satisfy the inequality (9). Add them together and notice that since $e_1 \neq e'_1$ and $e_2 \neq e'_2$, the sum of the left hand sides is at least $\alpha_1 + (1/2 - \alpha_1) + \alpha_2 + (1/2 - \alpha_2) = 1$, which violates the (summed) inequality and therefore gives a contradiction.

It follows from $\lambda_{d,e} = 0$ and $g = 0$ that $n + |e| - 1$ is even and $h_{d,e} = n - 3$. Thus $\mathcal{M}_{d,e} = S^h X = S^h\mathbb{P}^1 = \mathbb{P}^{n-3}$. The rest of the lemma follows from the the inequality (8), together with the following proposition, which we have chosen to state as it is of independent interest. $\square$

Proposition 3.4. *If $g = 0$, then the moduli space $\mathcal{M}_\alpha \neq \emptyset \Leftrightarrow$*

$$\sum_{i=1}^{n} e_i + (-1)^{e_i}\alpha_i < \frac{n + |e| - 1}{2}. \tag{10}$$

for every $e = (e_1, \ldots, e_n)$, $e_i \in \{0, 1\}$, with $n - |e| + 1$ even.

Remark. For $n = 3$, $\mathcal{M}_\alpha$ is either empty or a point. In this case, the proposition can be verified directly by comparing the inequalities (10) to the well-known fusion rules (or the quantum Clebsch-Gordan conditions):

$$\mathcal{M}_\alpha \neq \emptyset \Leftrightarrow |\alpha_1 - \alpha_2| \leq \alpha_3 \leq \min(\alpha_1 + \alpha_2, 1 - \alpha_1 - \alpha_2).$$

Proof. Like the proof of part (b) of the theorem, we shall use the techniques of [4]. Recall the weight space $W = \{\alpha \mid 0 \leq \alpha_i \leq 1/2\}$ and the hyperplanes $H_{d,e} = \{\alpha \mid d + \beta(\alpha, e) = 0\}$ defined earlier. We call connected components of $W \setminus \cup_{d,e} H_{d,e}$ *chambers*. A chamber C is called *null* if the associated moduli space $\mathcal{M}_\alpha$ is empty in genus 0 for every $\alpha \in C$. The proposition follows once we show that every null chamber is given by $C_{d,e} = \{\alpha \mid d + \beta(\alpha, e) > 0\}$, where $2d = 1 - n - |e|$.

Associated to the configuration of hyperplanes in W is a graph with one vertex for each chamber and an edge between two vertices whenever the two chambers are separated by a hyperplane. We shall see that in terms of this graph, null chambers have valency one. The (unique) hyperplane separating a null chamber from the rest of W is called a *vanishing wall.* If $\delta \in H_{d,e}$, a vanishing wall, and α, α' are nearby weights on either side of $H_{d,e}$, then the proof of Proposition 5.1 of [4] shows that $\mathcal{M}_\delta = \Sigma_\delta$ and, assuming that $\mathcal{M}_{\alpha'} = \emptyset$, the map ϕ is a fibration with fiber $\mathbb{P}^a$, where $a = h^1(M_*^\vee \otimes L_*) - 1$. Moreover, $h^1(L_*^\vee \otimes M_*) = 0$ and this last equation in fact characterizes vanishing walls.

We claim now that every vanishing hyperplane is given by $H_{d,e}$ for $2d = 1 - n - |e|$. Now if $d = \deg L$ and $e = e(L)$, then direct computation shows that $h^1(L_*^\vee \otimes M_*) = 2d + n + |e| - 1$. On the other hand, if $n + |e| - 1$ is even and $d = \frac{1-n-|e|}{2}$, then $H_{d,e}$ is a vanishing hyperplane.

Along $H_{d,e}$, the relevant line bundles of parabolic degree 0 are given by $L_* = \mathcal{O}_X(\frac{-n-|e|+1}{2})[-\beta]_*$ and $M_* = \mathcal{O}_X(\frac{-n+|e|-1}{2})[-\gamma]_*$, where $\delta \in H_{d,e}, \beta = \beta(\delta, e)$ and $\gamma_i = 1 - \beta_i$. Since $h^1(L_*^\vee \otimes M_*) = 0$ and $h^1(M_*^\vee \otimes L_*) = n - 2$, it follows that the null chamber is defined by $C_{d,e} = \{\alpha \mid \beta(\alpha, e) > \frac{n+|e|-1}{2}\}$. To verify this is indeed a chamber, we prove that no other hyperplane cuts through $C_{d,e}$. This will also show that null chambers have valency one in the graph associated to the configuration of hyperplanes.

