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VECTOR FIELDS ON CALABI–YAU MANIFOLDS IN CHARACTERISTIC p

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1. Statement of Main Results

A compact Kähler manifold X with numerically trivial canonical class splits into a product of a complex torus, symplectic manifolds and unitary manifolds after replacing X by an finite étale covering $\tilde{X}$ (Bogomolov decomposition, see [B]). In particular, any non-trivial vector field on X comes from the torus component of $\tilde{X}$.

In positive characteristic, however, very little is known on the vector field of a Calabi-Yau manifold, by which we mean a smooth projective variety with numerically trivial Chern class. The most important contribution to this topic so far is a theorem of A.N. Rudakov and I.R. Shafarevich to the effect that a K3 surface does not admit a non-zero global vector field [RS]. In a crucial step toward the main result, they showed that a K3 surface which carries a non-trivial vector field would necessarily be unirational [RS, Theorem 4]. The objective of this paper is to give a higher dimensional analogue of this result.

In this paper all varieties are defined over an algebraically closed field of arbitrary characteristic, but we are principally interested in positive characteristic cases.

Our result is this:

Theorem 1.1. *Let X be a smooth projective variety with numerically trivial canonical divisor (i.e. X is a Calabi-Yau manifold). Let F be the subsheaf of the tangent sheaf T_X generated by the global sections $H^0(X, T_X)$. Then exactly one of the following four cases occurs:*

(a) $F = 0$; namely X has no global vector field.
(b) $F = \mathcal{O}_X^{\oplus r} \neq 0$ and T_X/F is locally free.
(c) $F = \mathcal{O}_X^{\oplus r} \neq 0$, T_X/F is not locally free, and X is uniruled.
(d) $c_1(F) \neq 0$ and X is uniruled.

In case (c) or (d), there exists a purely inseparable cover $\tilde{Y}$ of X, such that $\tilde{Y}$ is normal with $-K_{\tilde{Y}}$ is numerically equivalent to 0 or to a non-zero, effective divisor, according as we are in case (c) or (d). The cases (c) (d) are automatically ruled out when the characteristic of the ground field is zero.

Remark. Over the complex numbers, our assertion immediately follows from the Bogomolov decomposition aforementioned.

Over an arbitrary ground field, we have the case (b) if X is a product of an r-dimensional abelian variety W and an $(n-r)$-dimensional Calabi-Yau manifold V. More generally, we come across this situation on an étale quotient X of $W \times V$ by a finite group action which leaves $\mathrm{pr}_W^* H^0(W, T_W)$ invariant. In dimension two, (b) happens if and only if X is an abelian surface or a bielliptic surface, which means X is indeed of the form $W \times V/$(finite étale action). The author does not know whether this passes over to higher dimension.

From Theorem follows:

Corollary 1.2. *Let X be a non-uniruled Calabi-Yau n-fold. If X admits r linearly independent vector fields, then any Chern polynomial $P(X) = P(c_1(X), \dots, c_n(X))$ of pure degree i is rationally equivalent to 0 for $i > n - r$. In particular, given an rational representation $\rho : GL(n,k) \to GL(s,k)$ and the associated vector bundle T_X^ρ, the Euler characteristic $\chi(X, T_X^\rho)$ vanishes. For example, $\chi(X, \mathcal{O}_X) = \chi(X, T_X) = \chi(X, \Omega_X^s) = 0, s \in \mathbb{N}$.*

For instance, when $c_n(X) \neq 0$, a Calabi-Yau manifold with non-trivial vector field is necessarily uniruled.

Corollary 1.3. *Let X be a smooth surface with numerically trivial canonical divisor and vanishing irregularity $q(X) = \dim H^1(X, \mathcal{O}_X)$. If X carries a non-zero vector field, then X is unirational.*

In fact, from our condition, $\chi(X, \mathcal{O}_X) = 1$ or 2 according as X is a K3 surface or an Enriques surface, and hence the Albanese variety of X is trivial, $c_2(X) \neq 0$. Hence X must be uniruled by Corollary 1.2. If case (d) in Theorem 1.1 occurs, then the purely inseparable cover $\tilde{Y}$ is ruled by the Enriques classification, with the Albanese variety being trivial. Hence $\tilde{Y}$ is rational so that X is unirational. When case (c) occurs, then $K_{\tilde{Y}}$ is numerically trivial, with singularities at the points where T_X/F is not locally free. Taking the minimal resolution of $\tilde{Y}$, we infer that $\tilde{Y}$ is rational unless all the singularities are rational double points. In the latter exceptional case, the minimal resolution must be either a K3 surface (when X is K3) or an Enriques surface (when X is Enriques), and hence the Euler number of the singular variety $\tilde{Y}$ is strictly smaller than 24 or 12. On the other hand, the purely inseparable morphism $\tilde{Y} \to X$ is a homeomorphism, so that $e(X) = e(\tilde{Y}) = 24$ or 12, a contradiction. We can thus recover Theorem 4 of Rudakov-Shafarevich.

2. Sheaves of Jets and Differential Operators

Let us briefly review the theory of differential operators and jets. The treatment here essentially follows that of A. Grothendieck [EGA I], to which the reader is referred for details.

