RESOLUTIONS OF ORBIFOLD SINGULARITIES AND REPRESENTATIONS OF THE MCKAY QUIVER

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0. INTRODUCTION

This talk deals with certain moduli spaces X_ζ (depending on a parameter ζ) which give partial resolutions of orbifold singularities (of the form $\mathbb{C}^n/\Gamma$ for a finite subgroup Γ of $GL(n)$). These were defined and studied in [SI94], as generalisations of Kronheimer's hyper-Kähler gravitational instantons [Kro89] to dimensions higher than 2. They can be viewed alternatively [SI94] as moduli spaces of stable bundles, as representations moduli of quivers, or as Kähler quotients. For subgroups of SU(2), Kronheimer [Kro89] showed that they give the minimal resolution of the singularity when ζ is generic.[1] For subgroups of SU(3), some evidence exists [SI94] that they coincide with the smooth crepant resolutions of $\mathbb{C}^3/\Gamma$ whose existence was predicted by Witten et al. [DHVW85, DHVW86], and established in the last decade by Markusevich, Roan and Itō [MOP87, Roa90, Mar93, Roa93, Itō94, Roa94].

In this talk I make a further contribution to the two-dimensional case by showing that Kronheimer's result also holds when Γ is a cyclic subgroup of GL(2):

Theorem 1. *The moduli spaces X_ζ coincide, for generic values of ζ, with the minimal resolution of $\mathbb{C}^2/\Gamma$, for the case where Γ is a cyclic subgroup of* GL(2).

I will rely heavily on [SI94] for background and for the theorems describing the moduli X_ζ. The plan of this paper is as follows. In §1 I make some preliminary remarks about geometric invariant theory (GIT) quotients. These enter in an essential way in the contruction of the moduli X_ζ, which is given in §2. In §3, I state a theorem which describes them in terms of toric geometry[2] in the case where Γ is a cyclic group. Finally, in §4 I sketch the proof of theorem 1.

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[1]Kronheimer actually shows a lot more than this.

[2]My version of the toric notation is given in Appendix A.

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1. VARIATION OF AFFINE GIT QUOTIENTS

This appendix contains some elementary remarks about Geometric Invariant Theory (GIT) quotients of affine varieties, and the way they depend on the linearisation.[3]

Recall that if X is an affine variety, $R = k[X]$ is its ring of regular functions, and G a reductive group acting linearly on X, the affine GIT quotient of X by G is the variety whose ring of regular functions is the G-invariant subring of R:

$$X /\!\!/ G = \operatorname{Spec} R^G.$$

More generally, if ζ is a character $\zeta \colon G \to \mathbb{C}^*$ of G (i.e. a linearisation of the trivial bundle over X), then the GIT quotient of X by G with respect to ζ is

$$X /\!\!/_\zeta G = \operatorname{Proj} R[z_\zeta]^G,$$

where G acts on the complex variable z_ζ via the character ζ.

Note that if ζ is the trivial linearisation (denoted by $0 \colon G \to 1$) the G-invariant part of $R[z_0]$ is nothing but $R^G[z_0]$ so

$$X /\!\!/_0 G = \operatorname{Proj} R[z_0]^G = \operatorname{Proj} R^G[z_0] = \operatorname{Spec} R^G = X /\!\!/ G.$$

For general ζ, grading $R[z_\zeta]$ by the powers of the variable z_ζ, the degree-zero part of $R[z_\zeta]^G$ is equal to R^G. This shows [Har77, Ex. II.4.8.1 and Cor. II.5.16] that $X /\!\!/_\zeta G$ is projective over $X /\!\!/ G$, and in particular [Har77, Theorem II.4.9] that the map $\rho_\zeta \colon X /\!\!/_\zeta G \to X /\!\!/_0 G$ is proper. This map is induced from the inclusion of the corresponding semi-stable sets, and is an isomorphism on the equivalence classes of stable points. In general, the semi-stable set $X^{ss}(\zeta)$ will be Zariski-open in X, so if X is irreducible, it will be dense if non-empty. If this is the case, ρ_ζ is an epimorphism which is one-one over the open set of stable points.

The results are summarized in the following theorem.

Theorem 2. *Let G be a reductive group acting linearly on an affine variety X. Then there are projective epimorphisms $X /\!\!/_\zeta G \to X /\!\!/_0 G$ which are isomorphisms on the equivalence classes of stable points.*

2. CONSTRUCTION OF THE MODULI X_ζ

In this section, I recall the definition and general properties of the moduli X_ζ [SI94]. Throughout the talk, Γ denotes a finite group of order r acting on $Q \cong \mathbb{C}^n$ via $\Gamma \subset \operatorname{GL}(n)$. The (left) regular representation of Γ is denoted $\phi_R \colon \Gamma \to \operatorname{Aut} R$

$$R = \operatorname{span}_\mathbb{C}(e_\gamma | \gamma \in \Gamma \text{ with action } \phi_R(\gamma)e_\delta := e_{\gamma\delta}).$$

Define $\mathcal{M} := Q \otimes_\mathbb{C} \operatorname{End}_\mathbb{C} R$. Let Γ act on both factors (on $\operatorname{End}_\mathbb{C} R$ by conjugation) and denote by $\mathcal{M}^\Gamma$ the invariant subspace.

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[3]The way general GIT quotients vary is studied in [GS89, Tha93, DH94].

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Choosing a basis $\{q_i\}$ for Q, an element $\alpha \in \mathcal{M}$ is represented by an n-tuple $(\alpha_1, \ldots, \alpha_n)$ of endomorphisms of R which satisfy $\alpha = \sum_i q_i \otimes \alpha_i$. An element α belongs to the invariant subspace $\mathcal{M}^\Gamma$ iff its components satisfy the equivariance condition

$$(2.1) \qquad \sum_l \phi_Q(\gamma)_{kl}\alpha_l = \phi_R(\gamma)\alpha_k\phi_R(\gamma)^{-1}, \qquad \forall k,$$

where $\phi_Q(\gamma)$ is the $n \times n$ matrix which gives the action of the element γ on Q with respect to the basis $\{q_i\}$.

The group $\mathrm{GL}(R)$ acts on $\mathcal{M}$, (trivially on Q, by conjugation on $\mathrm{End}\,R$):

$$(2.2) \qquad \alpha_i \mapsto g\alpha_i g^{-1}, \qquad g \in \mathrm{GL}(R).$$

The subgroup $\mathbb{C}^* \subset \mathrm{GL}(R)$ acts trivially, and one obtains a free action of $G := \mathrm{GL}(R)/\mathbb{C}^*$. The commutator subgroup $\mathrm{GL}^\Gamma(R)$ (i.e. the endomorphisms of R which commute with the action of Γ) acts on $\mathcal{M}^\Gamma$ and induces a free action of $G^\Gamma := \mathrm{GL}^\Gamma(R)/\mathbb{C}^*$.

