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KZB EQUATIONS OF HIGHER GENERA

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Introduction.

In the WZW model, the correlation functions are defined as horizontal sections of the vector bundle of conformal blocks over a family of algebraic curves with respect to a certain connection [TUY]. In genus 0 case, the vector bundle of conformal blocks is realized in a explicit way and we get the Knizhnik-Zamolodchikov (KZ) equations by rewriting the connection. Felder and Wieczerkowski generalized this picture to genus one case and got the elliptic analogue of the KZ equations [FW]. Our aim is to have a parallel description over a family of curves of arbitrary genus and to obtain the higher genus generalization of the KZ equations, which we call the Knizhnik-Zamolodchikov-Bernard (KZB) equations.

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Notations.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, G a Lie group with $Lie(G) = \mathfrak{g}$ and $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}c$ be the affine Lie algebra associated to $\mathfrak{g}$ and the normalized Killing form $(\,,\,)$.

Fix the level $\ell \in \mathbb{Z}_{\geq 0}$ and let P_ℓ be the set consisting of level ℓ dominant integral weights of $\widehat{\mathfrak{g}}$.

For a weight $\lambda \in P_\ell$, $\mathcal{H}_\lambda$ denotes the integrable highest weight irreducible left $\widehat{\mathfrak{g}}$-module with highest weight λ and V_λ the irreducible highest weight left $\mathfrak{g}$-module with highest weight λ. Note that the space V_λ can be considered as a subspace of $\mathcal{H}_\lambda$. We also use the following notations:

$$\mathcal{H}_\lambda^\dagger = \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_\lambda, \mathbb{C}),\ V_\lambda^\dagger = \mathrm{Hom}_{\mathbb{C}}(V_\lambda, \mathbb{C}),$$

and we regard them as right $\widehat{\mathfrak{g}}$ and $\mathfrak{g}$ module respectively. On $\mathcal{H}_\lambda$, generators of the Virasoro algebra act through the Sugawara construction:

$$L_n = \frac{1}{2(\ell + h^\vee)} \sum_{k \in \mathbb{Z}} \sum_{a=1}^{\dim_{\mathbb{C}} \mathfrak{g}} : J_a \otimes t^k \cdot J_a \otimes t^{n-k} : \ (n \in \mathbb{Z}).$$

Here $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, $\{J_a\}$ an orthonormal basis of $\mathfrak{g}$, and $: \ :$ denotes the normal ordering. It is easily checked that the above generators satisfy the following relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}(m^3 - m)\frac{c_v}{12},$$

where $c_v = \frac{\dim_{\mathbb{C}} \mathfrak{g}}{2(\ell + h^\vee)}$.

§1 WZW model on $\mathbb{P}^1$ and KZ equations.

First we recall the formulation of the WZW model on $\mathbb{P}^1$ following [TUY] and see the equivalence between the KZ equations and a certain connection on the vector bundle of conformal blocks.

In the sequel we fix an N-tuple of weights $\vec{\lambda} = (\lambda_1, \dots, \lambda_N) \in (P_\ell)^N$ and use the following notation:

$$W_{\vec{\lambda}} = W_{\lambda_1} \otimes \cdots \otimes W_{\lambda_N}$$

for vector spaces W_{λ_j}. Put $R = \mathbb{C}^N \backslash \Delta$ and regard R as a subspace of configuration space of N-points $(z_1, \dots, z_N)$ on $\mathbb{P}^1$. Denote by $\widehat{\mathfrak{g}}_N$ the Lie algebra obtained by identifying all the centers in the Lie algebra $\widehat{\mathfrak{g}}^{\oplus N}$. Namely, $\widehat{\mathfrak{g}}_N$ is the one dimensional central extension of the direct sum of N copies of the loop algebra: $\widehat{\mathfrak{g}}_N = (L\mathfrak{g})^{\oplus N} \oplus \mathbb{C}c$. For a point $(z_1, \dots, z_N)$, we denotes the divisor $\sum z_j$ by D, and consider the space $H^0(\mathbb{P}^1, \mathfrak{g} \otimes \mathcal{O}(*D))$ of $\mathfrak{g}$-valued meromorphic functions on $\mathbb{P}^1$ whose poles belong to D. The space $H^0(\mathbb{P}^1, \mathfrak{g} \otimes \mathcal{O}(*D))$ is regarded as a subspace of $L\mathfrak{g}^{\oplus N} \subset \widehat{\mathfrak{g}}_N$ through Laurent expansion at z_j's, and it is a Lie subalgebra of $\widehat{\mathfrak{g}}_N$ due to the residue theorem.

