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A note on scrolls of smallest embedded codimension

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§0. Introduction

The situation considered in this note is as follows. Let M be an algebraic submanifold of $\mathbf{P}^N$ with $n = \dim M$. M is said to be a scroll over S if there is a surjective morphism $\pi : M \to S$ such that every fiber $F_x = \pi^{-1}(x)$ over $x \in S$ is a linear subspace in $\mathbf{P}^N$ of dimension $r - 1 = n - s$, where $s = \dim S$. This is equivalent to saying that $M \cong \mathbf{P}_S(\mathcal{E})$ for some vector bundle $\mathcal{E}$ of rank $r = n - s + 1$ and the tautological bundle $H(\mathcal{E})$ is the hyperplane section bundle of M.

We have $b_i(M) = b_i(\mathbf{P}^N)$ for $i \le 2n - N$ by Barth-Lefschetz Theorem, hence $N \ge 2n - 1$ for scrolls, since otherwise $1 + b_2(S) = b_2(M) = b_2(\mathbf{P}^N) = 1$, contradiction. Thus we want to study scrolls such that $N = 2n - 1$.

The case $s = 1$ was studied by Lanteri-Turrini [LT], who showed that M is the Segre scroll over $\mathbf{P}^1$ in this case; namely $S \cong \mathbf{P}^1$, $\mathcal{E} \cong \mathcal{O}(1)^{\oplus n}$, $M \cong \mathbf{P}^1 \times \mathbf{P}^{n-1}$ and $M \subset \mathbf{P}^N$ is the Segre embedding. In this paper we are interested mainly in the case $s = 2$.

This problem was studied by Ottaviani [Ot] and Beltrametti-Schneider-Sommese [BSS], [BS] in case $n = 3$ and by Ionescu-Toma [IT] for general n. Their results are as follows.

Theorem. (cf. [Ot]). *Let $M \subset \mathbf{P}^5$ be a three-dimensional scroll over a surface S. Then one of the following conditions is satisfied.*
(1) (Segre scroll) $S \cong \mathbf{P}^2$, $\mathcal{E} \cong \mathcal{O}(1)^{\oplus 2}$, $d = \deg M = 3$.
(2) (Bordiga scroll) $S \cong \mathbf{P}^2$, $c_1(\mathcal{E}) = \mathcal{O}(4)$, $c_2(\mathcal{E}) = 10$ *and* $d = 6$.
(3) (Palatini scroll) *S is isomorphic to a cubic surface,* $c_1(\mathcal{E}) = \mathcal{O}_S(2)$, $c_2(\mathcal{E}) = 5$ *and* $d = 7$.
(4) (K3 scroll) *S is a K3-surface obtained as a linear section of the Grassman variety parametrizing lines in $\mathbf{P}^5$ embedded by Plücker, and $\mathcal{E}$ is the restriction of the tautological vector bundle.* $c_2(\mathcal{E}) = 5$ *and* $d = 9$.

All these four cases actually occur.

Theorem. (cf. [IT]). *Let $M \subset \mathbf{P}^N$ be a scroll over a surface S with $N = 2n - 1$, $n > 3$ and let $\mathcal{E}$ and $d = \deg M$ be as above. Then, M is one of the following types:*
(S_n) $S \cong \mathbf{P}^2$, $\mathcal{E} \cong \mathcal{O}(1)^{\oplus(n-1)}$, $M \cong \mathbf{P}^2 \times \mathbf{P}^{n-2}$, $d = n(n-1)/2$.
(B_n) $S \cong \mathbf{P}^2$, $c_1(\mathcal{E}) = \mathcal{O}(n+1)$, $c_2(\mathcal{E}) = (n+1)(n+2)/2$, $d = n(n+1)/2$.
(M_n) *S is a K3 surface,* $c_1(\mathcal{E})^2 = 2n^2 - 4$, $c_2(\mathcal{E}) = n^2 - 4$, $d = n^2$.
$(?_n)$ *S is a surface of general type.*

The existence of scrolls of the above type (S_n) is classical and is due to Segre. The case (B_n) is shown to exist for every n in [IT]. The existence of the type (M_n) is proved by Mukai ([Mu]). On the other hand, no example of type $(?_n)$ is found: [IT] suspect rather that there is no such scroll. Any way, such scrolls must satisfy several numerical relations among their invariants. Here I propose a conjecture concerning further classifications derived from these relations, which is verified for $n \le 11$ by hand and for $n \le 1100$ by a computer. In particular, a scroll of the type $(?_n)$ does not exist unless $n = 6, 10, 11, 12, 16, 18, 20, 24, 30, \cdots$. On the other hand, no example is known and the existence problem is unsettled for these n (and also for $n > 1100$, of course).