Define

$$(2.3) \qquad \mathcal{N} := \{\alpha \in \mathcal{M} \,|\, [\alpha_i, \alpha_j] = 0\}.$$

The points of $\mathcal{N}$ (resp. $\mathcal{N}^\Gamma = \mathcal{M}^\Gamma \cap \mathcal{N}$) consist of n-tuples (resp. equivariant n-tuples) of commuting endomorphisms, and is acted upon by G (resp. G^Γ).

Denoting by Z the space of characters of G^Γ,[4] one can form the quotients

$$X_\zeta := \mathcal{N}^\Gamma /\!\!/_\zeta G^\Gamma, \quad \text{for } \zeta \in Z.$$

Remark. One can identify $\mathcal{N}^\Gamma$ with complex structures on $Q^* \times R \to Q^*$ which are invariant under translations on the base and under the action of Γ. In this way, X_ζ can be viewed as moduli of (very special) stable bundles [SI94, Chapter IV].

Proposition 1 ([SI94, Thm.3.4]). *If Γ acts freely outside the origin, then $X_0 \cong Q/\Gamma$ as varieties.*

It turns out that the only non-stable class in X_0 is the origin, so applying theorem 2 gives the following corollary:

Corollary 1 (Partial Resolutions). *When Γ acts freely outside the origin, the quotients X_ζ are partial resolutions of the isolated singularity Q/Γ, i.e. they admit proper birational maps to Q/Γ which are isomorphisms outside the origin in Q/Γ.*

Remark. Other general properties of X_ζ (identification with Kähler quotients, existence of ALE metrics) are studied in [SI94, Chapter IV].

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[4]Z is an integer lattice of rank equal to the number of irreducible representations of Γ minus one.

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3. Moduli for Abelian groups and flows on the McKay quiver

A *quiver* is an oriented graph, possibly with multiple arrows and loops. A *representation of a quiver* is a realization of its diagram of vertices and arrows in some category: it corresponds to replacing the vertices by objects (e.g. vector spaces) and the arrows by morphisms between the objects. If we require the morphisms to satisfy certain relations between them, we say we have a *representation of a quiver with relations*. There is a natural notion of isomorphism of two representations, and one can thus form *representation moduli*: many problems in linear algebra[5] can then be expressed as representation moduli problems for quivers.

The only quiver I shall be concerned with is the *McKay quiver* $\mathcal{Q}_{\Gamma,Q}$, which is a quiver naturally associated to a representation Q of a finite group Γ. The spaces X_ζ can be viewed as representations moduli of the McKay quiver subject to certain relations [SI94, Chapter III]. The details will not concern us; it suffices to know that, applied to the case of abelian groups, this relates X_ζ (via toric geometry) to an n-dimensional convex polyhedron C_ζ. It turns out that this polyhedron is closely related to a classical linear programming problem which is known to network optimization theorists as the "transportation problem". In order to state the results, I will need to review some notions regarding the McKay quiver, and regarding flows in networks.

3.1. The McKay Quiver.
From now on, I write $\mathcal{Q}$ for the McKay quiver $\mathcal{Q}_{\Gamma,Q}$. Its vertices $\mathcal{Q}_0$ label the irreducible representations $R_0, \ldots, R_{r-1}$ of Γ and the arrows describe how the tensor product of Q with each irreducible decomposes into a sum of irreducibles. Specifically, there are a_{ij} arrows from the vertex i to the vertex j, where a_{ij} is the non-negative integer appearing in the following decomposition into irreducibles

$$(3.1) \qquad Q \otimes R_i = \bigoplus_j a_{ji} R_j.$$

In other words, the multiplicities (a_{ij}) form the *adjacency matrix* of $\mathcal{Q}$.

3.2. The Abelian Case.
In the abelian case, there are $r = |\Gamma|$ irreducible representations $R_0, \ldots, R_{r-1}$ of Γ, all of dimension one. I therefore identify the vertex set of $\mathcal{Q}$ with $\widehat{\Gamma}$, the dual group to Γ. I always identify Γ with a subgroup of $\mathbb{C}^{*n} \subset \mathrm{GL}(n)$, and therefore $\widehat{\Gamma}$ with a product of finite groups $\mathbb{Z}_{r_i}$. The group operation is denoted additively. Let us see an example.

Let $\Gamma = \mu_5 \subset \mathbb{C}^*$ be the group of 5-th roots of unity, acting on $\mathbb{C}^3$ with weights $w_1 = 1$, $w_2 = 2$ and $w_3 = 3$ (this action is denoted symbolically by $\frac{1}{5}(1,2,3)$). The character group $\widehat{\Gamma}$ is the additive group $\mathbb{Z}_5$ of integers modulo 5 and there are arrows from v to $v - w$ (mod 5) for $v \in \{0, \ldots, 4\}$ and $w \in \{1, 2, 3\}$. The quiver looks like that in figure 1 below:

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[5]For instance, the so called "four-subspace problem" [Rin80].

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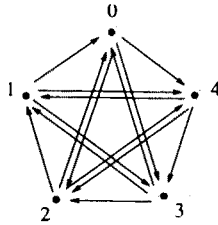
FIGURE 1. The McKay quiver for the action $\frac{1}{5}(1,2,3)$.

In general, the McKay quiver for a cyclic group μ_r acting with weights $w_1, \ldots, w_n$ is similar. There are r vertices $\{0, \ldots, r-1\}$ and nr arrows of the form $a_v^i := v \to v - w_i$ (mod r) for $v \in \{0, \ldots, r-1\}$ and $i = 1, \ldots, n$.

The McKay quiver for a cyclic group thus comes equipped with an extra structure in which each arrow is labelled by the (index of the) weight which corresponds to it; the arrows a_v^i for $v \in \{0, \ldots, r-1\}$ will be called, for short, *arrows of weight i*.

3.3. The Moduli X_ζ as representation moduli. I mentioned earlier that X_ζ could be identified with representation moduli[6] of $\mathcal{Q}_\Gamma$, and that this leads to a complete description of X_ζ in the case where Γ is abelian. I will outline the main steps of this identification below, and state the theorem which describes X_ζ for the simpler case of cyclic groups.

Firstly, the space $\mathcal{M}^\Gamma$ can be identified with the space $\mathbb{C}^{\mathcal{Q}_1} = \mathrm{Map}(\mathcal{Q}_1, \mathbb{C})$. In other words, an element α corresponds uniquely to an assignment $\tilde{\alpha}$ of complex numbers to the arrows of the McKay quiver. The subvariety $\mathcal{N}^\Gamma$ corresponds to assignments which satisfy certain quadratic relations between the arrows of the form

$$(3.2) \qquad \tilde{\alpha}(a_v^i)\tilde{\alpha}(a_{v-w_i}^j) - \tilde{\alpha}(a_v^j)\tilde{\alpha}(a_{v-w_j}^i) = 0, \quad \text{for all } v, i, j.$$

One might suspect at this point that the description of X_ζ will involve these quadratic relations. The surprising thing is that it doesn't.