So suppose to the contrary that $\alpha \in H_{d',e'} \cap C_{d,e}$. Then we have $\sum(-1)^{e_i}\alpha_i > \frac{n-|e|-1}{2}$ and $\sum(-1)^{e_i'}\alpha_i = -|e'| - d' = k \in \mathbb{Z}$. If $e_i = e_i' = 0$, then $((-1)^{e_i} + (-1)^{e_i'})\alpha_i < 1$ and in all other cases, $((-1)^{e_i} + (-1)^{e_i'})\alpha_i \leq 0$. Using a similar property for $e'' = 1 - e'$, we see

$$\begin{aligned}
\frac{n-|e|-1}{2} + k &< \sum_{i=1}^{n}((-1)^{e_i} + (-1)^{e_i'})\alpha_i < \sum_{e_i = e_i' = 0} 1, \\
\frac{n-|e|-1}{2} - k &< \sum_{i=1}^{n}((-1)^{e_i} + (-1)^{e_i''})\alpha_i < \sum_{e_i = e_i'' = 0} 1.
\end{aligned}$$

These are strict inequalities of integers, so after adding one to the left hand sides and summing the two inequalities (which are no longer strict), we see $n - |e| + 1 \leq \sum_{e_i=0} 1 = n - |e|$, a contradiction. $\square$

3.4. The Betti numbers of the moduli space of parabolic Higgs bundles. The results of the previous section show that the Betti numbers of $\mathcal{N}_\alpha^0$ depend only on the genus g and number n of parabolic points. In this section, we give a formula for the Poincaré polynomial of $\mathcal{N}_\alpha^0$. Such a general calculation is not possible for $P_t(\mathcal{M}_\alpha^0)$ without first specifying α, so take $\alpha = (\frac{1}{3}, \dots, \frac{1}{3^n})$. Using Proposition 3.4 (taking $e = (0, 1, \dots, 1)$) it is clear the α lies in a null chamber. We could calculate $P_t(\mathcal{M}_\alpha^0)$ using the Atiyah-Bott procedure for parabolic bundles as in [5], but there is an easier method which exploits the fact that α lies in a null chamber. First of all, using the results of §6.4 in [5], we get

$$P_t(\mathcal{M}^0_\alpha) = \frac{(1+t^2)^{n-1}(1+t^3)^{2g}}{(1-t^2)^2} - \frac{(1+t)^{2g}}{(1-t^2)}\sum_{\lambda,e} t^{2d_{\lambda,e}}.$$

Note that $d_{\lambda,e}$ depends on g ($d_{\lambda,e} = d_{\lambda,e}(g=0)+g$), but the indexing set $\{\lambda, e\}$ is independent of g. Since $\mathcal{M}^0_\alpha(g=0)=\emptyset$, this determines the sum and we see that

$$P_t(\mathcal{M}^0_\alpha) = (1+t^2)^{n-1}\left(\frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)^2}\right).$$

It follows from Theorem 3.1 that

$$P_t(\mathcal{N}^0_\alpha) = P_t(\mathcal{M}^0_\alpha) + \sum_{d,e} t^{\lambda_{d,e}} P_t(\mathcal{M}_{d,e}),$$

where the sum is taken over (d,e) satisfying (4), which, for our choice of α, is simply $e_1 - |e| \le d \le [g-1-\frac{|e|}{2}]$, where $[x]$ is the greatest integer less than x. Setting $j = 2d+n+|e|-1$, then j satisfies:

$$n+2e_1-|e|-1 \le j \le 2g+n-3 \quad \text{and} \quad j-n-|e|+1 \text{ is even.}$$

Also $\lambda_{d,e} = 2(g+j)$ and $h_{d,e} = 2g+n-j-3$.

Fixing e_1 and $|e|$, for each d, there are $\binom{n-1}{|e|-e_1}$ strata given by the choice of e. Thus, for each j, there are $q_j = \sum_{i=0}^{j}\binom{n-1}{i}$ strata (note that $q_j = 2^{n-1}$ for $j \ge n-1$) and we see

$$\begin{aligned}
\sum_{d,e} t^{\lambda_{d,e}} P_t(\mathcal{M}_{d,e}) &= \sum_{|e|=0}^{n}\binom{n-1}{|e|-e_1}\sum_{d=e_1-|e|}^{[g-1-|e|/2]} t^{\lambda_{d,e}} P_t(\tilde{S}^{h_{d,e}}X) \\
&= \sum_{j=0}^{2g+n-3} q_j t^{2(g+j)} P_t(\tilde{S}^{2g+n-j-3}X) \\
&= \sum_{j=0}^{n-2} q_j t^{2(g+j)} P_t(\tilde{S}^{2g+n-j-3}X) + \sum_{j=0}^{2g-2} 2^{n-1} t^{2(g+n+j-1)} P_t(\tilde{S}^{2g-j-2}X).
\end{aligned}$$

