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Higher direct images of canonical sheaves tensorized with semi-positive vector bundles by proper Kähler morphisms

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§1. Let $f:X\to Y$ be a proper surjective morphism of reduced analytic spaces with X non-singular, $\dim_{\mathbf{C}} X=n$ and $\dim_{\mathbf{C}} Y=m$ respectively. Let (E, h) be a holomorphic vector bundle on X. Our main purpose is to study of Leray spectral sequence:

$$H^p(Y, R^q f_* \Omega^n_X(E)) \Rightarrow H^{p+q}(X, \Omega^n_X(E)).$$

under certain conditions. The first contribution in this direction was achieved by Grauert and Riemennschneider.

Theorem G-R (cf.[G-R]). Let $f:X\to Y$ be a bimeromorphic morphism from a projective algebraic manifold X to a reduced analytic space Y and let (E, h) be a Nakano semi-positive holomorphic vector bundle on X. Then

$$R^q f_* \Omega^n_X(E)=0 \text{ for any } q\geqq 1$$

Here the notion of Nakano semi-positivity of vector bundle is defined as follows:

Definition 1 A holomorphic vector bundle (E, h) on X is said to be Nakano semi-positive (resp. positive) if the curvature form $\Theta_h\in C^{1,1}(X,\mathrm{Hom}(E,\overline{E}))$ of E relative to h is a positive semi-definite (resp. positive definite) quadratic form on each fibre of the vector bundle $E\otimes TX$, where TX is the holomorphic tangent bundle of X.

Remark 2 In the line bundle case the notion of Nakano semi-positivity coincides with the semi-positivity in the sense of Kodaira. However in the case of rank > 1 there is another notion of semi-positivity. A holomorphic vector bundle (E, h) on M is semi-positive (resp. positive) in the sense of Griffiths, i.e. Θ_h is a positive semi-definite (resp. positive definite) quadratic form on each fibre of the vector bundle $E \oplus TX$. The following facts are known:

(i) The Nakano semi-positivity (resp. positivity) implies the semi-positivity (resp. positivity) in the sense of Griffiths

(ii) If (E, h) is semi-positive (resp. positive) in the sense of Griffiths, then $(E \otimes \det E, h \otimes \det h)$ is Nakano semi-positive (resp. positive) (cf. [De-S])

After this , Kollár succeeded to get a large progress with respect to the degeneration of Leray spectral sequence and torsion freeness of higher direct images of canonical sheaves on projective algebraic varieties in [Ko-1] and [Ko-2]. His result is summarized in the following way:

Theorem K (cf. [Ko-1] and [Ko-2]). Let $f: X \to Y$ be a surjective morphism of projective morphism of projective algebraic varieties with X non-singular of dimension n and Y of dimension m respectively. Then the following assertions hold:

(i) Let L be a line bundle generated by global sections on X. Then the Leray spectral sequence:

$$H^p(Y, R^q f_* \Omega_X^n(L)) \Rightarrow H^{p+q}(X, \Omega_X^n(L))$$

degenerates at E_2

(ii) $R^q f_* \Omega_X^n$ is torsion free for any $q \geqq 0$ and vanishes if $q > n - m$

(iii) Let A be an ample line bundle on Y and set $E = f^* A$. Then the homomorphism

$$\mu_\sigma^q : H^q(X, \Omega_X^n(E^{\otimes k})) \to H^q(X, \Omega_X^n(E^{\otimes j+k}))$$

induced by the tensor product with a non-trivial holomorphic section σ on X of the j times tensor product $E^{\otimes j}$ of E is injective for any $q \geqq 0$, j and $k \geqq 1$.