For a finite set A, denote by $\mathbb{Q}^A$ the set of maps $A \to \mathbb{Q}$. This is a vectorspace with a natural basis given by the characteristic functions χ_a ($a \in A$) which take the value 1 on the element a and zero elsewhere.

In the quiver picture, the character group of $\mathrm{GL}^\Gamma(R)$ corresponds to the lattice $\mathbb{Z}^{\mathcal{Q}_0}$. The character group of G^Γ corresponds to the sublattice $\mathbb{Z}_0^{\mathcal{Q}_0} := \{\zeta \in \mathbb{Z}^{\mathcal{Q}_0} : \sum_{v \in \mathcal{Q}_0} \zeta(v) = 0\}$.

The moduli X_ζ correspond to toric varieties defined by a polyhedron $C_\zeta \subseteq \mathbb{Q}^n$, which is obtained from the McKay quiver and the value of ζ by a generalisation of the classical "transportation problem" familiar to network optimization and linear programming specialists. The important point is that it can be calculated without

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[6]Actually, one has to take representations which satisfy certain relations.

any reference whatsoever to equations (3.2). In order to describe C_ζ, I need to introduce the basic concepts of network flow theory.

3.4. Flows in Networks. Any quiver has a natural map $\partial \colon \mathbb{Q}^{\mathcal{Q}_1} \to \mathbb{Q}^{\mathcal{Q}_0}$, defined by

$$\partial(f)(v) := \sum_{a \in \mathrm{In}(v)} f(a) - \sum_{v \in \mathrm{Out}(v)} f(a).$$

This can be thought of as follows: imagine that the vertices $\mathcal{Q}_0$ represent the location where a certain comodity is produced or consumed, and the arrows between them the itineraries by which the commodity can be transported. For any flow f of commodities (i.e. assignment of integers to the arrows $\mathcal{Q}_1$) the quantity ∂f represents the net excess of goods which results at each given location (i.e. vertex).

Given an element $\zeta \in \mathbb{Z}^{\mathcal{Q}_0}$ one can ask which non-negative flows f satisfy the equation $\partial f = \zeta$. The answer is a convex polyhedron F_ζ called the solution polyhedron for the transportation problem on $\mathcal{Q}$.[7] The polyhedron C_ζ is the projection of this polyhedron by the map $\pi \colon \mathbb{Q}^{\mathcal{Q}_1} \to \mathbb{Q}^n$ which sends the basis vectors corresponding to the arrows of weight i ($\chi_{a_v^i}, v \in \mathcal{Q}_0$) to the i-th basis vector e_i of $\mathbb{Q}^n$. We will call C_ζ the ζ-solution polyhedron for the generalised transportation problem for $(\mathcal{Q}, \pi)$.

The theorem which describes X_ζ precisely is the following one:

Theorem 3. *Let Γ be the cyclic group of order r acting on Q with weights $(w_1, \dots, w_n)$. Then the character group Z of G^Γ is given by $\mathbb{Z}_0^{\mathcal{Q}_0}$. For any $\zeta \in \mathbb{Z}_0^{\mathcal{Q}_0}$, the moduli space X_ζ is the toric variety $\overline{T}^{M, C_\zeta}$, where M is the sublattice of $\mathbb{Z}^n$ defined by*

$$M := \{ m \in \mathbb{Z}^n \mid \frac{1}{r}(w_1, \dots, w_n) \cdot m \in \mathbb{Z} \},$$

and $C_\zeta = \pi(F_\zeta)$ is the ζ-solution polyhedron for the generalised transportation problem on the McKay quiver.

Remark. In Chapter V of [SI94] the polyhedra C_ζ are described in more detail. In particular, a recipe is given to determine the number of extreme points and singularities of *all* the C_ζ at once.

The polyhedron C_ζ for the value $\zeta = (-1, -1, -1, -1, 4)$ is shown in figure 2. The corresponding trees, flows and coordinates of the extreme points appear in figure 3.

It is immediately apparent from figure 2 that X_ζ has a singularity at the extreme point $(1, 3, 1)$: the tangent cone there has four generators. The other extreme points are the intersection of three faces: in order to determine whether they are in fact singular or not one must check whether the primitive generators in $M \subset \mathbb{Z}^3$ of the tangent cone actually generate M. In this case it turns out that they do, so they are smooth points. The moduli space X_ζ in this case is therefore a partial resolution, with a remaining singularity of the type "cone over a rational double point".

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[7]For obvious "conservation" reasons, it is non-empty only if $\sum_{v \in \mathcal{Q}_0} \zeta(v) = 0$.
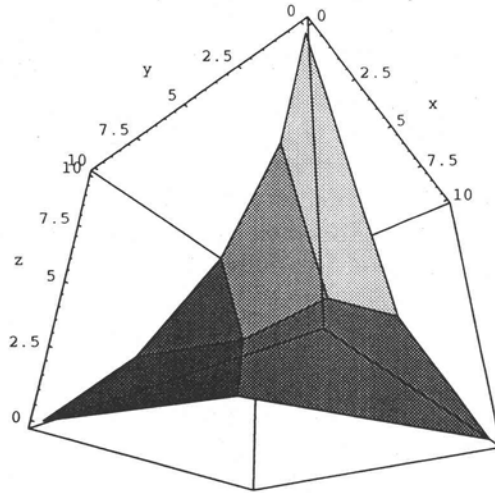
FIGURE 2. C_ζ for $\frac{1}{5}(1,2,3)$, $\zeta = (-1,-1,-1,-1,4)$, giving a partial resolution with a remaining ordinary double point singularity.
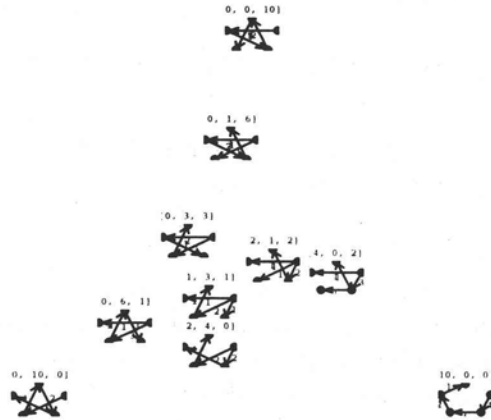

FIGURE 3. Extreme flows for $\frac{1}{5}(1,2,3)$, $\zeta = (-1,-1,-1,-1,4)$.

