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Mordell-Weil lattices and certain Calabi-Yau threefolds

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1 Mordell-Weil lattices for Jacobians

Let $K = k(C)$ be the rational function field of a smooth projective curve C defined over an algebraically closed field k. Let Γ be a smooth projective curve defined over K of genus $g \geq 1$ and let J_Γ be the Jacobian variety of Γ, which is an abelian variety defined over K. We denote by $\Gamma(K)$ (resp. $J_\Gamma(K)$) the set of K-rational points of Γ (resp. of J_Γ). The group structure of J_Γ induces the structure of an abelian group on $J_\Gamma(K)$, which is called *the Mordell-Weil group* of J_Γ (or *the Mordell-Weil group* of Γ/K by abuse of language).

The Mordell-Weil group can be considered geometrically as follows. By theory of smooth minimal models of algebraic surfaces, there exists a proper surjective morphism

$$f : X \longrightarrow C \tag{1.1}$$

from a smooth projective surface X to the curve C whose generic fiber X_η is isomorphic to Γ (over K). Moreover we can assume that there are no exceptional curves E of the first kind in any closed fiber of f. Such a model is unique up to isomorphism. By using this model, a K-rational point of Γ corresponds to a regular algebraic section $\sigma : C \longrightarrow X$ of f. Since J_Γ is an abelian variety over K, we can also obtain the unique good model, that is, the Néron model of J_Γ

$$h : \mathcal{J} \longrightarrow C, \tag{1.2}$$

which is a group scheme over C (and whose generic fiber is J_Γ). (Note that $h : \mathcal{J} \longrightarrow C$ is not necessarily proper.)

Let $\mathcal{S}(\mathcal{J}/C)$ denote the group of sections of $h : \mathcal{J} \longrightarrow C$. Then we have the canonical isomorphism

$$J_\Gamma(K) \simeq \mathcal{S}(\mathcal{J}/C). \tag{1.3}$$

It is known that the Mordell-Weil group $J_\Gamma(K)$ is finitely generated if the K/k-trace of J_Γ is trivial.

Shioda ([Sh1], [Sh2]) described the Mordell-Weil group $J_\Gamma(K)$ as follows. Let $\mathrm{NS}(X)$ be the Néron-Severi group of the surface X. Then it is known that $\mathrm{NS}(X)$ is a finitely generated abelian group with the intersection pairing

$$(\ , \ ) : \mathrm{NS}(X) \times \mathrm{NS}(X) \longrightarrow \mathbf{Z}.$$

From now on we assume that there exists a section $\sigma_0 : C \longrightarrow X$.

We set $O = \sigma_0(C)$ and denote by F a general closed fiber and consider them as elements in $\mathrm{NS}(X)$. We define subgroups U, T of $\mathrm{NS}(X)$ by

$$U =< O, F >, \quad T =< U, \text{ all irreducible components of closed fibers} > .$$

Clearly, we have $U \subset T \subset \mathrm{NS}(X)$. Moreover we set $L = T^\perp \subset \mathrm{NS}(X)$.

The following theorems are fundamental results of Shioda([Sh1], [Sh2]).

Theorem 1.1 *Assume that the K/k-trace of J_Γ is trivial. Then there exists a group isomorphism*

$$\mathrm{NS}(X)/T \simeq J_\Gamma(K). \tag{1.4}$$

Theorem 1.2 *Assume that the K/k-trace is trivial and* $\mathrm{NS}(X)$ *is torsion-free. Then we have the natural homomorphism*

$$\phi : J_\Gamma(K) \longrightarrow \mathrm{NS}(X) \otimes_{\mathbf{Z}} \mathbf{Q} \tag{1.5}$$

such that $\phi(Q) \perp T$. The kernel of ϕ is equal to the torsion part of $J_\Gamma(K)$ and $\mathrm{Im}\phi \subset L^* = \mathrm{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$.

By using theorem 1.2, one can define a pairing on $J_\Gamma(K)$ by $< P, Q >= -(\phi(P), \phi(Q)) \in \mathbf{Q}$ for $P, Q \in J_\Gamma(K)$. This gives a positive-definite symmetric bilinear form on $J_\Gamma(K)/J_\Gamma(K)_{tor}$, and Shioda called the pair $(J_\Gamma(K)/J_\Gamma(K)_{tor}, < \ , \ >)$ *the Mordell-Weil lattice.*

2 Upperbounds of Mordell-Weil rank

We denote by r the Mordell-Weil rank, i.e. $r = \dim_{\mathbf{Q}} J_{\Gamma}(K) \otimes \mathbf{Q}$. We have the following theorem which gives an upperbound of r. (See [Sa0], [Sa1].)

Theorem 2.1 *Assume that char. $k = 0$. Let $f : X \longrightarrow C$ be as above and assume that K/k-trace of J_Γ is trivial. Then we have*

$$r \leq (6 + 4/g)\chi(X, \mathcal{O}_X) + (1 - \pi)\{\frac{4g^2 - 2g - 4}{g}\}. \tag{2.1}$$

Here we set π = genus of C.

