代数幾何学シンポジウム記録
1994年度 pp.95-99

SOME BOUNDEDNESS THEOREMS AND THEIR APPLICATIONS

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1. Questions

1.1. For the most part, we are working over the field $\mathbb{C}$ of complex numbers.

Fact (A). The class of n-dimensional Fano manifolds, i.e. manifolds with ample $-K_X$, is bounded (Kollár – Miyaoka – Mori).

Fact (B). The class of n-dimensional manifolds of general type with ample K_X is bounded and $K_X^2 \leq C$ (by Matsusaka's Big Theorem).

Fact (C). The class of n-dimensional manifolds of general type with ample K_X and $K_X^2 = C$ is bounded.

Definition 1.2. Let k be a field, $\mathcal{C}$ be a class of projective schemes $/k$. One says that $\mathcal{C}$ is bounded if there exists a morphism of schemes $\mathcal{X} \xrightarrow{F} \mathcal{S}$ such that

(1) $\mathcal{S}$ has finite type,
(2) F is projective,
(3) all elements of $\mathcal{C}$ appear as some of the closed fibers of F (not necessarily in a one-to-one way and not necessarily all fibers are from $\mathcal{C}$).

1.3. Let X be a normal variety, $i : U \hookrightarrow X$ nonsingular locus. Define $\mathcal{O}(K_X) := i_*\mathcal{O}_U(K_U), \mathcal{O}(nK_X) := i_*\mathcal{O}_U(nK_U)$.

Question 1. Assume that $\mathcal{O}(nK_X)$ is an ample line bundle for some $n \in \mathbb{Z}$. Do (A) and (B) still hold? The answer is NO.

Example 1.4. For $G \subset PGL(2,k)$ finite, $X = \mathbb{P}^2/G$ is Fano, i.e. K_X is ample. Therefore, there are infinitely many types of Fanos with quotient singularities.

Example 1.5 (Blache). There exists an infinite sequence X_n of surfaces with quotient singularities and ample K_X such that

(1) $K^2_{X_n} \uparrow\uparrow$,
(2) $\lim_{n\to\infty} K^2_{X_n} = 1$.

So (B) also fails.

Question 2. What are the natural conditions on the singularities of $X \in \mathcal{C}$, under which (A), (B) and (C) still hold? We will answer this question in dimension 2, conjecture in dimension $n \geq 3$.

Application 1 (*Kollár - Shepherd-Barron*). **Compactification of the moduli space of surfaces of general type.**

The most basic question here is to find good "limits" of surfaces of general type. So, let $\mathcal{X} \longrightarrow \mathcal{S} - 0$ be a one-parameter family over a punctured disk, so that every fiber is a nonsingular surface with ample K_X and $K_X^2 = C$.

Completely in line with the construction for the case of curves, we first apply the semistable reduction theorem. After a base change $\mathcal{S}' \to \mathcal{S}$ we obtain a nonsingular 3-fold such that the central fiber over $0 \in \mathcal{S}'$ is reduced and has only normal crossings. At this point apply the Mori theory. The canonical model of this 3-fold over $\mathcal{S}'$ is the good family that we are looking for.

The central fiber has ample K_X, $K_X^2 = C$ and semi-log canonical singularities (which may be worse than quotient). X has normal crossings in codimension 1.

Let $X^\nu = \bigcup X_i \longrightarrow X$ be a normalization, $B_i \subset X_i$ be the double curves. Then

$$K_X^2 = \sum (K_{X_i} + B_i)^2.$$

So, if we knew something about the set $\{(K_{X_i} + B_i)^2\}$ and the boundedness of $\{(X_i, B_i)\}$, this would imply the boundedness of $\{X\}$. B_i is called the boundary.

Application 2 (*Alexeev*). In a similar vein, it is possible to construct the moduli of pairs (X, B), with $B = \sum_{i=1}^n B_i$ reduced divisor on X and ample $K + B$. This is a generalization of the moduli space $M_{g,n}$ of n-pointed curves to the case of surfaces. Moreover, instead of "absolute" surfaces, one can consider maps to a fixed projective scheme, with $K + B$ only relatively ample.

Application 3 (*Hurwitz, Xiao, Kollár*). **Bounds for automorphism groups.**

Let X be a manifold with ample K_X. By Iitaka, $G = \operatorname{Aut} X$ is finite. Is there a constant $c = c(\dim X)$ such that

$$|\operatorname{Aut} X| \le c \cdot K_X^{\dim X}?$$

Let $X \xrightarrow{\pi} X/G = Y$ be the quotient morphism, Y has quotient singularities.

$$K_X = \pi^* K_Y + \sum (m_i - 1) D_i$$

$$K_X = \pi^*(K_Y + \sum \frac{m_i - 1}{m_i} B_i)$$

$$K_X^n = |\operatorname{Aut} X| \cdot (K_Y + \sum \frac{m_i - 1}{m_i} B_i)^n = |\operatorname{Aut} X| \cdot (K_Y + B)^n$$

$$|Aut X| = \frac{K_X^n}{(K_Y + B)^n}$$

If $(K_Y + B)^n \ge 1/c$ then we are done.