Definition 1.1 For $(z_1, \dots, z_N) \in R$, we put

$$\mathcal{V}^\dagger(z_1,\dots,z_N) = \{ \ \psi \in \mathcal{H}^\dagger_{\vec{\lambda}} \mid \psi a = 0 \text{ for any } a \in H^0(\mathbb{P}^1, \mathfrak{g} \otimes \mathcal{O}(*D)) \ \}.$$

We call $\mathcal{V}^\dagger(z_1,\dots,z_N)$ the space of conformal blocks.

It turns out that the space $\mathcal{V}^\dagger(z_1,\dots,z_N)$ is a finite dimensional vector space over $\mathbb{C}$ and there exists a holomorphic vector bundle $\widetilde{\mathcal{V}}^\dagger$ on R whose fiber at $(z_1,\dots,z_N)$ is $\mathcal{V}^\dagger(z_1,\dots,z_N)$ and which carries the flat connection

$$d - \sum dz_j L^{(j)}_{-1}.$$

Here $L^{(j)}_{-1} = id \otimes \cdots \otimes id \otimes L_{-1} \otimes id \otimes \cdots \otimes id$ (L_{-1} at the j-th factor).

Now let us derive the KZ equations from the connection $d - \sum dz_j L^{(j)}_{-1}$. The point is the following lemma.

Lemma 1.2. *The map*

$$\mathcal{V}^\dagger(z_1,\dots,z_N) \ \rightarrow \ V^\dagger_{\vec{\lambda}}$$

given by the restriction to $V_{\vec{\lambda}} \subset \mathcal{H}_{\vec{\lambda}}$ is injective.

Thus the vector bundle $\widetilde{\mathcal{V}}^\dagger$ can be considered as a subbundle of of finite rank a trivial vector bundle $V^\dagger_{\vec{\lambda}} \times R$. We can write down the condition of horizontality with respect to the connection $d - \sum dz_j L^{(j)}_{-1}$ under this realization to get the KZ equations. In fact, if we have a section ψ of $\widetilde{\mathcal{V}}^\dagger$ satisfying $d\psi = \sum dz_j \psi L^{(j)}_{-1}$, then its restriction $\bar{\psi}$ to $V_{\vec{\lambda}}$ satisfies

$$\begin{aligned}(\ell + h^\vee)\frac{\partial}{\partial z_j}\bar{\psi} &= \bar{\psi}\sum_a (J_a \otimes t^{-1})^{(j)} (J_a \otimes t^0)^{(j)} \\ &= \bar{\psi}\sum_{i \neq j}\sum_a \frac{1}{z_j - z_j} J^{(i)}_a J^{(j)}_a \quad (j = 1,\dots,N).\end{aligned}$$

The second equality follows from the invariance with respect to $J_a \otimes \frac{1}{z_i - z_j} \in H^0(\mathbb{P}^1, \mathfrak{g} \otimes \mathcal{O}(*D))$.

§2 Higher genus case.

The construction of the vector bundle of conformal blocks naturally generalizes to higher genus case [TUY]. The difference is that we need not only N-pointed curves $(X; z_1, \dots, z_N)$ but also local coordinates ξ_j around the points z_j to give a meaning to the Laurent expansions. Thus the space $\mathcal{V}^\dagger(\mathfrak{X})$ of conformal blocks is associated to the data $\mathfrak{X} = (X; z_1, \dots, z_N; \xi_1, \dots, \xi_N)$ and the vector bundle $\widetilde{\mathcal{V}}^\dagger$ of conformal blocks is constructed over a family of N-pointed curves with coordinates.

A connection on the vector bundle $\widetilde{\mathcal{V}}^\dagger$ is defined via the Sugawara construction and turns out to be projectively flat in general. Our aim is to rewrite this connection to get the higher genus generalization of the KZ equations.