Every variety or scheme is defined over an algebraically closed field k. Unless otherwise mentioned, k has positive characteristic p.

Given an element $a \in \mathcal{O}_X$, the multiplication by a is a k-linear operator on $\mathcal{O}_X$. We define a (local) differential operator of order 0 as an element $\in \mathcal{O}_X = \mathrm{Diff}_X^0$. A *local differential operator of order* $\leq i$ is inductively defined as a locally defined k-linear operator $\xi : \mathcal{O}_X \to \mathcal{O}_X$ such that $[\xi, a] = \xi \circ a - a \circ \xi$ is a differential operator of order $\leq i - 1$

for arbitrary $a \in \mathrm{Diff}_X^0 = \mathcal{O}_X$. A local vector field $\in T_X$ is viewed as a differential operator of order exactly one. Conversely a differential operator of order ≤ 1 is a sum of a local vector field and a differential operator of order zero (i.e. a function). Denote by Diff_X^i the sheaf of local differential operators of order $\leq i$. The left multiplication by $a \in \mathcal{O}_X = \mathrm{Diff}_X^0$, $\xi \mapsto a\xi = a \circ \xi$ defines an left $\mathcal{O}_X$-module structure on Diff_X^i. The union $\mathrm{Diff}_X = \bigcup \mathrm{Diff}_X^i$ is thus a left $\mathcal{O}_X$-module as well as a non-commutative k-algebra, because $\mathrm{Diff}_X^i \mathrm{Diff}_X^j \subset \mathrm{Diff}_X^{i+j}$. Note, however, Diff_X is not an $\mathcal{O}_X$-algebra.

The sheaf of i-th jets is defined to be the sheaf $\overline{\mathrm{Jet}}_X^{[i]} = \mathcal{O}_X \otimes_k \mathcal{O}_X / \mathcal{I}_X^{i+1}$, where $\mathcal{I}_X$ is the defining ideal of the diagonal $\Delta_X \subset X \times X$. The projective limit

$$\overline{\mathrm{Jet}}_X = \varprojlim \mathcal{O}_X \otimes \mathcal{O}_X / \mathcal{I}_X^{i+1}$$

is called the sheaf of *infinite jets*. $\overline{\mathrm{Jet}}_X$ is a sheaf of rings, which admits a canonical direct sum (augumentation algebra) decomposition $\overline{\mathrm{Jet}}_X = \mathcal{O}_X \oplus \mathrm{Jet}_X$, where $\mathcal{O}_X$ is identified with $\mathcal{O}_X \otimes 1 \subset \mathcal{O}_X \otimes \mathcal{O}_X$, and Jet_X is (the completion of) the ideal generated by $\mathcal{I}_X$. The projection $\mathrm{pr} : \overline{\mathrm{Jet}}_X \to \mathrm{Jet}_X$ is defined by $\mathrm{pr}(a \otimes b) = a \otimes b - ab \otimes 1$.

Given $a \in \mathcal{O}_X$, we denote by $\overline{d}a$ and by da the elements $1 \otimes a \in \overline{\mathrm{Jet}}_X$ and $\mathrm{pr}(\overline{d}(a)) = 1 \otimes a - a \otimes 1 \in \mathrm{Jet}_X$. $\overline{d}$ is a ring homomorphism. The correspondences $a \mapsto \overline{d}a$ and $a \mapsto da$ define injective k-linear maps $\mathcal{O}_X \to \overline{\mathrm{Jet}}_X$ and $\mathcal{O}_X/k \to \mathrm{Jet}_X$, respectively. If X is smooth and we introduce a local coordinate $(x_1, \ldots, x_n)$, the sheaf of jets $\overline{\mathrm{Jet}}_X$ is locally isomorphic to $\mathcal{O}_X[[dx_1, \ldots, dx_n]]$, and Jet_X is the ideal generated by $dx_1, \ldots, dx_n$.

By definition, $\overline{\mathrm{Jet}}_X = \mathcal{O}_X \oplus \mathrm{Jet}_X$ has a structure of $\mathcal{O}_X$-bimodule. In what follows, we view $\overline{\mathrm{Jet}}_X$ as a left $\mathcal{O}_X$-module; namely, for $a \in \mathcal{O}_X$ and $b \otimes c \in \mathrm{Jet}_X$, the multiplication $a(b \otimes c)$ is understood as $ab \otimes c$. Then the projection $\mathrm{pr} : \overline{\mathrm{Jet}}_X \to \mathrm{Jet}_X$ is $\mathcal{O}_X$-linear.