Mathematical tools used here are almost the same as [IT], but we review them here for the convenience of the reader. When I started this study, I was not aware of this paper [IT]. I would like to express my hearty thanks to Professors Ottaviani who informed me of the result in [IT]. I thank Professor Mukai, who communicated to me the existence of scrolls of the type (M_n). I also thank Professors Lanteri and Schneider for their cooperations in e-mail correspondence

during the preparation of this paper.

§1. Computing Chern classes

(1.1) Throughout this paper let M be a scroll in $\mathbf{P}^{2n-1}$ over S as in §0. Let $\mathcal{E}$ be the vector bundle on S of rank $r = n - s + 1$, $s = \dim S$, such that $M \cong \mathbf{P}_S(\mathcal{E})$ and the tautological bundle $H(\mathcal{E})$ is the hyperplane section bundle $\mathcal{O}_M(1)$, which will be denoted simply by H from now on. We put $d = \deg M = H^n\{M\}$.

(1.2) *Fact. Via the ring homomorphism* $\pi^* : H^{\cdot}(S) \to H^{\cdot}(M)$, *the cohomology ring of* M *becomes a free* $H^{\cdot}(S)$*-module generated by* $1, h, h^2, \cdots, h^{r-1}$, *where* $h = c_1(H) \in H^2(M)$. *Moreover* $\sum_{i=0}^{r}(-h)^i e_{r-i} = 0$ *in* $H^{\cdot}(M)$, *where* $e_j = \pi^* c_j(\mathcal{E})$.

This is well known for general vector bundle $\mathcal{E}$ of rank r. Thus, the ring structure of $H^{\cdot}(M)$ is determined by $H^{\cdot}(S)$ and Chern classes of $\mathcal{E}$.

(1.3) *Fact.* Put $s_i(\mathcal{E}) = \pi_* h^{r-1+i} \in H^{2i}(S)$ and $s(\mathcal{E}) = \sum_{i=0}^{\infty} s_i(\mathcal{E}) \in H^{\cdot}(S)$. *Then* $s(\mathcal{E})c(\mathcal{E}^{\vee}) = 1$, *where* $c(\mathcal{E}^{\vee})$ *is the total Chern class of the dual bundle* $\mathcal{E}^{\vee}$ *of* $\mathcal{E}$.

This is also standard. $s_i(\mathcal{E})$ is called the i-th Segre class of $\mathcal{E}$ and $s(\mathcal{E})$ is called the total Sege class of $\mathcal{E}$. Moreover the following formulas are well known:

$$s_1(\mathcal{E}) = c_1(\mathcal{E}),$$
$$s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}),$$
$$\cdots$$

(1.4) **Corollary.** *For any vector bundle* E *on* X *with* $\operatorname{rank} E = r$ *and for any line bundle* L *on* X *with* $c_1(L) = \ell$, *we have* $s_i(E \otimes L) = \sum_j \binom{r-1+i}{j} s_{i-j}(E)\ell^j$.

Indeed, $s_i(E \otimes L) = \pi_*((h + \ell)^{(r-1+i)}) = \sum_j \binom{r-1+i}{j} \pi_* h^{r-1+i-j} \cdot \ell^j$ for $\pi : \mathbf{P}(E) \to X$.

(1.5) Let $\pi : M = \mathbf{P}(\mathcal{E}) \to S$ be the projection. Let $\mathcal{A}$ be the kernel of the natural surjection $\pi^*\mathcal{E} \to \mathcal{O}[H]$. Then we have an exact sequence $0 \to \mathcal{O} \to \mathcal{E}^{\vee} \otimes H \to \mathcal{A}^{\vee} \otimes H \to 0$. This is identified with the relative Euler sequence, and so we have the exact sequence $0 \to \mathcal{A}^{\vee} \otimes H \to \Theta_M \to \pi^*\Theta_S \to 0$, where Θ_X denotes the tangent bundle of X.

From these exact sequences we obtain the following relation $c(\Theta_M) = \pi^* c(\Theta_S) c(\pi^*\mathcal{E}^{\vee} \otimes H)$ of total Chern classes.