The difficulty for this purpose is that, in genus ≥ 1 cases, Lemma 1.2 does not hold any more, since there are no meromorphic functions on genus ≥ 1 curves which has only one simple pole. Thus we have to find the space, instead of $V_{\vec{\lambda}}^\dagger$, into which the space $\mathcal{V}^\dagger(\mathfrak{X})$ can be embedded.

Let us now construct such a space, where the moduli space of principal G-bundles enters. In the following we fix an N-pointed curves with local coordinates $\mathfrak{X} = (X; z_1, \dots, z_N; \xi_1 \dots \xi_N)$. Let $\mathcal{N}_{X,D}^{(*)}$ be the moduli space of principal G-bundles with trivializations at the points z_j and U an open subset of $\mathcal{N}_{X,D}^{(*)}$ which contains the point (P_0, η_0) corresponding to the trivial G-bundle P_0 and its canonical trivializations η_0.

For a point $(P, \eta) \in U$, consider the associated $\mathfrak{g}$-bundle $\mathfrak{g}_P = P \times_G \mathfrak{g}$ and embed the space $H^0(X, \mathfrak{g}_P(*D))$ to $\widehat{\mathfrak{g}}_N$. The space $\mathcal{V}^\dagger(\mathfrak{X}, P, \eta)$ of conformal blocks is defined in a similar way as in Definition 1.1. Clearly we have $\mathcal{V}^\dagger(\mathfrak{X}, P_0, \eta_0) = \mathcal{V}^\dagger(\mathfrak{X})$. There exists again a vector bundle $\widetilde{\mathcal{V}}^\dagger(\mathfrak{X})$ over U whose fiber at (P, η) is $\mathcal{V}^\dagger(\mathfrak{X}, P, \eta)$ and which carries a projectively flat connection ∇^G. We will see its definition later in §3.

Denote by $\mathcal{V}^\dagger(\mathfrak{X})(U)^{\nabla^G}$ the set of horizontal sections of $\widetilde{\mathcal{V}}^\dagger(\mathfrak{X})$. Then this space is in one-to-one orrespondence with $\mathcal{V}^\dagger(\mathfrak{X})$:

$$\widetilde{\mathcal{V}}^\dagger(\mathfrak{X})(U)^{\nabla^G} \cong \mathcal{V}^\dagger(\mathfrak{X}) \tag{2.1}$$

The key is the following proposition:

Proposition 2.2. *The restriction map*

$$\mathcal{V}^\dagger(\mathfrak{X})(U)^{\nabla^G} \to V_{\vec{\lambda}}^\dagger \otimes \mathcal{O}(U)$$

is injective.

By (2.1) and 2.2, we have $\mathcal{V}^\dagger(\mathfrak{X}) \subset V_{\vec{\lambda}}^\dagger \otimes \mathcal{O}(U)$. Under this realization we can rewrite the projectively flat connection on $\widetilde{\mathcal{V}}^\dagger(\mathfrak{X})$ and get the differential equations for $V_{\vec{\lambda}}^\dagger$-valued functions, which we consider as higher genus generalization of the KZ equations. Such equations were first considered by Bernard [Be1,2] and we call them the Knizhnik-Zamolodchikov-Bernard (KZB) equations.

§3 Connection defined by gauge symmetries.

We don't recall the definition of the projectively flat connection on the vector bundle $\widetilde{\mathcal{V}}^\dagger$ of conformal blocks and refer to [TUY], cf.the beginning of §2.

Here what is in question is the connection ∇^G defined by gauge symmetry on a parameter space of local universal family of principal G-bundles or on the moduli space of (semi-stable) principal G-bundles. It is, for example, parallel to the equations of flows generated by $(U(1)$-)currents in the conformal field theory of free fermions [KNTY].