The action of a differential operator on $\mathcal{O}_X$ is naturally extended to an $\mathcal{O}_X$-linear homomorphism $\overline{\mathrm{Jet}}_X \to \mathcal{O}_X$ by the formula $\xi(a \otimes b) = a\xi(b)$. Indeed, if a differential operator ξ is of order $\leq i$, then ξ turns out to kill Jet_X^i, so that $\xi(\sum a_I dx^I)$ is actually a finite sum. It is known that Diff_X^i, the sheaf of differential operators of order $\leq i$, is identical with $\mathcal{H}om_{\mathcal{O}_X}(\overline{\mathrm{Jet}}_X^{[i]}, \mathcal{O}_X)$. In particular, $\mathrm{Diff}_X^i / \mathrm{Diff}_X^{i-1}$ is the dual sheaf of $\mathrm{Jet}_X^i / \mathrm{Jet}_X^{i+1} \simeq \mathrm{Sym}_{\mathcal{O}_X}^i \Omega_X^1$. The equivalence class $[\xi] \in \mathrm{Diff}_X^i / \mathrm{Diff}_X^{i-1} = (\mathrm{Sym}_{\mathcal{O}_X}^i \Omega_X^1)^*$ of $\xi \in \mathrm{Diff}_X^i$ is nothing but the i-th *principal symbol* of the operator.

In characteristic zero, the ring of differential operators is generated by vector fields. On the contrary in positive characteristic p, Diff_X is not finitely generated, while Jet_X is essentially finitely generated. When X is smooth, generators of the left $\mathcal{O}_X$-algebra Diff_X is given as follows. Let $q = p^m$ be a power of p. As a local $\mathcal{O}_X$-basis of the locally free sheaf $\mathrm{Jet}_X^{[q]}$, we choose $dx_1^q, \ldots, dx_n^q, f_1, \ldots, f_N$ such that f_j is a monomial of degree q in the dx_i involving at least two factors. The $\mathcal{O}_X$-linear map $\partial_i^{(q)}$ defined by $\partial_i^{(q)}(dx_j^q) = \delta_{ij}$, $\delta_i(f_j) = 0$ is said to be the divided q-th power of $\partial_i = \partial / \partial x_i$. In this notation, Diff_X is generated by the polynomials in $\partial_i, \partial_i^{(p)}, \partial_i^{(p^2)}, \ldots$ as an left $\mathcal{O}_X$-module. We have

$$(\partial_i^{(q)})^j((dx)^{qj}) = j!,$$

so that $(\partial_i^{(q)})^p = 0$. The elements $dx^I, I = (i_1, \ldots, i_n), |i| \leq m$ form a local basis of the $\mathcal{O}_X$-module $\overline{\mathrm{Jet}}_X^{[m]} = \mathcal{O}_X \oplus \mathrm{Jet}_X^{[m]}$ of finite rank. For each index i_j, let $i_j = a_{0j} + a_{1j}p + a_{2j}p^2 + \cdots$

be the p-adic representation ($0 \leq a_{\nu j} < p$). The dual basis of the dual sheaf Diff_X^m is then given by

$$\prod_{\nu,j} \frac{(\partial_j^{(p^\nu)})^{a_{\nu j}}}{a_{\nu j}!}.$$

Let R be an $\mathcal{O}_X$-algebra. Thanks to the natural (left) $\mathcal{O}_X$-module structures on Jet_X and Diff_X, compatible with the paring, we can naturally define an R-bilinear pairing $R \otimes \mathrm{Diff}_X \times R \otimes \mathrm{Jet}_X \to R$, denoted by the symbol $\langle \ | \ \rangle$.

3. Proof of Main Theorem

General theory in the preveous section in mind, we prove Theorem 1.1. In this section the ground field k is always algebraically closed and of characteristic $p > 0$.

Let $F \subset T_X$ be an involutive, p-closed saturated subsheaf. The functions $\in \mathcal{O}_X$ that are killed by F form a subring $\subset \mathcal{O}_X$, denoted by $\mathcal{O}_Y$, which defines a normal variety $Y = X/F$ (cf. Rudakov-Shafarevich [RS], Ekedahl [E], Shepherd-Barron [SB]). There is an inclusion relation $\mathcal{O}_X \supset \mathcal{O}_Y \supset \mathcal{O}_X^{(p)}$. In particular, we have finite, purely inseparable morphisms $\pi : X \to Y$ and $\varphi : Y \to X'$, where $\varphi\pi : X \to X'$ is the geometric Frobenius. The mapping degree of π and φ are p^r and p^{n-r}, respectively, where $r = \mathrm{rank} F$.

Denote the smooth locus of Y by $Y^\circ \subset Y$, and take a local parameter system $(y_1, \ldots, y_n)$ of Y° in such a way that $y_1, \ldots, y_{n-r}, z_1, \ldots, z_r$ is a local parameter system on $X^\circ = \pi^{-1}(Y^\circ)$ and that $y_{n-r+i} = z_i^p$. Then F, when restricted on X°, is the subbundle of T_X generated by $\partial/\partial z_1, \ldots, \partial/\partial z_r$.