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The moduli space X_ζ in figure 4 on the other hand is easily checked to be non-singular: it is a full resolution of the singularity $\frac{1}{5}(1,2,3)$.
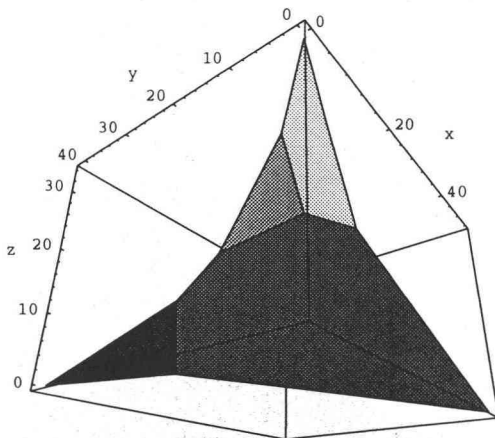

FIGURE 4. C_ζ for $\frac{1}{5}(1,2,3)$, $\zeta = (9,8,-3,-2,-12)$, giving a smooth resolution.
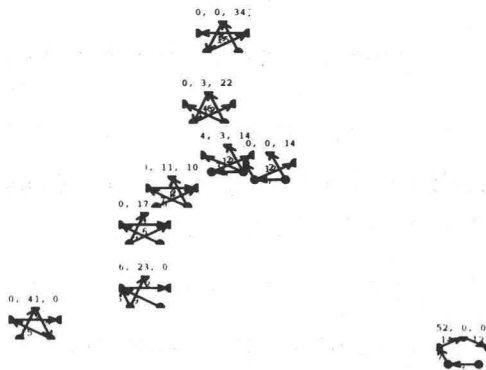

FIGURE 5. Extreme flows for $\frac{1}{5}(1,2,3)$, $\zeta = (9,8,-3,-2,-12)$.

Another example (this time for $\Gamma \subset SU(3)$) is shown in figure 6. In this case, the polyhedron is again non-singular. In fact, using a computational method given

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in [SI94, Chapter V], one shows that, for this action, X_ζ is non-singular (and further-more crepant) for any generic value of ζ.

$$\{\{\tfrac{1}{11}, \tfrac{4}{11}, \tfrac{6}{11}\}, \{-9, 2, 2, -13, 4, 4, 2, 2, 2, 2, 2\}\}$$
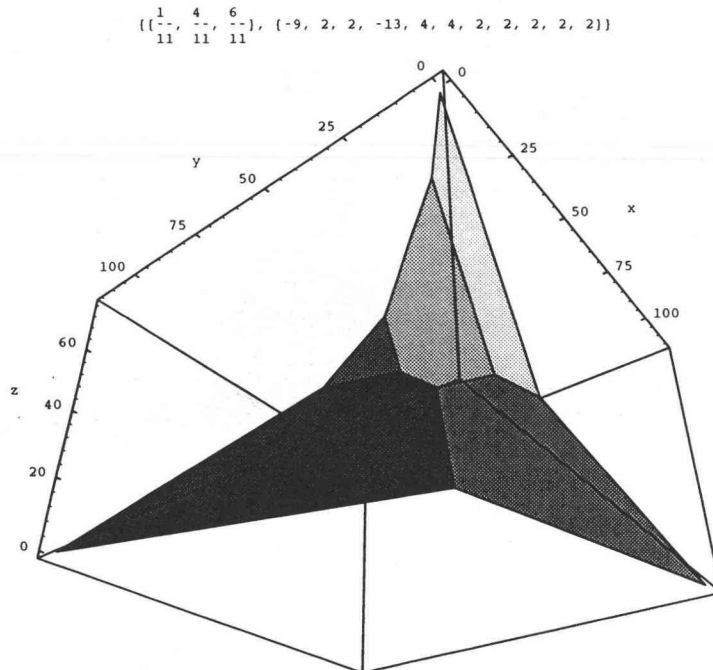

FIGURE 6. Smooth Crepant Resolution for $\frac{1}{11}(1,4,6)$

4. CYCLIC SUBGROUPS OF GL(2)

4.1. Orbifold Singularities. Let $\Gamma = \mu_r$, the cyclic group of order r, acting on $\mathbb{C}^n$ diagonally with weights $w_1, \ldots, w_n$. Let

$$M := \{m \in \mathbb{Z}^n | \tfrac{1}{r}(w_1, \ldots, w_n) \cdot m \in \mathbb{Z}\},$$

and denote by C_0 the first quadrant in $\mathbb{Q}^n = M_\mathbb{Q}$[8]. Dual to these are

$$N = \mathbb{Z}^{*n} + \mathbb{Z}\tfrac{1}{r}(w_1, \ldots, w_n),$$

and the first quadrant $\sigma = C_0^\vee$ in $\mathbb{Q}^{*n} = N_\mathbb{Q}$ (see Appendix A for the notation).

The moduli space X_0 is given in toric notation by $X_0 = \overline{T}^{M,C_0} = \overline{T}_{N,\sigma}$. This is of course isomorphic to the quotient $\mathbb{C}^n/\Gamma$.

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[8]Note that this is compatible with the definition of C_ζ given previously.

4.2. Minimal Resolutions in Dimension 2. In two dimensions, the objects above can be related as follows. Let $L\colon \mathbb{Q}^2 \to \mathbb{Q}^2$ be given r times the rotation of the plane by $\pi/2$, i.e. $L(x,y) = (-ry, rx)$. Then, identifying $\mathbb{Z}^2$ with $\mathbb{Z}^{2*}$ in the standard way, we have

$$M = L(N) \subset \mathbb{Z}^2 \subset N \subset \mathbb{Q}^2.$$

The minimal resolution of $\mathbb{C}^2/\Gamma$ is obtained by subdividing σ in the following way [Oda88, Prop. 1.19]. Let $\Theta \subset \sigma$ denote the convex hull of $\sigma \cap N \setminus \{0\}$. Denote by $\delta\Theta$ the boundary of Θ. Imagine traveling along $\delta\Theta$ from $(0,+\infty)$ to $(+\infty,0)$ and denote by $l_0, l_1, \ldots, l_{s+1}$ the lattice points of N encountered along the way, in that order (thus $\operatorname{ext}\Theta = \{l_0, \ldots, l_{s+1}\}$, with the points indexed according to their order along $\delta\Theta$). Let Σ denote the fan obtained by subdividing σ by the rays determined by the vectors l_i for $i = 1$ to s.[9] Then the minimal resolution of $X = \mathbb{C}^2/\Gamma$ is

$$\widetilde{X} = \overline{T}_{N,\Sigma}.$$

FIGURE 7. The minimal resolution of $\frac{1}{5}(1,2)$ constructed by toric means.

Using the isomorphism L, one can translate the statement above into the following lemma.