(iv) Let A be as above. Then

$$H^p(Y, O_Y(A) \otimes R^q f_* \Omega_X^n) = 0 \quad \text{if } p \geqq 1 \text{ and } q \geqq 0$$

Furthermore Kollár observed that the assertions (ii) and (iv) are equivalent to show the one (iii) and discussed the local freeness of $R^q f_* \Omega_X^n$ in [Ko-2]. His result was proved by Nakayama, [Ny-1] independently and was localized by Moriwaki,[Mo] in terms of projective morphisms. Their methods heavily depends on theory of variation of Hodge structures and several vanishing theorems of cohomology groups which were formulated in [Ny-2],[O] and [Ta-1]. On the other hand Morihiko Saito obtained those results as a part of his theory of Hodge modules (cf.[Sa-1],[Sa-2]).

Here we formulate those problems in the category of Kähler manifolds and generalize them to proper Kähler morphisms of analytic spaces. With respect to the torsion freeness and decomposition theorems for higher direct images of canonical sheaves, Saito generalized his theory to proper Kähler morphisms of analytic spaces in [Sa-3]. However our approach enables us to discuss the decomposition, torsion freeness, injectivity and vanishing theorems for higher direct images of canonical sheaves tensorized with semi-positive holomorphic vector bundles by proper Kähler morphisms. It is known that such a torsion freeness theorem for higher direct images of canonical sheaves does not always hold without Kähler condition (cf.[Nm]).

Our main result is summarized as follows ([Ta-2]):

Theorem Let $f: X \to Y$ be a proper surjective morphism from a complex manifold X of dimension n to a reduced analytic space Y of dimension m so that every connected component of X is mapped surjectively to Y. Suppose X admits a Kähler metric ω_X and (E, h_E) is a Nakano semi-positive holomorphic vector bundle on X. Then the following theorems hold:

I. **Decomposition Theorem.** The Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_* \Omega_X^n(E)) \Rightarrow H^{p+q}(X, \Omega_X^n(E))$$

for f degenarates at E_2.

Especially if (X, ω_X) is a compact connected Kähler manifold, then for any $r \geqq 0$ it holds that

$$\dim_{\mathbf{C}} H^r(X, \Omega_X^n(E)) = \sum_{p+q=r} \dim_{\mathbf{C}} H^p(Y, R^q f_* \Omega_X^n(E))$$

II. Torsion Freeness Theorem. For $q \geqq 1$ the sheaf homomorphism

$$\mathcal{L}^q : R^0 f_* \Omega_X^{n-q}(E) \to R^q f_* \Omega_X^n(E)$$

induced by the q-times left exterior product by ω_X admits a splitting sheaf homomorphism

$$\delta^q : R^q f_* \Omega_X^n(E) \to R^0 f_* \Omega_X^{n-q}(E)$$

with $\mathcal{L}^q \circ \delta^q = \mathrm{id}$. Especially $R^q f_* \Omega_X^n(E)$ is torsion free for $q \geqq 0$ and zero if $q > n-m$.

III. Injectivity Theorem. Let (F, h_F) be a semi-positive holomorphic line bundle on X such that $F^{\otimes j}$ admits a non-trivial holomorphic section σ with $j \geqq 1$. Then the sheaf homomorphism:

$$R^q f_*(\sigma) : R^q f_* \Omega_X^n(F^{\otimes k} \otimes E) \to R^q f_* \Omega_X^n(F^{\otimes j+k} \otimes E)$$

induced by the tensor product with σ is injective for any $q \geqq 0$ and $k \geqq 1$.

IV. Relative Vanishing Theorem. Let $g: Y \to Z$ be a surjective proper morphism of reduced analytic spaces. Then the Leray spectral sequence:

$$R^p g_* R^q f_* \Omega_X^n(E) \Rightarrow R^{p+q}(g \circ f)_* \Omega_X^n(E)$$

degenerates. Especially (i) if A is a g-ample line bundle on Y, then

$$R^r(g \circ f)_* \Omega_X^n(f^* A \otimes E) \simeq R^0 g_* O_Y(A) \otimes R^r f_* \Omega_X^n(E) \quad \text{for any } r \geqq 0$$

and (ii) if g is generically finite, then

$$R^r(g \circ f)_* \Omega_X^n(E) \simeq R^0 g_* R^r f_* \Omega_X^n(E) \quad \text{for any } r \geqq 0.$$

V. Local Freeness Theorem. Suppose X is connected and Y is non-singular. (i) If f has connected fibres, then the sheaf homomorphism

$$\mathcal{L}^{n-m} : \Omega_Y^m \to R^{n-m} f_* \Omega_X^n$$

yields an isomorphism. (ii) If f is a regular family outside a normal crossing divisor of Y, then $R^q f_* \Omega_X^n$ is locally free for any $q \geqq 0$.