(1.6) Let $\mathcal{N}$ be the normal bundle of M in $\mathbf{P}^{2n-1}$ and let Θ be the restriction of the tangent bundle of $\mathbf{P}^{2n-1}$ to M. Then we have $c(\Theta_M)c(\mathcal{N}) = c(\Theta) = (1 + h)^{2n}$, where $h = c_1(H)$.

Combining (1.5), we get $c(\mathcal{N}) = (1 + h)^{2n} c(\pi^*\mathcal{E}^{\vee} \otimes H)^{-1} \pi^* c(\Theta_S)^{-1} = (1 + h)^{2n} s(\pi^*\mathcal{E} \otimes [-H]) \pi^* s(\Omega_S)$.

(1.7) In the following computation, $\pi^*\alpha$ will be denoted simply by α for $\alpha \in H^{\cdot}(S)$.

By (1.4), we have $s(\mathcal{E}_M \otimes [-H]) = \sum_{i=0}^{\infty}(\sum_{j=0}^{r-1+i} \binom{r-1+i}{j} s_{i-j}(\mathcal{E})(-h)^j)$. Therefore the component of $(1 + h)^{2n} s(\mathcal{E}_M \otimes [-H])$ of degree $2k$ is $\sum_{i,j} \binom{2n}{k-i}\binom{r-1+i}{j}(-1)^j h^{k-i+j} s_{i-j}(\mathcal{E})$ $= \sum_{\ell}(\sum_j \binom{2n}{k-j-\ell}\binom{r-1+j+\ell}{j}(-1)^j) h^{k-\ell} s_{\ell}(\mathcal{E}) = \sum_{\ell} \binom{2n-r-\ell}{k-\ell} h^{k-\ell} s_{\ell}(\mathcal{E})$ by the following

(1.8) *Claim.* $\sum_{(a,b)|a+b=c} \binom{m-1+a}{a}(-1)^a \binom{p}{b} = \binom{p-m}{c}$.

To see this, use the Taylor expansion $(1 + T)^{-m} = \sum_{a \geq 0} \binom{m-1+a}{a}(-T)^a$ and compute the coefficients of T^c in $(1 + T)^{-m}(1 + T)^p = (1 + T)^{p-m}$.

(1.9) Combining (1.6) and (1.7), we get a formula for $c(\mathcal{N})$. On the other hand, we have $c_n(\mathcal{N}) = 0$ and $c_{n-1}(\mathcal{N}) = dh^{n-1}$ since M is of codimension $n - 1$ in $\mathbf{P}^N$. This gives non-trivial relations among Chern classes of $\mathcal{E}$ and Θ_S. In the next section we analyse them precisely in case $s = \dim S = 2$ and $r = n - 1$.

§2. Over a surface

From now on we assume $s = \dim S = 2$ and set $e_j = c_j(\mathcal{E})$, $\gamma_i = c_i(\Theta_S)$.

(2.1) From (1.6) and (1.7) we obtain
$c_{n-1}(\mathcal{N}) = \binom{n+1}{n-1}h^{n-1} + \binom{n}{n-2}h^{n-2}e_1 + \binom{n-1}{n-3}h^{n-3}(e_1^2 - e_2) - \binom{n+1}{n-2}h^{n-2}\gamma_1 - \binom{n}{n-3}h^{n-3}e_1\gamma_1 + \binom{n+1}{n-3}h^{n-3}(\gamma_1^2 - \gamma_2)$.
This is dh^{n-1}, while $h^{n-1} - h^{n-2}e_1 + h^{n-3}e_2 = 0$ is the unique relation in $H^{\cdot}(M)$ (cf. (1.2)). Hence, substituting $h^{n-1} = h^{n-2}e_1 - h^{n-3}e_2$ and comparing the coefficients of h^{n-2} and h^{n-3}, we get the following relations in $H^{\cdot}(S)$:

(i) $$(n^2 - d)e_1 = (n+1)n(n-1)\gamma_1/6, \qquad \text{and}$$

(ii) $$\tfrac{(n-1)(n-2)}{2}e_1^2 + (d - n^2 + n - 1)e_2 - \tfrac{n(n-1)(n-2)}{6}e_1\gamma_1 + \tfrac{(n+1)n(n-1)(n-2)}{24}(\gamma_1^2 - \gamma_2) = 0.$$