Lemma 1. *Let $C_0' = \mathbb{Q}_{\leq 0} \times \mathbb{Q}_{\geq 0}$ denote the second quadrant in $\mathbb{Q}^2$, and let Θ' denote the convex hull of $C_0' \cap M \setminus \{0\}$. Let $v_i := L(l_i)$ for $i = 0, \ldots, s+1$. Then $v_0, \ldots, v_{s+1}$ are the lattice points of M one encounters as one travels along the boundary $\delta\Theta'$ of Θ' from $(-\infty, 0)$ to $(0, +\infty)$.*

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[9]The rays corresponding to l_0 and l_{s+1} need not be considered since they are nothing but the generators of σ.

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Proof. Follows form the fact that C_0, N and Θ get mapped respectively to C_0', M and Θ' by L. $\square$
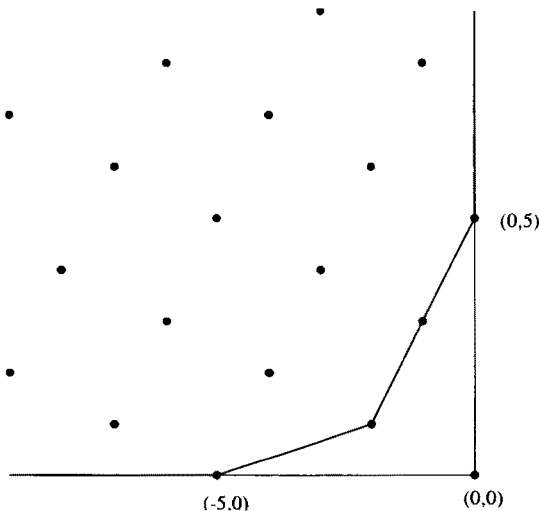

FIGURE 8. A picture of Θ' for the minimal resolution of $\frac{1}{5}(1,2)$.

4.3. Proving that X_ζ is the minimal resolutions for generic ζ. Let Σ_ζ denote the fan determined by the polyhedron $C_\zeta \subset M_\mathbb{Q}$. In dimension 2, the fan Σ_ζ is described by the following lemma, whose proof is again obvious.

Lemma 2. *Let $v_0^\zeta, \ldots, v_{t+1}^\zeta$ denote the primitive vectors (in M), representing the directions of the edges of the boundary of C_ζ as one travels along it from $(+\infty, 0)$ to $(0, +\infty)$ (note that this orientation is opposite to the orientation of $\delta\Theta$). Then Σ_ζ is the subdivision of σ by the rays corresponding to the vectors $l_i^\zeta := L^{-1}(v_i^\zeta) \in N$.*

The following proposition implies Theorem 1 and also the fact that the X_ζ are (partial) toric blowups of $\mathbb{C}^2/\Gamma$ for any ζ.

Proposition 2. *Let $n = 2$ and let Σ denote the fan corresponding to the minimal resolution of $\mathbb{C}^2/\Gamma$, as described above. Then, for any ζ, the fan Σ is a subdivision of Σ_ζ and the two fans coincide for generic ζ.*

Since it is slightly messy, a full proof of the proposition will be postponed till a coming paper; I will content myself here with a brief sketch.

By lemmas 1 and 2, it is sufficient to prove that

(1) the set $\{v_0^\zeta, \ldots, v_{t+1}^\zeta\}$ is a subset of $\text{ext}\,\Theta' = \{v_0, \ldots, v_{s+1}\}$ and,

(2) the two sets coincide for generic ζ.

4.3.1. *Sketch proof of (1) — Part 1.* Orient the boundary of C_ζ from $(+\infty, 0)$ to $(0, +\infty)$, and let v be a generator for one the boundary segments, chosen so that v is primitive in M. Because C_ζ is convex and contained in C_0, we have $v \in C_0' \setminus \{0\}$. Hence, it only remains to show that $v \in \text{ext}\,\Theta'$.

For this, recall that C_ζ is the image of $F_\zeta \subset \mathbb{Q}_{\geq 0}^{Q_1}$ under the projection $\pi \colon \mathbb{Q}^{Q_1} \to \mathbb{Q}^2$. Suppose that x is the extreme point of C_ζ out of which v emanates and let $f \in F_\zeta$ be any flow such that $\pi(f) = x$ (and $\partial f = \zeta$). Then v must be the image under π of a flow g satisfying $\partial g = 0$ (and which must be non-negative outside supp f). At this point, I will pause to explain the notion of a simple flow. The rest of the proof will consist in proving that g must be simple, and this will imply that $v \in \text{ext}\,\Theta'$.

4.3.2. *Simple flows.* Any flow g such that $\partial g = 0$ must have a support which is a union of cycles. More precisely, it is possible to decompose g into a sum of non-zero flows g^i which satisfy $\partial g^i = 0$.

Definition. Two flows h, h' are called *conformal* if they do not take values of opposite signs[10] on any arrow.

If one insists in a *conformal decomposition*, namely one in which the g^i are conformal to g, then one obtains a finite number of flows g^i which cannot be decomposed (conformally) any further. We call the resulting flows *simple* flows. There may in general be *many* conformal decompositions of g into simple flows, but that will not concern us.

4.3.3. *Sketch Proof of (1) — Part 2.*

Claim. The flow g is simple.

Proof. Decompose g into a positive integral linear combination of conformal non-zero simple flows g_i such that $\partial g^i = 0$. Since all the g_i can be added to f without making it negative, F_ζ contains the convex polytope

$$\{f + \sum \lambda_i g_i | 0 \leq \lambda_i \leq 1\}.$$

Applying π and writing $u_i := \pi(g_i)$, one deduces that C_ζ must contain the convex polytope

$$\{x + \sum \lambda_i u_i | 0 \leq \lambda_i \leq 1\},$$

which itself contains the element $x + v$. If all the u_i are not equal, then the inclusion above would imply that $x + v$ is in the interior of C_ζ, contradicting the assumption that v is an edge of C_ζ. Hence all the simple flows appearing in the decomposition of g have the same image — say u — under π and v must be a multiple of u. But since v is assumed to be a primitive vector, this multiple must be one and there can be only one such simple flow. Hence the flow g is itself was already simple. $\square$

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[10]The sign of zero is considered neutral, i.e. not opposite to anything.

The proof of (1) is now completed by the following lemma.

Lemma 3. *Let g be a simple flow such that $\partial g = 0$ and $\pi(g) \in C_0'$. Then $\pi(g) \in \text{ext}\,\Theta'$.*

Proof. (Sketch) Suppose that $v^0 := \pi(g) \notin \text{ext}\,\Theta'$. Since two adjacent lattice points of M on the boundary of Θ' form a basis of M, there exist v^1 and v^2 in $\text{ext}\,\Theta'$ such that $v^0 = v^1 + v^2$. We know that v^0, v^1, v^2 belong to the lattice M. There must be a flow g^1 which is conformal to g and such that $\pi(g^1) = v^1$. Thus g admits a conformal decomposition, so cannot be simple. $\square$

This completes the sketch of the proof of (1).