Consider the left ideal $\mathrm{Diff}_X F \subset \mathrm{Diff}_X$ generated by F, which determines the null-submodule

$$(\mathrm{Diff}_X F)^\perp = \{a \in \overline{\mathrm{Jet}}_X | \ \langle \mathrm{Diff}_X F | a\rangle = 0\} \subset \overline{\mathrm{Jet}}_X.$$

It is easy to see that, on the open subset X°, $(\mathrm{Diff}_X F)^\perp$ is the completion of the $\mathcal{O}_X$-subalgebra of $\overline{\mathrm{Jet}}_X$ generated by $dy_1, \ldots, dy_r, dz_1^p, \ldots, dz_{n-r}^p$:

$$(\mathrm{Diff}_{X^\circ} F)^\perp = \mathcal{O}_{X^\circ}[[dy_1, \ldots, dy_{n-r}, dz_1^p, \ldots, dz_r^p]] = \mathcal{O}_{X^\circ} \overline{\mathrm{Jet}}_{Y^\circ} = \pi^* \overline{\mathrm{Jet}}_{Y^\circ}.$$

Let C be a smooth irreducible curve and $g_0 : C \to X$ a morphism. Then $g_0^* : \mathcal{O}_X \to \mathcal{O}_C$ defines an $\mathcal{O}_X$-algebra structure (and hence an $\mathcal{O}_Y$-algebra structure via $(\pi g_0)^*$) on $\mathcal{O}_C$. Consider $\mathcal{O}_C(\mathrm{Diff}_X F) \subset \mathcal{O}_C \otimes_{\mathcal{O}_X} (\mathrm{Diff}_X)$ and the associated null-submodule $(\mathcal{O}_C(\mathrm{Diff}_X F))^\perp \subset \mathcal{O}_C \otimes \overline{\mathrm{Jet}}_X$. On the open subset $C^\circ = f^{-1}(X^\circ)$, we easily check that

$$(\mathcal{O}_{C^\circ} \mathrm{Diff}_X F))^\perp = \mathcal{O}_{C^\circ}[[dy_1, \ldots, dy_{n-r}, dz_1^p, \ldots, dz_r^p]] = \mathcal{O}_{C^\circ} \overline{\mathrm{Jet}}_Y.$$

The subsheaf $\mathcal{O}_C F \subset \mathcal{O}_C T_X = \mathcal{H}om_{\mathcal{O}_C}(\mathcal{O}_C \Omega_X^1, \mathcal{O}_C)$ gives rise to a saturated subsheaf $(\mathcal{O}_C F)^\perp \subset \mathcal{O}_C \Omega_X^1$. The subsheaf $(\mathcal{O}_C F)^\perp$ is a subbundle of $\mathcal{O}_C \Omega_X^1$ by the smoothness of the curve C. Let $\eta_1, \ldots, \eta_{n-r}$ be a local basis of $(\mathcal{O}_C F)^\perp$ such that $\eta_1, \ldots, \eta_{n-r}, \xi_1, \ldots, \xi_r$ form a local basis of $\mathcal{O}_C \Omega^1$.

Lemma 3.1. *Let C be a smooth irreducible curve and $g_0 : C \to X$ a morphism such that $C^\circ = g_0^{-1}(X^\circ)$ is open dense in C. Put $\widetilde{\mathcal{O}_C\mathrm{Jet}_Y} = (\mathcal{O}_C\mathrm{Diff}_X F)^\perp \subset \mathcal{O}_C\overline{\mathrm{Jet}}_X$, and $\widetilde{\mathcal{O}_C\mathrm{Jet}_Y}^{\langle m\rangle} = \widetilde{\mathcal{O}_C\mathrm{Jet}_Y} \cap \mathcal{O}_C\mathrm{Jet}_X^m$. ($\mathrm{Jet}_X^m$ denotes the m-th power of the ideal Jet_X; do not confuse it with $\mathrm{Jet}_X^{[m]} = \mathrm{Jet}_X/\mathrm{Jet}_X^{m+1}$.) Then*

(1) $\widetilde{\mathcal{O}_C\mathrm{Jet}_Y}$ is an $\mathcal{O}_C$-subalgebra of $\mathcal{O}_C\overline{\mathrm{Jet}}_X$.

(2) $\widetilde{\mathcal{O}_C\mathrm{Jet}_Y}^{\langle m\rangle}/\widetilde{\mathcal{O}_C\mathrm{Jet}}^{\langle m+1\rangle}$ is a locally free $\mathcal{O}_C$-module, of which a local basis is given by the monomials of degree m in the η_i and the ξ_i^p, where the degree of ξ_i^p is counted as p.

Proof. (1) In fact, $k(C)\overline{\mathrm{Jet}}_Y \subset k(C)\overline{\mathrm{Jet}}_X$ is a $k(C)$-subalgebra isomorphic to the formal power series ring $k(C)[[dy_1, \ldots, dy_{n-r}, dz_1^p, \ldots, dz_r^p]]$. Hence $\widetilde{\mathcal{O}_C\mathrm{Jet}_Y} = k(C)\overline{\mathrm{Jet}}_Y \cap \mathcal{O}_C\overline{\mathrm{Jet}}_X$ is an $\mathcal{O}_C$-algebra.