We may say that the above theorem is a relative version of Theorem K in the category of Kähler manifolds. The above assertoins are linked togather as the compact case by Kollár. In fact the following indications hold:

(I)+(III)+Nakano-Fujiki's vanishing theorem $\Rightarrow$ (IV)

(II)+(IV) $\Rightarrow$ (V)

Hence we have only to show (I),(II) and (III).

Remark. In the case of rank of $E>1$, we can formulate the above result in terms of the semi-positivity of the dual of tautological line bundle of $P(E^*)$. In this case we can obtain the same results as above for the vector bundles $E \otimes \det E$ and $\det E$.

§2. Let X be a holomorphically convex manifold of dimension n and let (E, h) be a holomorphic vector bundle on X. Our main result follows from a structure theorem for $H^q(X, \Omega_X^n(E))$ as a ring of holomorphic functions module. In order to explain this theorem we fix the following situation and notations:

(X, ω_X) ; a complex manifold with the hermitian metric ω_X of dimension n

$f: X \to S$; a proper surjective morphism from X to a connected (reduced) Stein space S of dimension m such that any connected component of X is mapped surjectively to S

$\pi: S \to C^d$; a proper embedding of S into a complex number space of dimension d with a global coordinate system $(t^1, \ldots, t^d)$

$\Psi: S \to [0, +\infty)$; a smooth strictly plurisubharmonic exhaustion function on S defined by $\Psi := \pi^*\left(\sum_{j=1}^{d} |t^j|^2\right)$

$\Phi := f^* \Psi$; the pull back of Ψ by f

$X(W)$; the inverse image $f^{-1}(W)$ of a subset W of S.

Proposition Let (E, h) be a holomorphic vector bundle on X. For any p and $q \geqq 0$ we define the following space of E-valued $\bar\partial$-closed harmonic (p, q) forms relative to ω_X and h:

$$H^{p,q}(X, E, \Phi) := \{ u \in C^{p,q}(X, E) \ ; \ \bar\partial u = \vartheta_h u = 0 \text{ and } e(\bar\partial\Phi)^* u = 0 \text{ on } X \}$$

where ϑ_h and $e(\bar\partial\Phi)^*$ are the adjoint operators of $\bar\partial$ and the left exterior product $e(\bar\partial\Phi)$ by $\bar\partial\Phi$ respectively.

Suppose ω_X is Kähler and (E, h) is Nakano semi-positive. Then the following assertions hold for $p = n$ and any $q \geqq 1$:

(i) Assume $u \in C^{n,q}(X, E)$ satisfies $e(\bar\partial\Phi)^* u = 0$ on X. Then u satisfies $\bar\partial u = \vartheta_h u = 0$ if and only if $\bar\vartheta u = 0$ and $< i\ e(\Theta_h + \partial\bar\partial\Phi)\Lambda u, u >_h = 0$ on X, where $\bar\vartheta = - * \bar\partial *$, Θ_h is the curvature form of E relative to h and $< \ , \ >_h$ is the pointwise inner product relative to the metrics ω_X and h

Remark If (E, h) is Nakano semi-positive and Φ is as above, then

$$< i\ e(\Theta_h)\Lambda u, u >_h \ \geqq\ 0 \quad \text{and} \quad < i\ e(\partial\bar\partial\Phi)\Lambda u, u >_h \ \geqq\ 0 \quad \text{on} \quad X$$

for any $u \in C^{n,q}(X, E)$ and $q \geqq 1$

(ii) The Hodge's star operator $*$ relative to ω_X yields an injective homomorphism from $H^{n,q}(X, E, \Phi)$ into $\Gamma(X, \Omega_X^{n-q}(E))$ which induces a structure of Frechet space onto $H^{n,q}(X, E, \Phi)$. Especially $H^{n,q}(X, E, \Phi)$ is a torsoin free $O(S)$-module.