4.3.4. *Sketch Proof of (2).* Let x be an extreme point of C_ζ, where the ingoing edge of δC_ζ (i.e. to the left of x) is v^i and the outgoing edge (to the right of x) is some vector $v \in \text{ext}\,\Theta'$. We show that if v is anything but v^{i+1}, then ζ is not generic.

Let $f \in F_\zeta$ be a representative for x, i.e. a flow such that $\pi(f) = x$. This f can be chosen so that $\text{supp}\,f$ forms a spanning tree in the McKay quiver by [SI94, Lemma V.3.24]. Since x is an extreme point of C_ζ, there exists a simple flow g^i such that $\partial g^i = 0$, $\pi(g^i) = v^i$ and $f + g^i$ is not admissible. This means that $f(a) = 0$ for some arrow a on which g^i is negative. In fact, f must vanish on exactly one such arrow, because if not, some component of ζ must be zero, contradicting the genericity assumption. The flow $f + g^i$ then, is negative on exactly one arrow. Therefore there exists a simple flow g such that $\pi(f + g) \in \delta C_\zeta$. But the fact that g is simple again
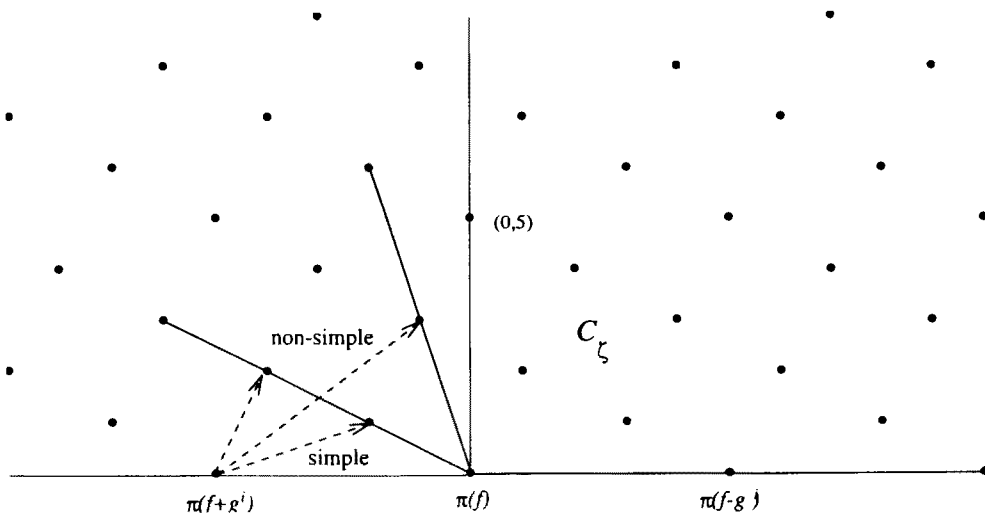

FIGURE 9. The situation at the first extreme point of C_ζ for the case $\frac{1}{5}(1,2)$.

forces $\pi(g)$ to coincide with v^{i+1} (see figure 9). $\square$

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Appendix A. Toric Varieties

Toric varieties provide a dictionary between compactifications of n-dimensional complex algebraic tori and the convex geometry of polyhedral sets in $\mathbb{Q}^n$. There are two dual approaches to studying toric varieties and many different notations. The purpose of this appendix is to give an introduction in notation which I believe is convenient (it's also largely compatible with the notation in [Oda88]). Standard references for proofs and further details are the survey article [Dan78], the introductory book [Ful93], or the references [Oda88, KKMSD73].

Let M denote an integer lattice of rank n and N its dual lattice $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.

A.1. Affine Toric Varieties.
Every complex algebraic torus has a group of characters which is an integer lattice M. The torus can be written as

$$T^M := \mathrm{Spec}\,\mathbb{C}[M],$$

which, as a set of (closed) points, can be thought of as the group homomorphisms $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$. For convenience, I also introduce the dual notation $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^* = T^M$.

Let S be a finitely generated semi-group that generates the lattice M as a group. Then its group algebra $\mathbb{C}[S]$ is finitely generated and the inclusion $\mathbb{C}[S] \hookrightarrow \mathbb{C}[M]$ induces an equivariant affine embedding of the torus T^M:

$$(A.1) \qquad\qquad T^M \hookrightarrow \overline{T}^S := \mathrm{Spec}\,\mathbb{C}[S].$$

In fact, since S generates M, T^M acts freely on $\overline{T}^S$ with a dense orbit, i.e. $\overline{T}^S$ is an equivariant compactification of T^M. A variety is called a *toric variety* if it has an action of a complex algebraic torus with a dense orbit; the variety $\overline{T}^S$ is called the *affine toric variety* determined by S. Thus one has a correspondence

$$\begin{array}{ccc} \text{Finitely Generated Semi-Groups} & \to & \text{(Abstract) Affine Toric Varieties} \\ S & \mapsto & \overline{T}^S. \end{array}$$

The (closed) points of the scheme $\mathrm{Spec}\,\mathbb{C}[S]$ correspond to maximal ideals of $\mathbb{C}[S]$, and hence to (kernels of) $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[S] \to \mathbb{C}$. Any such morphism is induced by a semi-group homomorphism from $(S, +)$ to the multiplicative semi-group of complex numbers $(\mathbb{C}, \cdot)$. Thus, on the point level,

$$\mathrm{Spec}\,\mathbb{C}[S] = \mathrm{Hom}_{\text{s-group}}(S, \mathbb{C}).$$

Giving generators $m_1, \ldots, m_r$ for the semi-group S subject to relations

$$\sum_j a_{ij} m_j = \sum_j b_{ij} m_j, \quad i = 1, \ldots, k$$

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for $a_{ij}, b_{ij} \in \mathbb{Z}_+$ identifies the points of $\mathrm{Spec}\,\mathbb{C}[S]$ with the subvariety of $\mathbb{C}^r$ given by the monomial equations

$$(A.2) \qquad \prod_j X_j^{a_{ij}} = \prod_j X_j^{b_{ij}}, \quad i = 1, \ldots, k.$$

In other words (A.2) determines an ideal of $\mathbb{C}[X_1, \ldots, X_r]$ and $\mathrm{Spec}\,\mathbb{C}[S]$ just corresponds to the subvariety defined by this ideal. From this description it is easy to see the following criterion for the smoothness of $\overline{T}^S$:

Lemma 4. *The affine toric variety $\overline{T}^S$ is smooth if and only if S is a free semi-group.*

When the semi-group S is *saturated* (i.e. $ns \in S, n > 0 \implies s \in S$), then $\mathbb{C}[S]$ is *normal*, i.e. integrally closed in its field of fractions, and we get a normal affine variety $\overline{T}^S$; in most cases this is taken to be part of the definition of a toric variety. The reason is that the condition of being saturated allows one to obtain such semi-groups by intersecting the lattice M with the convex polyhedral cone $C \subset M_\mathbb{Q}$ generated by the elements of S [Oda88, Prop. 1.1]. This gives a functor

$$\begin{array}{ccc} \text{Convex Rational Polyhedral Cones} & \to & \text{(Normal) Affine Toric Varieties} \\ C & \mapsto & \overline{T}^{C \cap M}. \end{array}$$

The toric variety $\overline{T}^{C \cap M}$ is smooth if and only if C can be generated by a basis of M. Such cones will be called *basic* or *non-singular* with respect to M.