For later use, we describe the connection ∇^G on the vector bundle $\widetilde{\mathcal{V}}^\dagger(\mathfrak{X}) \to U \subset \mathcal{N}_{X,D}^{(*)}$. For each $(P,\eta) \in U$, from the exact sequence

$$0 \longrightarrow \mathfrak{g}_P(-mD) \longrightarrow \mathfrak{g}_P(*D) \longrightarrow \oplus_{j=1}^N \mathfrak{g} \otimes \mathbb{C}[\xi_j^{-1}]\xi_j^{m-1} \longrightarrow 0,$$

we have

$$0 \longrightarrow H^0(X, \mathfrak{g}_P(*D)) \longrightarrow \oplus_{j=1}^N \mathfrak{g} \otimes \mathbb{C}((\xi_j)) \xrightarrow{\theta_P} T_{(P,\eta)}\mathcal{N}_{X,D}^{(*)} \longrightarrow 0. \tag{3.1}$$

Here we identified the space $\lim_{m\to\infty} H^1(X, \mathfrak{g}_P(-mD))$ with the tangent space $T_{(P,\eta)}\mathcal{N}_{X,D}^{(*)}$ of the moduli space of $\mathcal{N}_{X,D}^{(*)}$.

Then we can lift the homomorphism θ_P to the following:

$$0 \longrightarrow H^0(X, \mathfrak{g}_P(*D)) \longrightarrow \widehat{\mathfrak{g}}_N \xrightarrow{\tilde{\theta}_P} D_{\mathcal{L}^{(*)}}^{\leq 1}(P,\eta) \longrightarrow 0. \tag{3.2}$$

Here $D_{\mathcal{L}^{(*)}}^{\leq 1}$ denotes the sheaf of differential operators of order ≤ 1 acting on sections of the determinant line bundle $\mathcal{L}^{(*)}$ for the universal adjoint bundle $\mathfrak{g}_{\mathbb{P}}$ over $\mathcal{N}_{X,D}^{(*)}$ (associated to the universal principal G-bundle $\mathcal{P}$ on $\mathcal{N}_{X,D}$). $D_{\mathcal{L}^{(*)}}^{\leq 1}(P,\eta)$ is its fiber.

Recall that $D^{\leq 1}_{\mathcal{L}^{(\bullet)}}$ is an extension of $T_{\mathcal{N}^{(\bullet)}_{X,D}}$ by the structure sheaf of $\mathcal{N}^{(*)}_{X,D}$.

Definition 3.3 For $v \in D^{\leq 1}_{\mathcal{L}^{(\bullet)}}(P,\eta)$ we put

$$\nabla^G_v = v - w.$$

Here w is an element of $\widehat{\mathfrak{g}}_N$ such that $\tilde{\theta_P}(w) = v$.

On $\mathcal{H}^\dagger_{\vec{\lambda}} \otimes \mathcal{O}(U)$, the vector v acts as a first order differential operator, and w acts as an element of $\widehat{\mathfrak{g}}_N$. Then the action of w is unambiguously defined on $\widetilde{\mathcal{V}}^\dagger(\mathfrak{X})$. It can be proved that ∇^G defined by Definition 3.3 gives a projectively flat connection on $\widetilde{\mathcal{V}}^\dagger(\mathfrak{X})$.

To write down the connection on $\widetilde{\mathcal{V}}^\dagger$ explicitly, we want to know how some elements of $\widehat{\mathfrak{g}}_N$ act on $\widetilde{\mathcal{V}}^\dagger$ or on $V^\dagger_{\vec{\lambda}} \otimes \mathcal{O}(U)$ as elements of the tangent sheaf $T_{\mathcal{N}^{(\bullet)}_{X,D}}$. Namely, we want to know their images in $T_{\mathcal{N}^{(\bullet)}_{X,D}}$ by θ_P (composed with the projection from $\widehat{\mathfrak{g}}_N$ to $\oplus_{j=1}^N \mathfrak{g} \otimes \mathbb{C}((\xi_j))$).

§4 Method of calculation.

In this §, we explain the strategy to express the operator $\theta_P(\alpha)(\alpha \in \widehat{\mathfrak{g}}_N)$ explicitly in some sense.