(iii) If $u \in H^{n,q}(X, E, \Phi)$, then $< i\ e(\partial\bar\partial(\exp\varphi))\Lambda u, u >_h \equiv 0$ on X for any smooth plurisubharmonic function φ on X. In particular $H^{n,q}(X, E, \Phi)$ is independent of the choice of Φ.

(iv) The canonical homomorphism

$$\iota : \Gamma(X, \bar\partial A^{n,q-1}(E)) \to H^q(X, \Omega_X^n(E))$$

defined by Dolbeault's theorem induces an injective homomorphism

$$\iota : H^{n,q}(X, E, \Phi) \to H^q(X, \Omega_X^n(E))$$

where $\Gamma(X, \bar{\partial} A^{n,q-1}(E))$ is the space of $\bar{\partial}$-closed E-valued smooth (n, q) forms on X.

(v) Let Σ be the union of the set of singular points of S and the image of the degeneracy set of df by f. If there is a non-vanishing holomorphic m-form θ on $S_0 := S \backslash \Sigma$ and $1 \leqq q \leqq n-m$, then the restriction of any form u of $H^{n,q}(X, E, \Phi)$ onto $X(S_0)$ can be written as follows:

$$u = L^q(\sigma_u \wedge f^* \theta) \quad \text{for} \quad \sigma_u \in \Gamma(X(S_0), \Omega_{X/S_0}^{n-m-q}(E))$$

where Ω_{X/S_0}^p is the sheaf of germs of relative holomorphic p-forms on $X(S_0)$.

(vi) $H^{n,q}(X, E, \Phi) = 0$ vanishes for any $q > n - m$.

(vii) If V is a Stein open subset of S provided with a smooth strictly plurisubharmonic exhaustion function ψ, then the restriction homomorphism

$$r_V : H^{n,q}(X, E, \Phi) \to H^{n,q}(X(V), E, f^* \psi)$$

is well-defined.

This proposition can be shown only by a harmonic integral theory for weakly 1-complete Kähler manifolds. Hence the problem is to show that the homomorphism

$$\iota : H^{n,q}(X, E, \Phi) \to H^q(X, \Omega_X^n(E))$$

is actually surjective whenever E is Nakano semi-positive. This essential part can be done by using an L^2-theory for the $\bar{\partial}$-operator and bounded plurisubharmonic functions on complete Kähler manioflds. Namely we can show the following theorem.

Theorem Let the situation be the same as above. Then the following assertions hold for any integer $q \geqq 1$:

(i) The following space of harmonic forms relative to ω_X and h

$$H^{n,q}(X, E, \Phi) := \{\ u \in C^{n,q}(X, E)\ ;\ \bar{\partial} u = \vartheta_h u = 0 \text{ and } e(\bar{\partial}\Phi)^* u = 0 \text{ on } X\ \}$$

represents $H^q(X, \Omega_X^n(E))$ as a torsion free $O(S)$-module and the Hodge's star operator $*$ relative to ω_X yields an injective homomorphism from $H^{n,q}(X, E, \Phi)$ into $\Gamma(X, \Omega_X^{n-q}(E))$ which induces a splitting homomorphism

$$\delta^q\ :\ H^q(X, \Omega_X^n(E)) \to \Gamma(X, \Omega_X^{n-q}(E))$$

with $L^q \circ \delta^q = \mathrm{id}$ for the homomorphism

$$L^q\ :\ \Gamma(X, \Omega_X^{n-q}(E)) \to H^q(X, \Omega_X^n(E))$$

(ii) If V is a Stein open subset of S provided with a smooth strictly plurisubharmonic exhaustion function ψ, then the following diagram is commutative :

$$\begin{array}{ccc} H^q(X, \Omega_X^n(E)) & \xrightarrow{\delta^q} & \Gamma(X, \Omega_X^{n-q}(E)) \\ r_V \downarrow & & r_V \downarrow \\ H^q(X(V), \Omega_X^n(E)) & \xrightarrow[\delta^q]{} & \Gamma(X(V), \Omega_X^{n-q}(E)) \end{array}$$