A.2. General Toric Varieties. In fact, because of the contravariant nature of the above functor, it is more common to consider the dual picture in $N_\mathbb{Q}$. The *dual cone* to $C \subseteq M_\mathbb{Q}$ is the cone

$$C^\vee := \{n \in N_\mathbb{Q} : \langle n, m \rangle \geq 0, \forall m \in C\}.$$

Taking the dual again gives back the original cone. We define in this way a covariant functor

$$\begin{array}{ccc} \text{Convex Rational Polyhedral Cones in } N_\mathbb{Q} & \to & \text{Affine Toric Varieties} \\ \sigma & \mapsto & \overline{T}_{N,\sigma} := \overline{T}^{\sigma^\vee \cap M}. \end{array}$$

This functor satisfies the following key property.

Lemma 5. *If $\tau \leq \sigma$, i.e. if τ is a face of σ (given, say, by taking the intersection with the hyper-plane determined by an element $m \in M$) then $\overline{T}_{N,\tau}$ is naturally an open subset of $\overline{T}_{N,\sigma}$ (defined by the condition $u(m) \neq 0$ for $u \in \mathrm{Hom}_{s\text{-}group}(\sigma^\vee \cap M, \mathbb{C})$).*

In this way, one can glue together collections of cones in $N_\mathbb{Q}$ with appropriate compatibility conditions under the operations of intersection and restricting to a face: these collections are called *fans*. The compatibility conditions required for the cones of a fan Σ are

(1) Every face of a cone in Σ is also a cone in Σ.

(2) The intersection of two cones in Σ is a face of each.

These conditions ensure that one can glue together any $\overline{T}_\sigma$ and $\overline{T}_{\sigma'}$ along $\overline{T}_{\sigma \cap \sigma'}$, since the latter is an open set of each, and furthermore, that the diagonal map $\overline{T}_{\sigma \cap \sigma'} \hookrightarrow \overline{T}_\sigma \times \overline{T}_{\sigma'}$ is a closed embedding. The result is that we obtain a normal irreducible toric variety $\overline{T}_\Sigma = \overline{T}_{N,\Sigma}$ which is Hausdorff as an analytic space.

We denote by $|\Sigma| \subseteq N_{\mathbb{Q}}$ the union of the cones of Σ and call it the *support* of Σ.

A.3. Toric Morphisms.

Definition. A *map of fans* $\phi \colon (N', \Sigma') \to (N, \Sigma)$ is a $\mathbb{Z}$-linear homomorphism $\phi \colon N' \to N$ whose scalar extension $\phi_{\mathbb{Q}} \colon N'_{\mathbb{Q}} \to N_{\mathbb{Q}}$ satisfies the following property: for each $\sigma' \in \Sigma'$, there exists $\sigma \in \Sigma$ such that $\phi_{\mathbb{Q}}(\sigma') \subseteq \sigma$.

Lemma 6 ([Oda88, Thm. 1.13]). *A map of fans $\phi \colon N' \to N$ determines a morphism of toric varieties $\overline{T}_\phi \colon \overline{T}_{N',\Sigma'} \to \overline{T}_{N,\Sigma}$ which is equivariant with respect to the actions of $T_{N'}$ and T_N.*

Lemma 6 provides a dictionary for studying maps between toric varieties by studying the corresponding linear maps of lattices.

Proposition 3 ([Oda88, Thm. 1.15]). *The map*

$$\overline{T}_\phi \colon \overline{T}_{N',\Sigma'} \to \overline{T}_{N,\Sigma}$$

is proper if and only if

$$\phi^{-1}(|\Sigma|) = |\Sigma'|.$$

Taking the second lattice to be zero, we see that a toric variety $\overline{T}_{N',\Sigma'}$ is compact if and only if $|\Sigma'| = N'_{\mathbb{Q}}$, in which case the fan Σ' itself is called *complete*. Let us see some examples of toric morphisms.

A.3.1. *Finite Quotients.* Consider the case when N' is a $\mathbb{Z}$-submodule of N of finite index and $\Sigma' = \Sigma$. We write X' and X for the corresponding varieties:

Lemma 7 ([Oda88, Cor. 1.16, p.22],[Ful93, §2.2]). *With the data as above, we have*

$$X = X'/\Gamma,$$

i.e. $X' \to X$ coincides with the projection of X' with respect to natural action of the finite group

$$\Gamma = N/N' \cong \mathrm{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^*) = \ker[T_{N'} \to T_N].$$

Here M' is the dual of N' and is naturally an over-lattice of M. There is a unique pairing $M' \times N \to \mathbb{Q}/\mathbb{Z}$ which extends the pairings $M \times N \to \mathbb{Z}$ and $M' \times N' \to \mathbb{Z}$; this gives rise to a canonical pairing

$$M'/M \times N/N' \to \mathbb{Q}/\mathbb{Z},$$

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which we compose with the homomorphism $\mathbb{Q}/\mathbb{Z} \xrightarrow{\exp 2\pi i(\ \cdot\ )} \mathbb{C}^*$ *to identify* Γ *with* $\mathrm{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^*)$. *If we identify* $\gamma \in \Gamma$ *with the morphism* $n_\gamma \colon M' \to \mathbb{Q}/\mathbb{Z}$ *such that* $n_\gamma(M) \subset \mathbb{Z}$, *the action is given by*

$$(A.3) \qquad \gamma \cdot x'(m') = \exp(2\pi i \langle n_\gamma, m' \rangle) x'(m'),$$

for $x' \in \overline{T}_{\sigma'}$.

Example 1. Take the cone σ generated by the vectors $n_1 = (1, 0)$ and $n_2 = (1, 2)$ in $N = \mathbb{Z}^2$. This is non-basic, since $\det\left(\begin{smallmatrix} 1 & 0 \\ 1 & 2 \end{smallmatrix}\right) = 2$. The dual cone $\sigma^\vee$ in M is generated by the column vectors $m_1 = (0, 1)^T$ and $m_2 = (2, -1)^T$. These do not generate $M \cap \sigma^\vee$ as a semi-group: a third generator $m_3 = (1, 0)^T$ is needed. The relation $m_1 + m_2 = 2m_3$ shows that $\overline{T}_{N,\sigma}$ is isomorphic to the singularity $X_1 X_2 = X_3^2$ in $\mathbb{C}^3$.