Application 4 (*Fujita, Kawamata, Nakayama; Kollár, Oguiso–Peternell*). **Elliptic 3-folds.**

Let $X \xrightarrow{\pi} Z$ be a fibration with a general fiber which is an elliptic curve. Then (perhaps, after a birational modification) $K_X = \pi^*(K_Z+B)$, $h^0(nK_X) = h^0([n(K_Z+B)])$. Here $B = \sum b_i B_i$, $B_i \subset Z$ is a $\mathbb{Q}$- divisor on Z,

$$b_i \in \{\frac{1}{12}, \frac{2}{12}, \dots, \frac{11}{12}, 1-\frac{1}{n} | n \in \mathbb{N}\}.$$

Coefficients $1 - 1/n$ correspond to multiple fibers.

If $K_X \equiv 0$ (f.e. X is a Calabi-Yau manifold) then $K_Z + B \equiv 0$. The boundedness of (Z, B) in this case was used by K.Oguiso and Peternell to prove that there are only finitely many elliptic fibrations on a CY 3-fold of general type.

If the Kodaira $k(X) = 2$ then $k(K_Z + B) = 2$. Kollár proved that if there exists a universal bound on $(K + B)^2$ for ample $(K_Z + B)$, then there exists a universal constant $N > 0$ such that $h^0(NK_X) \neq 0$.

2. Results

2.1. Conclusions from above the discussion.

(1) Instead of the canonical divisor K_X, we need to consider a "log canonical" divisor $K_X + B = K_X + \sum b_j B_j$, $B_j \subset X$. Coefficients $b_j \in \mathcal{A} \subset [0,1]$ should, perhaps, be not arbitrary but satisfy additional conditions.
(2) The set $\{(K_X + B)^2\}$ is of special interest.

2.2. Singularities.

2.1. Let X be normal, $f : Y \longrightarrow X$ be a desingularization. Look at the formula

$$K_Y + f^{-1}B + \sum F_i = f^*(K_Y + B) + \sum a_i F_i$$

in which F_i are exceptional divisors. The coefficients a_i are called log discrepancies. We can even assume X to be only (S_2) and Gorenstein in codimension 1, with $B_i \not\subset \operatorname{Sing} X$.

Then either for all resolutions of singularities all $a_i \geq 0$, or $\inf_{Y \to X} a_i = -\infty$.

Definition 2.2. One says that the pair (X, B) is

(1) log canonical if $a_i \geq 0, b_j \leq 1$,
(2) Kawamata log terminal if $a_i > 0, b_j < 1$,
(3) canonical if $a_i \geq 0, b_j \leq 0$,
(4) terminal if $a_i \geq 0, b_j < 0$,
(5) ε-log canonical if $a_i \geq \varepsilon \geq 0, b_i \leq 1 - \varepsilon$,
(6) ε-log terminal if $a_i > \varepsilon > 0, b_i < 1 - \varepsilon$.

2.3. It is well known that the quotient singularities are (Kawamata) log terminal.

2.3. Theorems.

2.4. In what follows, $\dim X = 2$, $k = \bar{k}$, $\operatorname{char} k \geq 0$.

Theorem 2.5. *(A) holds for ε-log terminal. Moreover, $-(K_X+B)$ may be assumed to be only nef, modulo trivial exceptions.*

Theorem 2.6. *(B) holds for ε-log terminal if $\mathcal{A} = \{b_j\}$ satisfies the descending chain condition.*

Definition 2.7. $\mathcal{A} \subset [0,1]$ satisfies D.C.C. if any sequence $x_i \in \mathcal{A}$ with $x_i \Downarrow$ is finite.

Theorem 2.8. *(C) holds for semi-log canonical singularities if $\mathcal{A} = \{b_j\}$ satisfies the descending chain condition. Moreover, $\{(K_X + B)^2\}$ also satisfies D.C.C.*

2.9. All conditions in the theorems above are sharp.

3. On the proofs

3.1. There is a standard method for proving boundedness (Matsusaka, Kollár–Matsusaka). It basically says that every time when every variety in the class C has a polarization, i.e. an ample Cartier divisor H, and we can bound $H^{\dim X}$ and $H^{\dim X - 1} K_X$, the class is bounded.

The main difficulty is that in our situation we can take $H = \pm N K_X$ but N cannot be bounded locally, because log terminal singularities can have arbitrarily large index.

In the case of non-positive K_X it turns out that to bound the index N of X it is sufficient to bound the rank of the Picard group of the minimal desingularization $\widetilde{X}$.

In the case of positive K_X we use a very different method, we prove boundedness by obtaining a contradiction, applying the following

Lemma 3.2. *If for any infinite sequences $\{X_i \in C\}$ there exist an infinite subsequence $\{X_{i_k}\}$ which is bounded, then the class C is bounded.*

3.1. Higher dimensions. .

Conjecture 3.3. *The direct analogs of theorems* 2.5, 2.6, 2.8 *hold in arbitrary dimensions.*

3.4. The only known results in this direction are:

(1) ε-log terminal toric Fanos ($B = \emptyset$) are bounded (Borisovs),
(2) 1-log terminal Fano 3-folds ($B = \emptyset$) are bounded (Kawamata: $\mathbb{Q}$ -factorial case, Miyaoka–Mori–Kollár: general case).

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