(2) By definition, $\widetilde{\mathcal{O}_C\mathrm{Jet}_Y}^{\langle m\rangle}/\widetilde{\mathcal{O}_C\mathrm{Jet}_Y}^{\langle m+1\rangle}$ is an $\mathcal{O}_C$-submodule of $\mathcal{O}_C\mathrm{Jet}_X^m/\mathcal{O}_C\mathrm{Jet}_X^{m+1} \simeq \mathrm{Sym}^m_{\mathcal{O}_C}\mathcal{O}_C\Omega_X^1$. Its element is annihilated by $\mathrm{Diff}_X^{m-1}F + \mathrm{Diff}_X^{m-1} \subset \mathrm{Diff}_X^m$, of which the action is given by the multiplication of m-th principal symbols. It follows that $\widetilde{\mathcal{O}_C\mathrm{Jet}_Y}^{\langle m\rangle}/\widetilde{\mathcal{O}_C\mathrm{Jet}_Y}^{\langle m+1\rangle}$ is contained in the subsheaf $\subset \mathrm{Sym}^m_{\mathcal{O}_C}(\mathcal{O}_C\Omega_X^1)$ generated by the η_i and ξ_j^p.

Let us show the converse inclusion relation. Take $\zeta \in \mathcal{O}_C\mathrm{Jet}_X^m$ such that

$$(*) \qquad \zeta \mod \mathcal{O}_C\mathrm{Jet}_X^{m+1} \in \langle \eta_1, \ldots, \eta_r, \xi_1^p, \ldots, \xi_{n-r}^p\rangle \subset \mathrm{Sym}^m_{\mathcal{O}_C}(\mathcal{O}_C\Omega_X^1).$$

It suffices to show that there exists $\tilde{\zeta} \in \widetilde{\mathcal{O}_C\mathrm{Jet}_Y}$ such that $\tilde{\zeta} \equiv \zeta \mod \mathcal{O}_C\mathrm{Jet}_X^{m+1}$.

The condition (*) is equivalent to:

$$\langle \mathcal{O}_C\mathrm{Diff}_X^{m-1}F + \mathcal{O}_C\mathrm{Diff}_X^{m-1}|\zeta\rangle = \langle k(C)\mathrm{Diff}_X^{m-1}F + k(C)\mathrm{Diff}_X^{m-1}|\zeta\rangle = 0.$$

Thus we have a well-defined $\mathcal{O}_C$-linear map

$$|\zeta\rangle : (\mathcal{O}_C\mathrm{Diff}_X^{m+1} + k(C)\mathrm{Diff}_X^{m-1}F)/k(C)\mathrm{Diff}_X^{m-1}F \to \mathcal{O}_C.$$

Consider natural injections

$$k(C)\mathrm{Diff}_X^m F/k(C)\mathrm{Diff}_X^{m-1}F \hookrightarrow k(C)\mathrm{Diff}_X^{m+1}/k(C)\mathrm{Diff}_X^m,$$
$$\mathcal{O}_C\mathrm{Diff}_X^{m+1}/\mathcal{O}_C\mathrm{Diff}_X^m \hookrightarrow k(C)\mathrm{Diff}_X^{m+1}/k(C)\mathrm{Diff}_X^m.$$

Then $\mathcal{G} = \big(k(C)\mathrm{Diff}_X^m F/k(C)\mathrm{Diff}_X^{m-1}F\big) \cap \big(\mathcal{O}_C\mathrm{Diff}_X^{m+1}/\mathcal{O}_C\mathrm{Diff}_X^m\big)$ is a subbundle of

$$\begin{aligned}
&\mathcal{O}_C\mathrm{Diff}_X^{m+1}/\mathcal{O}_C\mathrm{Diff}_X^m \\
&\qquad \simeq \big(\mathcal{O}_C\mathrm{Diff}_X^{m+1} + k(C)\mathrm{Diff}_X^{m-1}F\big)/\big(\mathcal{O}_C\mathrm{Diff}_X^m + k(C)\mathrm{Diff}_X^{m-1}F\big) \\
&\qquad \simeq \big(\mathcal{O}_C\mathrm{Diff}_X^{m+1} + k(C)\mathrm{Diff}_X^m\big)/k(C)\mathrm{Diff}_X^m.
\end{aligned}$$

Take $\alpha_i \in k(C)\mathrm{Diff}_X^m F$ and $\beta_i \in \mathcal{O}_C\mathrm{Diff}_X^{m+1}$ such that $\alpha_i \equiv \beta_i \mod k(C)\mathrm{Diff}^{m-1}F + \mathcal{O}_C\mathrm{Diff}_X^m$ and that their equivalence classes form a basis of $\mathcal{G}$. Then we see that

$$\langle \alpha_i|\zeta\rangle \in \langle \beta_i|\zeta\rangle + \langle k(C)\mathrm{Diff}^{m-1}F|\zeta\rangle + \langle \mathcal{O}_C\mathrm{Diff}_X^m|\zeta\rangle \subset \mathcal{O}_C.$$

For a homogeneous polynomial ω of degree $\geq m+1$ in the dx_i with coefficients in $\mathcal{O}_C$, we have $\langle\beta_i|\omega\rangle = \langle\alpha_i|\omega\rangle$, because $k(C)\mathrm{Diff}^m$ kills $\omega \in \mathrm{Jet}_X^{m+1}$. By the natural isomorphism

$$\mathcal{O}_C\mathrm{Sym}^{m+1}\Omega_X^1 \simeq \mathcal{O}_C\mathrm{Jet}_X^{m+1}/\mathcal{O}_C\mathrm{Jet}_X^{m+2} \simeq \mathcal{H}om_{\mathcal{O}_C}(\mathcal{O}_C\mathrm{Diff}_X^{m+1}/\mathcal{O}_C\mathrm{Diff}_X^m, \mathcal{O}_C)$$

and by the fact that $\{\beta_i\}$ is a part of a local basis of $\mathcal{O}_C\mathrm{Diff}_X^{m+1}/\mathcal{O}_C\mathrm{Diff}_X^m$, we can find $\omega_1 \in \mathcal{O}_C\mathrm{Jet}_X^{m+1}$ such that $\beta_i(\omega_1) = \alpha_i(\omega_1) = \alpha_i(\zeta), i = 1, 2, \ldots$. Then $\zeta' = \zeta - \omega_1 \in \mathcal{O}_C\mathrm{Jet}_X$ is annihilated by $\mathrm{Diff}_X^m F$.