We remark that if $p_g(X) = \dim H^2(X, \mathcal{O}_X) > 0$ it is rather difficult to check that the inequality (2.1) is sharp. On the other hand, if $p_g(X) = q(X)(:= \dim H^1(X, \mathcal{O}_X)) = 0$ (e.g. X is a rational surface) we have the following theorem. ([Sa-Sak].)

Theorem 2.2 *Let $f : X \longrightarrow C$ be as above and assume that $p_g(X) = q(X) = 0$. Then we have*

(2.2). $$r \leq 4g + 4$$

Moreover there exist examples of fibrations $f : X \longrightarrow \mathbf{P}^1$ with $p_g(X) = q(X) = 0$ and $r = 4g + 4$.

We can also determine the structure of the fibration of curves of genus $g \geq 2$ with $p_g(X) = q(X) = 0$ and $r = 4g + 4$([Sa-Sak]).

Theorem 2.3 *Let $f : X \longrightarrow \mathbf{P}^1$ be as in Theorem 2.2 and assume that $r = 4g + 4$ and $g \geq 2$. Then there exists a finite double covering map $f : X \longrightarrow \mathbf{P}^1 \times \mathbf{P}^1$ whose branch locus $B \subset \mathbf{P}^1 \times \mathbf{P}^1$ is a smooth curve of bidegree $(2, 2g + 2)$.*

Remark 2.1 *In [Sa-Sak], we assume that X is a rational surface to obtain the upperbounds $r \leq 4g + 4$. However the upperbound of Mordell-Weil rank in Theorem 2.1 is a consequence of Xiao's slope inequality ([Xiao]) and hence the assumption $p_g(X) = q(X) = 0$ is enough to obtain the upperbound in (2.2). However the structure theorem 2.3 says that if $p_g(X) = q(X) = 0$ and $r = 4g + 4$ then X must be a rational surface.*

3 Maximal Mordell-Weil lattices D_{4g+4}^{+}

As far as the structure of maximal Mordell-Weil lattices is concerned, we can obtain the following theorem ([Sa-Sak]), which is a corollary of Theorem 2.3.

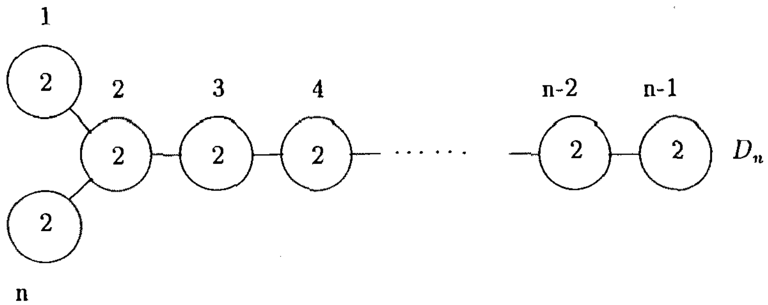
Theorem 3.1 *Let $f : X \longrightarrow \mathbf{P}^1$ be the fibration of curves of genus $g \geq 2$ with $p_g(X) = q(X) = 0$ and $r = 4g+4$. Then the Mordell-Weil lattice $J_\Gamma(K)$ is torsion-free and isometric to the positive-definite unimodular lattice D_{4g+4}^{+} (see bellow for the notation).*

Let us explain about the lattice D_{4g+4}^{+}. The lattice D_{4g+4}^{+} is a positive-definite unimodular lattice which is an overlattice of the lattice D_{4g+4} such that $[D_{4g+4}^{+} : D_{4g+4}] = 2$. Following Conway-Sloane's book (cf. [7, Ch. 4, C-S]), we will review the lattices D_n and D_{4m}^{+}.

For $n \geq 3$, we can embed D_n into the Euclidean lattice $\mathbf{Z}^n$ as

$$D_n = \{(x_1, \cdots, x_n) \in \mathbf{Z}^n : x_1 + \cdots + x_n \quad \text{even}\}. \tag{3.1}$$

The standard integral basis is given as usual (see [7, Ch. 4, C-S]) and its intersection diagram is given by the Coxeter-Dynkin diagram of type D_n:



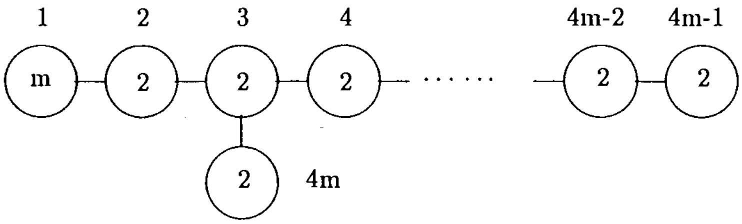
For $n = 4m \geq 4$, we take a vector

$$[1] = (1/2, 1/2, \cdots, 1/2) \in \mathbf{Q}^{4m},$$

and set

$$D_{4m}^{+} = D_{4m} \cup (D_{4m} + [1]). \tag{3.2}$$

The lattice D_{4m}^{+} is a positive-definite integral unimodular lattice and has the standard integral basis with the Coxeter-Dynkin diagram:



It should be noted here that $D_8^+ \simeq E_8$, hence one may regard $D_{4g+4}^+ (g \geq 2)$ as generalization of the lattice E_8. (One may recall that in Shioda's theory of Mordell-Weil lattices for rational elliptic surfaces E_8 arises as the frame lattice.)