The basic idea of explicit expression is to use the tangent space at the trivial bundle of the moduli space of (semi-stable) principal G-bundles as a space with coordinates. For this purpose, we construct a (local universal) family of principal G-bundles on X on an open neighbourhood $S^{(*)}$ of the origin in $T_{(P,\eta)}\mathcal{N}^{(*)}_{X,D}$ or its truncated analog on an open $S^{(1)}$ in $T_{(P,\eta)}\mathcal{N}^{(1)}_{X,D} \simeq H^1(X, \mathfrak{g}_{P_0} \otimes \mathcal{O}(-D))$. $S^{(1)}$ is isomorphic to an open U in $\mathcal{N}^{(1)}_{X,D}$ via the resulting classifyinig morphism $S^{(1)} \to \mathcal{N}^{(1)}_{X,D}$.

The calculation of θ_P is based on the exact sequence (3.1). Let us take the basis $\{v_i\}_i$ of $H^1(X, \mathfrak{g}_P \otimes \mathcal{O}(-D))$ dual to a basis $\{u_i\}_i$ of the Serre dual $H^0(X, \mathfrak{g}_P \otimes \omega_X(D))$. ($\mathfrak{g}_P^* \simeq \mathfrak{g}_P$ induced from $\mathfrak{g}^* \simeq \mathfrak{g}$ by the (normalized) Killing form.)

Then, for $\alpha \in \oplus_{j=1}^N \mathfrak{g} \otimes \mathbb{C}((\xi_j)) \subset \widehat{\mathfrak{g}}_N$, $\theta_P(\alpha)$ is calculated as

$$\theta_P(\alpha) = \sum_i \langle \theta_P(\alpha), u_i \rangle v_i$$

When the G-bundle P is the trivial one P_0, we have

$$H^1(X, \mathfrak{g}_{P_0} \otimes \mathcal{O}(-D)) \simeq \mathfrak{g} \otimes H^1(X, \mathcal{O}(-D))$$
$$H^0(X, \mathfrak{g}_{P_0} \otimes \omega_X(D)) \simeq \mathfrak{g} \otimes H^0(X, \omega_X(D)).$$

Thus we can choose a basis of the form $\{J^a \otimes \omega_i\}_{a,i}$ ($J^a \in \mathfrak{g}, \omega_i \in H^0(X, \omega_X(D))$). Accordingly, for $\alpha = (X_j \otimes f_j)_{j=1,\cdots,N}$, we have the expression

$$\theta_{P_0}(\alpha) = \sum_{j,(a,i)} (X_j, J^a)\mathrm{Res}_{z_j}(f_j\omega_i)\frac{\partial}{\partial v_{a,i}} \tag{4.1}$$

Here $\{v_{a,i}\}_{a,i}$ is the dual basis of $\{J^a \otimes \omega_i\}_{a,i}$ and is considered as coordinates of U. For explicit description, we can choose ω_i using a suitable differentials of the third kind.

By this simple observation, we are led to compare $T_{(P,\eta)}\mathcal{N}^{(*)}_{X,D}$ and $T_{(P_0,\eta_0)}\mathcal{N}^{(*)}_{X,D}$ for (P,η) (sufficiently) close to (P_0, η_0).

If we choose a path from (P,η) to (P_0,η_0) in $S^{(*)}$ or in U, we have the following diagram:

(4.2)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(X, \mathfrak{g}_{P_0} \otimes \mathcal{O}(*D)) & \longrightarrow & \oplus_{j=1}^N \mathfrak{g} \otimes \mathbb{C}((\xi_j)) & \longrightarrow & T_{(P_0,\eta_0)}\mathcal{N}^{(*)}_{X,D} & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & H^0(X, \mathfrak{g}_{P} \otimes \mathcal{O}(*D)) & \longrightarrow & \oplus_{j=1}^N \mathfrak{g} \otimes \mathbb{C}((\xi_j)) & \longrightarrow & T_{(P,\eta)}\mathcal{N}^{(*)}_{X,D} & \longrightarrow & 0
\end{array}$$

Remark We can equally consider the finite (truncated) version (e.g the one involving $T_{(P,\eta)}\mathcal{N}^{(1)}_{X,D}$) of the short exact sequence 3.1. But the vertical isomorphism can be given only for the infinite version as above.