Reiterating similar procedure of adjusting ζ by $\omega_\nu \in \mathcal{O}_C\mathrm{Jet}_X^{m+\nu}, \nu = 1, 2, \ldots$, we can find $\tilde{\zeta} \in \mathcal{O}_C\mathrm{Jet}_X = \mathcal{O}_C[[dx_1, \ldots, dx_n]]$ such that

(i) $\tilde{\zeta} \equiv \zeta \mod \mathcal{O}_C\mathrm{Jet}_X^{m+1}$, and that

(ii) $\tilde{\zeta} \in (\mathcal{O}_C\mathrm{Diff}_X F)^\perp = \widetilde{\mathcal{O}_C\mathrm{Jet}_Y}$.

This completes the proof of (2). □

Corollary 3.2. *Let C be a smooth irreducible curve and $g_0 : C \to X$ a morphism such that $C^\circ = g_0^{-1}(X^\circ)$ is non-empty.*

(1) Choose $\zeta_1, \ldots, \zeta_{n-r} \in \widetilde{\mathcal{O}_C\mathrm{Jet}_Y}$ such that they form a local $\mathcal{O}_C$-basis of

$$\widetilde{\mathcal{O}_C\mathrm{Jet}_Y}^{\langle 1\rangle}/\widetilde{\mathcal{O}_C\mathrm{Jet}}^{\langle 2\rangle} \simeq (\mathcal{O}_C F)^\perp \subset \mathcal{O}_C\Omega_X^1.$$

Then we have a local isomorphism

$$\widetilde{\mathcal{O}_C\mathrm{Jet}_Y} \simeq \mathcal{O}_C[[\zeta_1, \ldots, \zeta_{n-r}, dx_1^p, \ldots, dx_r^p]],$$

where $\{\zeta_1, \ldots, \zeta_{n-r}, dx_1, \ldots, dx_r\}$ is a basis of $\mathcal{O}_C\Omega_X^1$.

(2) Put

$$\widetilde{\mathcal{O}_C\Omega_Y}^1 = \widetilde{\mathcal{O}_C\mathrm{Jet}_Y}^{\langle 1\rangle}/(\widetilde{\mathcal{O}_C\mathrm{Jet}_Y}^{\langle 1\rangle})^2.$$

We have a canonical $\mathcal{O}_C$-homomorphism $\mathcal{O}_C\Omega_Y^1 \to \widetilde{\mathcal{O}_C\Omega_Y^1}$ which is an isomorphism on the open subset C°. There is a natural and globally defined exact sequence

$$(**) \qquad 0 \to \mathcal{O}_C(\mathcal{O}_C\Omega_X^1/(\mathcal{O}_C F)^\perp)^{(p)} \to \widetilde{\mathcal{O}_C\Omega_Y}^1 \to (\mathcal{O}_C F)^\perp \to 0.$$

Here $(\mathcal{O}_C\Omega^1/(\mathcal{O}_C F)^\perp)^{(p)}$ stands for the $\mathcal{O}_C^{(p)}$-module freely generated by $(dx_i)^p$, where $\{dx_i\}$ is an $\mathcal{O}_C$-basis of $\mathcal{O}_C\Omega_X^1/(\mathcal{O}_C F)^\perp$.

Let (A, M) be an artinian local k-algebra. Fix a morphism $f_0 : C \to Y$, which gives rise to a ring homomorphism $f_0^* : \mathcal{O}_Y \to \mathcal{O}_C \subset A \otimes \mathcal{O}_C$. A ring homomorphism $\psi : \mathcal{O}_Y \to A \otimes \mathcal{O}_C$ induces a ring homomorphsm $(f_0^*, \psi) : \mathcal{O}_Y \otimes \mathcal{O}_Y \to A \otimes \mathcal{O}_C$, defined by $(f_0^*, \psi)(a \otimes b) = f_0^*(a)\psi(b) \in A \otimes \mathcal{O}_C$. ψ is said to be (the ring homomorphism attached to) a *deformation* of f_0 parametrized by A if $(f_0^*, \psi)(I_{\Delta_Y}) \subset M \otimes \mathcal{O}_C$, or, equivalently, ψ mod $M \otimes \mathcal{O}_C$ coincides with f_0^*. When ψ is a deformation of f_0, the ring homomorphism (f_0^*, ψ) factors through $\mathcal{O}_Y \otimes \mathcal{O}_Y/I_{\Delta_X}^N$ for sufficiently large N, so that we can view it as an $\mathcal{O}_Y$-algebra homomorphism $\overline{\mathrm{Jet}}_Y \to \mathcal{O}_C$ or an $\mathcal{O}_C$-algebra homomorphism $\mathcal{O}_C \otimes_{\mathcal{O}_X} \overline{\mathrm{Jet}}_Y \to \mathcal{O}_C$. In particular, (f_0^*, ψ) defines a $k(C)$-algebra homomorphism $k(C)\overline{\mathrm{Jet}}_Y \to A \otimes k(C)$.

