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On Brauer-Manin equivalence for zero-cycles on varieties over local fields

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In this article we discuss a certain Diophantine problem of zero-cycles on a variety defined over an arithmetic field. Let k be a fixed ground field. In what follows k is either a number field (i.e. $[k:\mathbf{Q}]<\infty$) or a local field (i.e. $[k:\mathbf{Q}_p]<\infty$). Let X be a proper smooth variety over k. Let

$$Z_0(X)=\{\omega=\sum_{x\in X_0} n_x[x] | n_x\in\mathbf{Z},\ n_x=0 \text{ for almost all } x\in X_0\},$$

$$CH_0(X)=Z_0(X)/Z_0(X)_{rat}$$

be the group of zero-cycles on X and the Chow group of zero-cycles on X modulo rational equivalence respectively, where X_0 denotes the set of all closed points on X. Let

$$deg\ :\ CH_0(X)\to\mathbf{Z}\ ;\ \omega\to\sum_{x\in X_0} n_x[\kappa(x):k]$$

be the degree map with $\kappa(x)$ the residue field of $x\in X_0$. Let

$$I(X)=G.C.D.\{[\kappa(x):k] | x\in X_0\}.$$

By definition the image of the degree map is the subgroup $I(X)\mathbf{Z}\subset\mathbf{Z}$. $I(X)$ is an important invariant in the Diophantine problem of X. For example, if X is a quadric then $I(X)=1$ or 2 and X has a k-rationalpoint if and only if $I(X)=1$. Define $A_0(X)$ by the exact sequence

$$0\to A_0(X)\to CH_0(X)\xrightarrow{deg}\mathbf{Z}.$$

We are interested in the following problems.

$(Q1)$ What sort of structure does $A_0(X)$ carry? More precisely, is it finitely generated in case $[k:\mathbf{Q}]<\infty$? What kind of topology can it be given in case $[k:\mathbf{Q}_p]<\infty$?

$(Q2)$ Is there any algorithm to compute $I(X)$ or any criterion to have $I(X)=1$?

($Q2$) For a simple variety such as a rational variety, is there any algorithm to compute $A_0(X)$?

§1 The case of curves

First we recall the known results in case $\dim(X) = 1$. By the theory of Jacobian variety J_X of the curve X one knows that $A_0(X)$ is identified with $J_X(k)$, the group of k-rational points if $I(X) = 1$ and with a subgroup of finite index in $J_X(k)$ in general. In particular $A_0(X)$ is finitely generated in case $[k : \mathbf{Q}] < \infty$ by the theorem of Mordell-Weil. In case $[k : \mathbf{Q}_p] < \infty$ one uses a theorem of Mattuck [Mat] to deduce an isomorphism

$$A_0(X) \simeq \mathbf{Z}_p^{\oplus r} \oplus F,$$

for some integer $r \geq 0$ and a finite abelian group F.

Concerning $I(X)$, in case $[k : \mathbf{Q}_p] < \infty$ there exists a method to compute it even in arbitrary dimensional case. This will be explained later in Theorem D of §4. In case $[k : \mathbf{Q}] < \infty$ one may ask if the following Hasse principle holds in this context.

(H) Let P be the set of all places of k and let k_v be the completion of k at v. Put $X_v = X \times_k k_v$. Does $I(X_v) = 1$ for every $v \in P$ imply $I(X) = 1$?

Manin [Ma] defined a certain obstruction to (H) by using the Brauer-Grothendieck group $Br(X)$ [Gr] of X (this obstruction is called Brauer-Manin obstruction and it will be explained later in §6) and was able to produce counter examples to (H). Now the important question to arise is the uniqueness of the obstruction. The precise meaning of the 'uniqueness' will be explained later in §6. The following theorem is due to Manin in case that the genus of X is equal to 1 and generalized to arbitrary genus case by the author [Sa-2].