(iii) $H^q(X, \Omega_X^n(E))$ vanishes for any $q > n - m$.

Applying this theorem to the morphism $f : X \to Y$ locally one can see that for a set of pairs $\{V, \psi_V\}$, V is a Stein open subset of Y admitting a smooth plurisubharmonic exhaustion function ψ_V of V the data $\{H^{n,q}(X(V), E, \psi_V), r^V{}_W\}$ with the restriction homomorphism $r^V{}_W : H^{n,q}(X(V), E, \psi_V) \to H^{n,q}(X(W), E, \psi_W)$, $W \subset V$, is a presheaf and its sheafification $R^0 f_* H^{n,q}(E)$ is a torsion free O_Y-module such that the canonical homomorphism $R^0 f_* : H^{n,q}(X(V), E, \psi_V) \to \Gamma(X(V), R^0 f_* H^{n,q}(E))$ is an isomorphism. Therefore $R^0 f_* H^{n,q}(E))$ is isomorphic to $R^q f_* \Omega_X^n(E)$. This implies the degeneration of the Leray spectral sequence. The results (II) and (iii) are more or less induced from this fact.

References

[De-S] Demailly, J.P. and Skoda, H. : Relations entre les notions de positivités de P.A.Griffiths et de S.Nakano pour les fibrés vectoriels, Seminaire Pierre Lelong-Henri Skoda,(Analyse) Années 1978/79, Lecture notes in Math. 822, 304-309 (1980)

[G-R] Grauert, H. and Riemennschneider, O. : Verschwindungssätze für analytische Kohomologiegruppen, Invent. Math. 11, 263-292 (1970)

[Ko-1] Kollár, J. : Higher direct images of dualizing sheaves I, Ann. of Math. 123, 11-42 (1986)

[Ko-2] Kollár, J. : Higher direct images of dualizing sheaves II, Ann. of Math. 124, 171-202 (1986)

[Mo] Moriwaki, A. : Torsion freeness of higher direct images of canonical bundles, Math. Ann. 276), 385-398 (1987)

[Nm] Nakamura, I. : Complex parallelisable manifolds and their small deformations, J. Differential Geometry 10, 85-112 (1975)

[Ny-1] Nakayama, N. :Hodge filtrations and the higher direct images of canonical sheaves, Invent. Math. 86, 217-221 (1986)

[Ny-2] Nakayama, N. : The lower semi-continuity of the plurigenera of complex varieties, Advanced Studies in Pure Math. 10, 551-590, 1987

[O] Ohsawa, T. : Vanishing theorems on complete Kähler manifolds, Publ. RIMS. Kyoto Univ. 20, 21-38 (1984)

[Sa-1] Saito, M. : Modules de Hodge polarisable, Publ. RIMS. Kyoto Univ. 24, 849-995 (1988)

[Sa-2] Saito, M. : Mixed Hodge Modules, Publ. RIMS. Kyoto Univ. 26, 221-333 (1990)

[Sa-3] Saito, M. : Decomposition Theorem for Proper Kähler morphisms, Tohoku Math. J. 42, 127-148 (1990)

[Ta-1] Takegoshi, K. : Relative vanishing theorems in analytic spaces, Duke Math. J. 52, 273-279 (1985)

[Ta-2] Takegoshi, K. : Higher direct images of canonical sheaves with tensorized semi-positive vector bundles by proper Kähler morphisms, Preprint.