Assume that there is a morphism $g_0 : C \to X$ with $f_0 = \pi g_0$ and that $C^\circ = f_0^{-1}(Y^\circ) = g_0^{-1}(X^\circ)$ is dense. A deformation ψ of f_0^* is called *admissible* if $(f_0^*, \psi)(\widetilde{\mathcal{O}_C\mathrm{Jet}_Y}) \subset A \otimes \mathcal{O}_C$. This definition makes sense because $\widetilde{\mathcal{O}_C\mathrm{Jet}_Y} = (\mathcal{O}_C\mathrm{Diff}_X F)^\perp = (g_0^*\mathrm{Diff}_X F)^\perp$ is an $\mathcal{O}_C$-subalgebra of $k(C)\overline{\mathrm{Jet}}_Y = k(C)\overline{\mathrm{Jet}}_{Y^\circ}$.

Lemma 3.3. *Let C be a smooth affine curve. Let $g_0 : C \to X$ be a morphism such that $C^\circ = g_0^{-1}(X^\circ)$ is non-empty and that $(\mathcal{O}_C \mathrm{Diff}_X F)^\perp$ is the completion of the algebra locally generated by $\zeta_1, \dots, \zeta_{n-r}, dx_1^p, \dots, dx_r^p$. Then:*

(1) Given $\psi : \mathcal{O}_Y \to A/J \otimes \mathcal{O}_C$, an admissible deformation of $f_0 = \pi g_0$ parametrized by A/J, we can find an admissible deformation $\tilde{\psi} : \mathcal{O}_Y \to A \otimes \mathcal{O}_C$ such that $\tilde{\psi} \mod J = \psi$, called an admissible lift of ψ to A.

(2) Fix an arbitrary admissible lift $\tilde{\psi} : \mathcal{O}_Y \to A \otimes \mathcal{O}_C$ above. There is a natural one-to-one correspondence between the set of admissible lifts of ψ to A and the k-vector space $\mathcal{H}om_{\mathcal{O}_C}(\widetilde{\mathcal{O}_C \Omega}^1_Y, \mathcal{O}_C)$.

Proof. (1) $\widetilde{\mathcal{O}_C \mathrm{Jet}}_Y$ is isomorphic to $\mathcal{O}_C[[\zeta_1, \dots, \zeta_r, dx_1^p, \dots, dx_{n-r}^p]]$. Hence the $\mathcal{O}_C$-algebra homomorphism $(f_0^*, \psi) : \widetilde{\mathcal{O}_C \mathrm{Jet}}_Y \to A/J \otimes \mathcal{O}_C$ can be lifted to a homorphism $\tilde{\Psi} : \widetilde{\mathcal{O}_C \mathrm{Jet}}_Y \to A \otimes \mathcal{O}_C$. Then the natural homomorphism $\mathcal{O}_Y \to \widetilde{\mathcal{O}_C \mathrm{Jet}}_Y, a \mapsto 1 \otimes a$ determines a ring homomorphism $\tilde{\psi} : \mathcal{O}_Y \to A \otimes \mathcal{O}_C$, and it is clear that $\tilde{\Psi} = (f_0^*, \tilde{\psi})$.

(2) Once a specific lift is fixed, the identification above is obtained by a standard argument [SGA]. □

Corollary 3.4. *Let C be a smooth irreducible curve and $g_0 : C \to X$ a morphism such that $C^\circ \neq \emptyset$. Let $\psi : \mathcal{O}_Y \to A/J \otimes \mathcal{O}_C$ be an admissible deformation of $f_0 = \pi g_0$. Then the obstruction for admissible lifting of ψ to A lies in $H^1(C, \mathcal{H}om_{\mathcal{O}_C}(\widetilde{\mathcal{O}_C \Omega}^1_Y, \mathcal{O}_C))$. When the obstruction vanishes, the set of liftings is given by* $\mathrm{Hom}_{\mathcal{O}_C}(\widetilde{\mathcal{O}_C \Omega}^1_Y, \mathcal{O}_C))$.

Corollary 3.5. *Let C be smooth and projective and $g_0 : C \to X$ a morphism with C° non-empty. Then the quasi-projective scheme $\mathcal{H}om(C, Y)$ has dimension at least*

$$(p-1)\deg \widetilde{\mathcal{O}_C F} + \deg g_0^*(-K_X) + n(1 - g(C))$$

at $f_0 = \pi g_0$, where $\widetilde{\mathcal{O}_C F}$ stands for the saturation of $g_0^* F \subset g_0^* T_X$.