Theorem(A). *For a curve over number field, the Brauer-Manin obstruction to (H) is the unique one if and only if the Tate-Shafarevich group of the Jacobian of the curve is finite.*

§2 Brauer-Manin equivalence for zero-cycles on varieties over local fields

In case $\dim(X) > 1$ the study of the structure of $A_0(X)$ turns out to be very difficult. Manin introduced a new equivalence for zero-cycles which is coarser than the rational equivalence. It will be called the Brauer-Manin equivalence (BM equivalence) on $Z_0(X)$. A main point of our result is that one can obtain a fairly reasonable description of the structure of $Z_0(X)$ modulo the BM equivalence in case that the ground field k is a local field. Thus we assume $[k : \mathbf{Q}_p] < \infty$. Let $Br(X)$ be the Brauer-Grothendieck group of X [Gr], which is the group of certain equivalence classes $[\mathcal{A}]$ of sheaves of $\mathcal{O}_X$-algebras $\mathcal{A}$ locally free as $\mathcal{O}_X$-modules such that $\mathcal{A}_x := \mathcal{A} \otimes_{\mathcal{O}_X} \kappa(x)$ is a central simple algebra over $\kappa(x)$ for any $x \in X$. By definition $Br(\mathrm{Spec}(F))$ for a field F is the usual Brauer group $Br(F)$ of F. Now we define a pairing

$$< \ , \ > \ : Z_0(X) \times Br(X) \to Br(k)$$

by the formula

$$< \sum_{x \in X_0} n_x[x], [\mathcal{A}] >:= \sum_{x \in X_0} n_x N_{\kappa(x)/k}([\mathcal{A}_x]),$$

where $N_{\kappa(x)/k} \ : \ Br(\kappa(x)) \to Br(k)$ is the norm map for the finite extension $\kappa(x)/k$. Manin showed that the pairing factors through the rational equivalence on $Z_0(X)$. Combined with the isomorphism $Br(k) \simeq \mathbf{Q}/\mathbf{Z}$ following from the local class field theory of k, this gives rise to the canonical pairing

$$< \ , \ > \ : CH_0(X) \times Br(X) \to \mathbf{Q}/\mathbf{Z}.$$

We introduce the new equivalence on zero-cycles by

$$\omega \in Z_0(X), \ \omega \underset{BM}{\sim} 0 \Leftrightarrow < \omega, [\mathcal{A}] >= 0 \text{ for every } [\mathcal{A}] \in Br(X).$$

Let $CH_0(X)_{BM} \subset CH_0(X)$ be the subgroup of cycle classes of $\omega \underset{BM}{\sim} 0$ and write

$$CH_0(X)/BM = CH_0(X)/CH_0(X)_{BM},$$

$$A_0(X)/BM = A_0(X)/CH_0(X)_{BM}.$$

(It is easy to see $CH_0(X)_{BM} \subset A_0(X)$.) Our study of $A_0(X)$ is now divided into the two steps.

Problem A Study $A_0(X)/BM$.

Problem B Ask if $CH_0(X)_{BM} = 0$ or equivalently if the BM-equivalence coincides with the rational equivalence.

Concerning Problem B, in case $\dim(X) = 1$ Lichtenbaum showed that the BM-equivalence coincides with the rational equivalence. In case $\dim(X) > 1$ very little is known except the following.

Theorem(B). ([Sa-3]) *Assume* $\dim(X) = 2$, $H^2(X, \mathcal{O}_X) = 0$, X *is not of general type and the Albanese variety of* X *has potentially good reduction. Then the BM-equivalence coincides with the rational equivalence on* $Z_0(X)$.

§3 The structure of $A_0(X)/BM$.

Let the assumption and the notation be as in §2. In this section we state a result on the structure of $A_0(X)/BM$. For this we assume that there exists a regular model $\mathcal{X}$ of X over $\mathcal{O}_k$, the ring of integers of k. By definition $\mathcal{X}$ is a regular scheme endowed with a proper flat morphism

$$f \ : \ \mathcal{X} \to S = \mathrm{Spec}(\mathcal{O}_k)$$

with $\mathcal{X} \times_S \mathrm{Spec}(k) \simeq X$. Let $Y = f^{-1}(s)$ be the special fiber with the closed point $s \in S$. The following result is obtained by a joint work with J.-L. Colliot-Thélène.