We refer to the last two sums by $\tilde{S}_1$ and $\tilde{S}_2$. Using the Binomial Theorem and the general formula (p. 98 of [13]) $\mathrm{P}_t(\tilde{S}^h X) = (2^{2g}-1)\binom{2g-2}{h} t^h + \mathrm{P}_t(S^h X)$, we see that

$$\begin{aligned}
\tilde{S}_1 &= \sum_{j=0}^{n-2} q_j t^{2(g+j)} P_t(S^{2g+n-j-3}X) = S_1, \\
\tilde{S}_2 &= \sum_{j=0}^{2g-2} 2^{n-1} t^{2(g+n+j-1)} P_t(S^{2g-j-2}X) + \sum_{j=0}^{2g-2} 2^{n-1}(2^{2g}-1)\binom{2g-2}{j} t^{4g+2n+j-4} \\
&= S_2 + 2^{n-1}(2^{2g}-1)t^{2(2g+n-2)}(1+t)^{2g-2},
\end{aligned}$$

where S_1 and S_2 are the sums obtained by removing the tildes from the summands of $\widetilde{S}_1$ and $\widetilde{S}_2$. According to a result of [16], $P_t(S^hX)$ is the coefficient of x^h in

$$\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)}.$$

This allows us to evaluate S_i as follows:

$$\begin{aligned} S_1 &= \operatorname*{Res}_{x=0}\left(\sum_{j=0}^{n-2}\frac{q_j t^{2(g+j)}(1+xt)^{2g}}{x^{2g+n-j-2}(1-x)(1-xt^2)}\right),\\ S_2 &= \operatorname*{Res}_{x=0}\left(\frac{2^{n-1}t^{2(g+n-1)}(1+xt)^{2g}}{x^{2g-1}(1-x)(1-xt^2)^2}\right).\end{aligned}$$

But each of these rational functions is analytic at $x=\infty$, so we can use the Cauchy Residue Formula to evaluate instead at the poles $x=1$ and $x=1/t^2$. Letting $Q_n(t)=\sum_{k=0}^{n-2}q_kt^{2k}$ and noticing that $Q_n(1)=\sum_{k=0}^{n-2}q_k=2^{n-2}(n-1)$, we get

$$\begin{aligned} S_1 &= \left(Q_n(t)t^{2g}-2^{n-2}(n-1)t^{2(2g+n-2)}\right)\frac{(1+t)^{2g}}{(1-t^2)},\\ S_2 &= 2^{n-1}\left(t^{2(g+n-1)}+t^{4g+2n-3}((2g-1)t-2g)\right)\frac{(1+t)^{2g}}{(1-t^2)^2}.\end{aligned}$$

But since $Q_n(t)(1-t^2)+2^{n-1}t^{2(n-1)}=(1+t^2)^{n-1}$, it follows that

$$\begin{aligned} P_t(\mathcal{N}_\alpha^0) &= P_t(\mathcal{M}_\alpha^0)+\widetilde{S}_1+\widetilde{S}_2\\ &= P_t(\mathcal{M}_\alpha^0)+S_1+S_2+2^{n-1}(2^{2g}-1)t^{2(2g+n-2)}(1+t)^{2g-2}\\ &= \frac{(1+t^3)^{2g}(1+t^2)^{n-1}+2^{n-1}t^{2n+4g-3}(1+t)^{2g}[(2g-1)t-2g]}{(1-t^2)^2}\\ &\quad -\frac{2^{n-2}(n-1)t^{2n+4g-4}(1+t)^{2g}}{1-t^2}+2^{n-1}(2^{2g}-1)t^{4g+2n-4}(1+t)^{2g-2}.\end{aligned}$$

Evaluating this at $t=-1$ shows that the Euler characteristic of $\mathcal{N}_\alpha^0$ is given by

$$\chi(\mathcal{N}_\alpha^0)=\begin{cases}(n-1)(n-2)2^{n-4} & \text{if } g=0,\\ 3\cdot 2^n & \text{if } g=1,\\ 0 & \text{if } g\geq 2\end{cases}$$

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