Proof. By the exact sequence (**), we have

$$\begin{aligned} \deg \mathcal{H}om_{\mathcal{O}_C}(\widetilde{\mathcal{O}_C \Omega}^1_Y, \mathcal{O}_C) &= -p \deg(\mathcal{O}_C \Omega_X / (\mathcal{O}_C F)^\perp) - \deg(\mathcal{O}_C F)^\perp \\ &= -p \deg(\widetilde{\mathcal{O}_C F})^* - \deg(\mathcal{O}_C T_X / (\widetilde{\mathcal{O}_C F})^*) \\ &= p \deg(\widetilde{\mathcal{O}_C F}) + \big(\deg(\mathcal{O}_C T_X) - \deg(\widetilde{\mathcal{O}_C F})\big) \\ &= (p-1)\deg(\widetilde{\mathcal{O}_C F}) + \deg g_0^*(-K_X). \end{aligned}$$

It is well-known [M] that the dimension of admissible deformation is more than or equal to

$$\dim H^0(C, \mathcal{H}om_{\mathcal{O}_C}(\widetilde{\mathcal{O}_C \Omega}^1_Y, \mathcal{O}_C)) - \dim H^1(C, \mathcal{H}om_{\mathcal{O}_C}(\widetilde{\mathcal{O}_C \Omega}^1_Y, \mathcal{O}_C))$$

and the Riemann-Roch yields our estimate.

Proof of Main Theorem. Let $F \subset T_X$ be the subsheaf generated by the global vector fields. F is clearly closed under Lie bracket and p-th power. Let $F^\sharp \subset T_X$ be the saturation of

F, i.e. the kernel of the projection $T_X \to (T_X/F)/(\text{torsion})$. Then $F^\sharp$ is again closed under Lie bracket and p-th power. By definition, $c_1(F)$ is an effective divisor on X, and $c_1(F^\sharp) \geq c_1(F)$. $c_1(F) = 0$ if and only if $F \simeq H^0(X, T_X) \otimes \mathcal{O}_X \simeq \mathcal{O}_X^{\oplus r}$. $c_1(F^\sharp) = 0$ if and only if $c_1(F) = 0$ and $F^\sharp = F$ in codimension one. When in addition $F^\sharp$ is a subbundle of T_X, then an isomorphism between the two locally free sheaves F and $F^\sharp$ automatically extends to a global isomorphism on X, yielding $F^\sharp = F$.

We have thus three cases:

Case 1. $F = F^\sharp \simeq \mathcal{O}_X^{\oplus r}$ is a subbundle in T_X.
Case 2. $c_1(F^\sharp) = c_1(F) = 0$, but $F^\sharp \subset T_X$ is not a subbundle.
Case 3. $c_1(F^\sharp) > 0$.

As for Case 1, there is nothing to prove.

In Case 3, we follow arguments by Rudakov-Shafarevich [RS] and Shepherd-Barron [SB]. The canonical divisor K_Y of the quotient variety $Y = X/F^\sharp$ is calculated by

$$\pi^* \det \Omega_Y^1 = \pi^* \det(\mathrm{Jet}_Y/\mathrm{Jet}_Y^2) = (\det F^\sharp)^{\otimes 1-p} \otimes K_X.$$

Indeed in codimension one, $F^\sharp$ is a subbundle of T_X, thereby inducing a natural exact sequence

$$0 \to \mathcal{O}_X(\Omega_X^1/F^\perp)^{(p)} \to \mathrm{Jet}_Y/\mathrm{Jet}_Y^2 \to F^\perp \to 0,$$

and we have $F^\perp = (T_X/F^\sharp)^*, \Omega_X^1/F^\perp = (F^\sharp)^*$. Noting that $K_X \approx 0$ and $\det F > 0$, we see that $-\pi^* K_Y$ is numerically equivalent to a non-zero effective divisor. Hence Y is uniruled by Miyaoka-Mori [MM], and so is X, which is a purely inseparable cover of Y.

Thus we have only to show that X is uniruled in Case 2. Choose a smooth curve $\Gamma \subset X$ such that $\Gamma \cap X^\circ \neq \emptyset$ and that $F^\sharp$ is not a subbundle at one or more points on Γ. Take a purely inseparable morphism $C \to \Gamma$ of sufficiently high degree $q = p^m$. C is assumed to be smooth. Denote by g_0 the induced morphism $C \to X$. The saturation $\widetilde{\mathcal{O}_\Gamma F^\sharp} \subset \mathcal{O}_\Gamma T_X$ is strictly bigger than $\mathcal{O}_\Gamma F^\sharp/(\text{torsion})$, whence follows that $\deg(\widetilde{\mathcal{O}_C F^\sharp}) \geq q$. On the other hand, the dimension of $\mathrm{Hom}(C, Y)$ at $f_0 = \pi g_0$ is at least $(p-1)\deg(\widetilde{\mathcal{O}_C F^\sharp}) - \deg g_0^* T_X + n(1 - g(C))$, which is very large whenever q is sufficiently large. Hence for each point on $\pi(\Gamma)$, we can find a rational curve on Y passing through the point by [MM]. We have ample choice of $\Gamma \subset X$, which shows the uniruledness of Y, and hence of X. □

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