Theorem(C). *(1) We have a canonical isomorphism*

$$A_0(X)/BM \simeq \mathrm{Hom}(Br(X)/Br(\mathcal{X}) + Br(k), \mathbf{Q}/\mathbf{Z}).$$

Moreover the groups are isomorphic to $\mathbf{Z}_p^{\oplus r} \oplus F$ *for some integer* $r \geq 0$ *and a finite abelian group* F. *Thus* $A_0(X)/BM$ *is endowed with p-adic topology which makes it a compact topological group.*

(2) Take a dense open subset $U \subset Y$ *and put*

$$X_U^{reg} = \{x \in X_0 | \overline{\{x\}} \text{ is regular and } \overline{\{x\}} \cap U \neq \emptyset\},$$

where $\overline{\{x\}}$ *denotes the closure of* $x \in X$ *in* $\mathcal{X}$. *Let* $\Sigma_U \subset CH_0(X)/BM$ *be the subgroup generated by cycles classes of* $x \in X_U^{reg}$. *Then* $\Sigma_U^0 := \Sigma_U \cap A_0(X)/BM$ *is dense in* $A_0(X)/BM$.

Here we include some words about the proof. For the proof of the first isomorphism in (1) one needs a close analysis of the behavior of the "ramification along Y" of $\mathcal{A} \in Br(X)$. This is done by using Kato's theory of Brauer groups of higher dimensional local fields. For the proof of the second statement of (1) one uses the finiteness theorem of generalized idele class group in the higher dimensional class field theory of varieties over finite fields.

§4. Application I (An explicit calculation of $I(X)$)

Let the assumption and the notation be as in §3. From Theorem C we deduce the following.

Theorem(D). *The following three numbers are all equal.*

(1) $I(X)$.

(2) The order of $\mathrm{Ker}(Br(k) \to Br(X)/Br(\mathcal{X}))$.

(3) G.C.D.$\{m_i e_i | 1 \le i \le n\}$, *where* Y_i $(1 \le i \le n)$ *are the irreducible components of the special fiber* $Y = f^{-1}(s)$, m_i *is the multiplicity of* Y_i *in* Y *and* $e_i = [F_i : F]$ *with* $F = \kappa(s)$ *the residue field of* k *and* F_i *the algebraic closure of* F *in the function field* $F(Y_i)$ *of* Y_i.

The coincidence of the first and second numbers immediately follows from the following commutative diagram

$$\begin{array}{ccccc} & & Br(k) & \rightarrow & Br(X)/Br(\mathcal{X}) \\ & & \downarrow \cong & & \downarrow \\ 0 \rightarrow (\mathbf{Z}/I(X)\mathbf{Z})^* & \rightarrow & \mathbf{Z}^* & \xrightarrow{deg^*} & CH_0(X)^* \end{array}$$

with the right vertical arrow injective by Theorem C, where we use the convention $A^* = \mathrm{Hom}(A, \mathbf{Q}/\mathbf{Z})$. The proof of the coincidence of the first and second numbers requires some argument using the class field theoretic study of the structure of Brauer group of a henselian local ring with finite residue field in [Sa-1].

§5. Application II (An explicit calculation of $A_0(X)$)

Let k be a local field. Let X be a rational surface over k, namely a projective smooth surface such that $X \times_k L$ is birational to $\mathbf{P}^2_L$ for some finite extension L/k. By Theorem B and C we have the isomorphism

$$A_0(X) \simeq \mathrm{Hom}(Br(X)/Br(\mathcal{X}) + Br(k), \mathbf{Q}/\mathbf{Z}).$$

On the other hand we have the isomorphism for such a surface

$$Br(X)/Br(k) \simeq H^1(k, \mathrm{Pic}(\overline{X})),$$

where $\overline{X} = X \times_k \overline{k}$ with $\overline{k}$ an algebraic closure of k. If one can give an explicit generator of $\mathrm{Pic}(\overline{X})$ with the Galois action one can compute $Br(X)/Br(k)$. For example Manin has done it for the cubic surface

$$X \ : \ X_0^3 + X_1^3 + X_2^3 + aX_3^3 = 0, \quad a \in k^\times.$$

Note X is birational to $\mathbf{P}^2_k$ if and only if $a \in (k^\times)^3$. Thus, to compute $A_0(X)$ it suffices to determine which elements of $H^1(k, \mathrm{Pic}(\overline{X}))$ is 'unramified' along the special fiber Y. This is actually done to obtain the following.