If we consider $N' = \mathbb{Z}n_1 \oplus \mathbb{Z}n_2 \subset N$, then $M' = \frac{1}{2}\mathbb{Z}m_1 \oplus \frac{1}{2}\mathbb{Z}m_2$; the semi-group $M' \cap \sigma^\vee$ is generated by the basis $m'_1 = \frac{1}{2}m_1$ and $m'_2 = \frac{1}{2}m_2$ of M'; and $X' = \overline{T}_{N',\sigma} = \mathrm{Spec}\,\mathbb{C}[X'_1, X'_2] = \mathbb{C}^2$. The relations

$$m_1 = 2m'_1$$
$$m_2 = 2m'_2$$
$$m_3 = m'_1 + m'_2$$

show that the quotient map $X' \to X$ is induced by

$$
\begin{array}{ccc}
\mathbb{C}^2 & \to & \mathbb{C}^3 \\
\begin{pmatrix} X'_1 \\ X'_2 \end{pmatrix} & \mapsto & \begin{pmatrix} X_1 = X'^2_1 \\ X_2 = X'^2_2 \\ X_3 = X'_1 X'_2 \end{pmatrix},
\end{array}
$$

and this exhibits the variety $X_1 X_2 = X_3^2$ as the quotient of $\mathbb{C}^2$ by $\mathbb{Z}_2 = N/N'$. The generator $\gamma = (1, 1)$ acts, according to (A.3) by

$$\gamma X'_1 = \exp(2\pi i (1, 1) \left(\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right)) X'_1 = -X'_1$$
$$\gamma X'_2 = \exp(2\pi i (1, 1) \left(\begin{smallmatrix} 1 \\ -1/2 \end{smallmatrix}\right)) X'_2 = -X'_2.$$

Note that X'^2_1, X'^2_2 and $X'_1 X'_2$ are indeed generators of the sub-ring of $\mathbb{Z}_2$-invariant polynomials in $\mathbb{C}[X'_1, X'_2, X'_3]$.

Example 2 (Cyclic singularities). More generally, suppose $\Gamma = \mu_r$, a cyclic group of order r, acting with weights $w_1, \ldots, w_n$. Let

$$M := \{ m \in \mathbb{Z}^n \,|\, \frac{1}{r}(w_1, \ldots, w_n) \cdot m \in \mathbb{Z} \},$$

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and denote by C_0 the first quadrant in $\mathbb{Q}^n = M_{\mathbb{Q}}$. Dual to these are

$$N = \mathbb{Z}^{*n} + \mathbb{Z}\frac{1}{r}(w_1, \ldots, w_n),$$

and the first quadrant $\sigma = C_0^{\vee}$ in $\mathbb{Q}^{*n} = N_{\mathbb{Q}}$. Then the singularity $X_0 = \mathbb{C}^n/\Gamma$ is isomorphic to $\overline{T}^{M,C_0}$. If Σ_0 denotes the fan in N consisting of the cone C_0 and all its' faces, then $X_0 = \overline{T}_{N,\Sigma_0}$.

A.3.2. *Resolutions.* Instead of keeping the fan fixed and refining the lattice, we can keep the lattice fixed and *subdivide* the fan, namely replace Σ by a fan Σ' in such a way that the cones of Σ are unions of cones in Σ'. Proposition 3 implies that the morphism $X' := \overline{T}_{N,\Sigma'} \to X := \overline{T}_{N,\Sigma}$ is proper, and since it is an isomorphism on the big torus $\overline{T}_{N,0} \cong T_N$, it is actually *birational* as well.

Example 3. Take X as in example 1, and split σ into two cones by drawing the half-line from the origin through the point $(1,1)$. This gives a variety X' which is non-singular and is the resolution of X. The map $X' \to X$ blows down the curve corresponding to the one-dimensional cone generated by $(1,1)$.

In §4 I describe a well-known procedure for computing the minimal resolution of cyclic abelian singularities in dimension 2 (and then show that it coincides with X_ζ).

A.4. Toric Varieties and Convex Polyhedra. Let us go back to the original vectorspace $M_{\mathbb{Q}}$. Any convex polyhedron $P \subseteq M_{\mathbb{Q}}$ of full dimension determines a unique fan Σ_P and we define $\overline{T}^{M,P} := \overline{T}_{N,\Sigma_P}$. The cones in the fan Σ_P are the duals in $N_{\mathbb{Q}}$ of the tangent cones to P at its faces F. If F is a face of P then the *tangent cone* to P at F is the convex cone

$$T_F P := \mathbb{Q}_+(P - F) = \{\lambda(p - f) : p \in P, f \in F, \lambda \in \mathbb{Q}_+\}, \quad \text{for any } f \in \text{int } F.$$

When $F = \{p\} \in \text{ext } P$, the tangent cone of P at p coincides with the usual notion. It is easy to check that in fact, the fan Σ_P consists of the duals of the tangent cones to P at its extreme points along with all their faces. From this it follows that if $P = C$ is a convex cone in $M_{\mathbb{Q}}$ then $\overline{T}^{M,C} = \overline{T}^{C \cap M}$.

Remark. The correspondence between polyhedra and varieties is many-to-one: multiples and translates of the same polyhedron determine the same fan; the extra data specified by P turns out to correspond to an ample T^M-equivariant line bundle $L^P \to \overline{T}^P$ [Oda88]. (Actually, only polyhedra whose vertices are integral points of M give rise to genuine line bundles: rational polyhedra can however be viewed as formal fractional powers of line bundles.)

A.5. The Proj Construction. In fact, there is a direct construction of $\overline{T}^{M,P}$ which avoids introducing fans and piecewise linear functions.

Given a polyhedron $P \subset M_{\mathbb{Q}}$, consider the lattice $\widetilde{M} := \mathbb{Z} \times M$ and the polyhedral cone $CP \subset \widetilde{M}_{\mathbb{Q}}$ associated to P, i.e. the closure of the cone on P:

$$CP := \overline{\mathbb{Q}_{\geq 0}(\{1\} \times P)} \subset \widetilde{M}_{\mathbb{Q}} = \mathbb{Q} \times M.$$

The group algebra of the corresponding semi-group is $\mathbb{C}[CP \cap \widetilde{M}]$ and is a finitely generated graded $\mathbb{C}$-algebra, where the grading is induced by the first coordinate. Finally, define

(A.4) $$\overline{T}^{M,P} := \operatorname{Proj} \mathbb{C}[CP \cap \widetilde{M}].$$

If P is a convex cone, this notation is compatible with the one originally defined for cones. Also, the line bundle L^P is nothing but $\mathcal{O}(1)$.

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