Next from $c_n(\mathcal{N}) = 0$ we get

(iii) $$3e_1^2 - 2e_2 - ne_1\gamma_1 + \tfrac{(n+1)(n-1)}{6}(\gamma_1^2 - \gamma_2) = 0.$$

(2.2) Eliminating γ_2 from (ii) and (iii), we get $-3(n^2-4)e_1^2 + 6(2d - n^2 - 2)e_2 + n(n-2)(n+2)e_1\gamma_1 = 0$. Eliminating γ_1 further using (i) and noting $d = (e_1^2 - e_2)\{S\}$, we obtain

(*) $$\{2(q+2)d - q(q+5)\}\{(q+2)(q-4)d - (q+2)^2e_2 + (q-1)(q-4)\} = -q(q-1)(q-4)(q+5)$$

for $q = n^2$.

(2.3) Thus, $2(q+2)d - q(q+5)$ is one of the finitely many divisors of the right hand side, so there are only finitely many numerical possibilities for d and e_2. Note that they are positive integers since $\mathcal{E}$ is ample.

(2.4) For each (d, e_2) satisfying (*), we examine the relation (i). By the result [IT], it suffices to consider the case where the canonical bundle of S is ample, or equivalently, $d > q = n^2$. Let $6(d-n^2)/(n+1)n(n-1) = a/b$ for coprime integers a, b. Then $ae_1 \sim -b\gamma_1$, so $e_1 \sim bA$ and $K_S \sim -\gamma_1 \sim aA$ for some ample line bundle A on S. Therefore

$$e_1^2 = d + e_2 \quad \text{is divided by} \quad b^2$$

(2.5) In the above case, we have $\gamma_1^2 = a^2(d+e_2)/b^2$ and $e_1\gamma_1 = -a(d+e_2)/b$. Using (iii), we solve γ_2. The result has to satisfy the Noether relation:

$$\gamma_1^2 + \gamma_2 \equiv 0 \text{ modulo } 12.$$

(2.6) It is easy to produce a computer programm to enumerate pairs (n, d) satisfying the numerical conditions (2.2), (2.4) and (2.5). In view of the result of our experiment, we make the following

Conjecture. *Any pair (n, d) with $n > 3$ as above is one of the following types:*
(1) $n \equiv 0, 2, 6, 12$ *or* 16 *modulo* 18 *and* $d = q(q+5)/6$ *for* $q = n^2$. *Moreover* $e_2 = (q-4)(q+3)/6$, $K_S \sim ne_1$, $K_S^2 = q(q^2 + 2q - 6)/3$, $\gamma_2 = e(S) = (q^3 + 8q^2 + 24q + 36)/3$ *and* $\chi(\mathcal{O}_S) = (q^3 + 5q^2 + 9q + 18)/18$ *for the corresponding scrolls.*
(2) $n = 10$ *and* $d = 595$. *Moreover* $e_2 = 561$, $K_S \sim 3e_1$, $K_S^2 = 10404$, $e(S) = 12648$ *and* $\chi(\mathcal{O}_S) = 1921$.
(3) $n = 11$ *and* $d = 231$. *Moreover* $e_2 = 221$, $2K_S \sim e_1$, $K_S^2 = 113$, $e(S) = 283$ *and* $\chi(\mathcal{O}_S) = 33$.

(2.7) *Remarks.*
(1) The above conjecture is verified for $n \leq 11$ by my hand, and for $n \leq 1100$ by a personal

computer. In particular, the case with smallest n is $(n, d) = (6, 246)$. [IT] apparently claims that such a case can be ruled out by "divisibility manipulations", but I cannot understand the reasoning.
(2) The above condition for n of the type (1) is equivalent to $q \equiv 0$ or 4 modulo 18.
(3) It is perhaps a delicate problem whether scrolls of the type $(?_n)$ exist or not for a pair (n, d) in (2.6). I find no example at present. To settle the problem, we need some more geometric observations. I feel that the case (2.6.3) might be of particular interest, since this is the unique case with odd n and moreover the invariants are relatively small.
(4) The sectional genus $g = g(M, \mathcal{O}(1))$ can be computed by using the relation $2g - 2 = (K_S + e_1)e_1\{S\}$, and g is bounded by the Castelnuovo inequality. But it turns out that no pair (n, d) in (2.6) is thus ruled out.

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