To write down the vertical isomorphism in 4.2, we reduce it to the calculation of Kodaira-Spencer map KS_P in the following diagram :

$$\begin{array}{ccc}
T_{W_0}S^{(*)} & \xrightarrow{KS_{P_0}} & T_{(P_0,\eta_0)}\mathcal{N}^{(*)}_{X,D} \\
{\scriptstyle id}\downarrow & & \downarrow \\
T_W S^{(*)} & \xrightarrow[KS_P]{} & T_{(P,\eta)}\mathcal{N}^{(*)}_{X,D}
\end{array}$$

Here W (resp. W_0) $\in S^{(*)}$ corresponds to (P,η) (resp. (P_0,η_0)) $\in N^{(*)}$ by the classifying morphism. The left vertical arrow and KS_{P_0} are identities. The calculation of the right vertical arrow is equivalent to that of KS_P and its calculation by Čech cohomology provides the vertical arrows of the diagram 4.2.

The construction of a local universal family on an open set $S(\ni 0)$ of $T_{(P_0,\eta_0)}\mathcal{N}^{(1)}_{X,D}$ or $T_{(P_0,\eta_0)}\mathcal{N}^{(*)}_{X,D}$, which is the last ingredient, is done as follows. Choose a(n orthonormal) basis $\{J^a\}$ of $\mathfrak{g}$ consisting of nilpotent elements. Then, for $W = \{W_j\} = \{\sum_a J^a \otimes f_{aj}(\xi_j)\}_j \in \oplus_{j=1}^N \mathfrak{g} \otimes \mathbb{C}((\xi_j))$,

$$\exp W_j := \prod \exp(f_{aj} J^a)$$

becomes a G-valued meromorphic function for each j. The transition functions $\exp W = \{\exp W_j\}_j$ in a punctured neighbourhood of z_j define a principal G-bundle P_W. It depends only on W modulo $\mathfrak{g} \otimes H^0(X, \mathcal{O}(*D))$, i.e., the class of W in $T_{(P_0,\eta_0)}\mathcal{N}^{(*)}_{X,D}$. Then the sought-for formula for $\theta_{P_W}(\alpha)$ is given by

$$\theta_{P_W}(\alpha) = \theta_{P_0}(\mathrm{Ad}(\exp W)(\alpha)). \tag{4.4}$$

Here we skip the detailed explanation of the symbol $\mathrm{Ad}(\exp W)$, but it is the vertical isomorphism in the diagram 4.2.

The KZB equation for marked points.

We show the KZB equations for marked points on a fixed curve as an illustration of our discussion.

The correlation function $\langle \Phi | \in \widetilde{\mathcal{V}}^\dagger$ satisfies

$$(\ell + h^\vee)\frac{\partial}{\partial z_j}\langle \Phi| \;=\; \langle \Phi| \sum_a (J_a \otimes \xi_j^{-1})^{(j)} \cdot (J_a \otimes 1)^{(j)}.$$

Here $J_a \otimes \xi_j^{-1}$ and $J_a \otimes 1$ act on $\langle \Phi|$ as an element of the trivial bundle with fiber $V_{\vec{\lambda}}^\dagger \otimes \mathcal{O}(U)$ by Proposition 2.2, cf.§1. Note that, applying 4.1, we have

$$\theta_{P_0}(J_a \otimes \xi_j^{-1}) = \sum_i \omega_i(z_j) \frac{\partial}{\partial v_{a,i}}.$$

REFERENCES

[Be1] D. Bernard, *On the Wess-Zumino-Witten models on the torus*, Nucl. Phys. **B303** (1988), 77–93.

[Be2] D. Bernard, *On the Wess-Zumino-Witten models on Riemann surfaces*, Nucl. Phys. **B309** (1988), 145–174.

[FW] G. Felder and C. Wieczerkowski, *Conformal blocks on elliptic curves and the Knizhnik-Zamolodchikov-Bernard equations*, hep-th/941104 (1994).

[KNTY] N.Kawamoto, Y.Namikawa, A.Tsuchiya, Y.Yamada, *Geometric realization of conformal field theory on Riemann surfaces*, Commun. Math. Phys. **116** (1988), 247–308.

[TUY] A. Tsuchiya, K. Ueno and Y. Yamada, *Conformal field theory on universal family of stable curves with gauge symmetries*, Adv. Stud. in Pure Math. **19** (1989), 459–566.