Theorem(E). *Let X be as above and assume $a \notin (k^\times)^3$. Then*

$$A_0(X) \simeq \begin{cases} \mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z} & \text{if } \zeta_3 \in k, \\ \mathbf{Z}/3\mathbf{Z} & \text{if } \zeta_3 \in k \end{cases}$$

where ζ_3 denotes a primitive cubic root of unity.

§6. Application III (Global case)

In this section (contrary to the previous ones) k denotes a number field, namely $[k : \mathbf{Q}] < \infty$. Let X be a proper smooth variety over k with a regular model $\mathcal{X}$ over the ring $\mathcal{O}_k$ of integers in k. Let P be the set of all places of k and k_v the completion of k at $v \in P$. Put $X_v = X \times_k k_v$. One has the following conjecture ([K-S] and [CT]).

Conjecture(F). The following sequence is exact

$$\varprojlim_n CH_0(X)/n \to \prod_{v \in P} CH_0(X_v)/BM \xrightarrow{\alpha} \mathrm{Hom}(Br(X)/Br(\mathcal{X}), \mathbf{Q}/\mathbf{Z}) \to 0.$$

Here the map α is induced by the pairings for $v \in P$

$$< \; , \; >_v \;\; : CH_0(X_v) \times Br(X_v) \to \mathbf{Q}/\mathbf{Z}.$$

The product $\prod_{v \in P}$ is the restricted product with respect to the compact subgroups $A_0(X_v)/BM \subset CH_0(X_v)/BM$ (cf. Theorem C).

There are a couple of evidences for the conjecture.

(1) In case $\dim(X) = 1$ the conjecture is a consequence of the finiteness of the Tate-Shafarevich group of the Jacobian of X (cf. Theorem A of §1).

(2) In [Sa-2] a general conjecture is made on various motivic cohomology group of arithmetic schemes. For example the higher dimensional class field theory for arithmetic schemes by Parshin, Bloch, Kato-Saito fits nicely into this picture. The above conjecture is implied by this general conjecture.

As corollaries of Theorem C in §3 we obtain the following.

Theorem(G). *The map*

$$\prod_{v \in P} CH_0(X_v) \xrightarrow{\alpha} \mathrm{Hom}(Br(X)/Br(\mathcal{X}), \mathbf{Q}/\mathbf{Z})$$

is surjective.

Theorem(H). *Assume Conjecture(F). Assume that $I(X) > 1$ and that there exists a place w of k such that $I(X_w) = 1$. Equivalently we assume that there is no zero-cycle of degree one on X and that there exist $w \in P$ such that there exists a zero-cycle of degree one on X_w. Then there exists $\mathcal{A} \in Br(X)$ such that*

$$< \omega_v, \mathcal{A} >_v = 0 \quad \textit{for any } \omega_v \in Z_0(X_v) \textit{ with } v \neq w$$

and that

$$< \omega_w, \mathcal{A} >_w = 1/I \quad \textit{for any } \omega_w \in Z_0(X_w) \textit{ with } deg(\omega_w) = 1.$$

Recall that the (conjectural) uniqueness of the Brauer-Manin obstruction for zero-cycles of degree one on X asserts that for a given collection $\{\omega_v\}_{v \in P}$ of zero-cycles of degree one on X_v, the existence of $\mathcal{A} \in Br(X)$ such that

$$\sum_{v \in P} < \omega_v, \mathcal{A} >_v \neq 0$$

implies $I(X) > 1$. Now the conclusion of Theorem H is much stronger than this.

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