In the connection with Mirror symmetry conjecture for related Calabi-Yau three-folds which will appear in the next section, it is interesting to consider the *theta series* of lattices. We also follow the notation in [4, Ch. 4, C-S].

Let L be a positive-definite lattice. For each positive integer m, we set

$$N_L(m) = \#\{x \in L | < x, x >= m\}. \tag{3.3}$$

Then the theta series of L is defined by

$$\theta_L(z) = \sum_{x \in L} q^{<x,x>} = \sum_{m=1}^{\infty} N(m) q^m, \tag{3.4}$$

where $q = \exp(\pi i z)$.

In order to write the theta series of D_{4g+4}^+, we introduce the following Jacobi's theta functions:

$$\theta_2(z) = 2q^{1/4} \prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m})^2,$$

$$\theta_3(z) = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m-1})^2,$$

$$\theta_4(z) = \prod_{m=1}^{\infty} (1 - q^{2m})(1 - q^{2m-1})^2.$$

Then the theta series for D_{4g+4}^+ can be written as (see 7.3, Ch. 4 in [C-S]):

$$\theta_{D_{4g+4}^+}(z) = 1/2(\theta_2^{4g+4}(z) + \theta_3^{4g+4}(z) + \theta_4^{4g+4}(z)). \tag{3.5}$$

Expanding the right hand side of (3.5) in the powers of q, we obtain the explicit number $N_{4g+4}(m) = N_{D_{4g+4}^+}(m)$ of elements in D_{4g+4}^+ with length m.

For example, if $g = 2$, then we have the following expansion up to order q^{15}:

(3.6)

$$\theta_{D_{12}^+}(z) = 1 + 264\,q^2 + 2048\,q^3 + 7944\,q^4 + 24576\,q^5 + 64416\,q^6 + 135168\,q^7 +253704\,q^8 + 475136\,q^9 + 825264\,q^{10} + 1284096\,q^{11} + 1938336\,q^{12} +2973696\,q^{13} + 4437312\,q^{14} + 6107136\,q^{15} + \cdots.$$

This expansion gives us the number $N_{12}(m)$ up to $m = 15$. (The above expansion was done by Mathematica.)

4 Certain Calabi-Yau threefolds

In this section, we will assume that $k = \mathbf{C}$. Let X be a rational surface with fibration $f : X \longrightarrow \mathbf{P}^1$ of curves of genus 2. We assume that the fibration f is a *Lefschetz pencil*, that is, all singular fibers are reduced and have only one node. Then the Mordell-Weil lattice for such a fibration is isometric to D_{12}^+.

Theorem 4.1 *([Sa2].) Under the above notation and assumption, let $\mathcal{J} \longrightarrow \mathbf{P}^1$ be the Néron model of J_Γ. Then there exists a smooth projective threefold Y with a fibration $h : Y \longrightarrow \mathbf{P}^1$ which gives a relative compactification of Néron model $\mathcal{J}/\mathbf{P}^1$. Moreover Y has a trivial canonical bundle and $h^{2,0}(Y) = h^{1,0}(Y) = 0$, that is, Y is a Calabi-Yau threefold. Other Hodge numbers are given as follows: $h^{1,1}(Y) = h^{2,1}(Y) = 14$, hence the Hodge diamond of Y is self-mirror, that is, invariant under the $\pi/2$ rotation.*

Here we only remark that under the assumption of Lefschetz pencil $f : X \longrightarrow \mathbf{P}^1$ the smooth relative compactification $h : Y \longrightarrow \mathbf{P}^1$ of the Néron model $\mathcal{J} \longrightarrow \mathbf{P}^1$ is constructed by Nakamura [N]. Moreover one can relatively embed $X \longrightarrow \mathbf{P}^1$ into $h : Y \longrightarrow \mathbf{P}^1$ as a relative principal theta divisor. Therefore X can be considered as a smooth divisor in Y.

In connection with Mirror symmetry conjecture for the above Calabi-Yau threefolds, the following theorem may be interesting.

Theorem 4.2 *([Sa2].) Let $L = X + X^- + 2F$ be a divisor class on the Calabi-Yau threefold Y where X^- is the minus of X and F is a class of general closed fiber of $h : Y \longrightarrow \mathbf{P}^1$. Then L is nef and big divisor on Y. Moreover for any section $\sigma \in S(\mathcal{J}/\mathbf{P}^1) \simeq J_\Gamma(K)$, one has*

$$\deg L_{|\sigma(\mathbf{P}^1)} = <\sigma, \sigma> . \tag{4.1}$$

Here $< \sigma, \sigma >$ denotes the height of the section σ with respect to the Mordell-Weil lattice. Hence the number of rational curves coming from sections of $\mathcal{J}/\mathbf{P}^1$ with fixed degree m with respect to L is equal to $N_{12}(m)$ in §3, hence can be calculated by (3.6).

The detail will be published in [Sa2].

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