代数幾何学シンポジウム記録
1996年度 pp.160-167

Simultaneous minimal model of homogeneous toric deformation

Daisuke Matsushita *

Abstract

For a flat family of Du Val singularities, we can take a simultaneous resolution after finite base change. It is an interesting problem to consider this analogy for a flat family of higher dimensional canonical singularities. In this note, we consider an existence of simultaneous terminalization for K. Altmann's homogeneous toric deformation whose central fibre is an affine Gorenstein toric singularity. We obtain examples that there are no simultaneous terminalization even after finite base change and give a sufficient condition for an existence of simultaneous terminalization. Some examples of 4-dimensional flop are obtained through its application.

1 Introduction

For a flat family of surfaces $f : X \to S$, it is an interesting problem whether exists a birational morphism $\tau : \widetilde{X} \to X$ such that

(1) $f \circ \tau$ is a flat morphism

(2) The fibre $\widetilde{X}_s := (f \circ \tau)^{-1}(s)$ $(s \in S)$ is the minimal resolution of X_s.

If there exists such τ, we say f admits a simultaneous resolution. Let $f : X \to S$ be a flat morphism whose fibres have only Du Val singularities. Then Brieskorn [3, 4] and Tyurina[7] shows that there exists a finite surjective morphism $S' \to S$ and a flat morphism $f' : X \times_S S' \to S'$ admits a simultaneous resolution. It is a key fact for Minimal Model theory. Thus it is natural to consider an analogy for higher dimensional singularity. According to Minimal Model theory, it is suitable to consider an existence of "simultaneous terminalization" for a flat family of higher dimensional singularity.

DEFINITION. For a flat morphism $f : X \to S$ whose fibres are n-fold $(n \geq 3)$, we say f admits a p simultaneous terminalization if there exists a birational morphism $\tau : \widetilde{X} \to X$ which satisfies the following conditions:

1991 Mathematical Subject Classification. Primary 14E30; Secondary 14B07, 14M25

(1) $f \circ \tau$ is a flat morphism.

(2) The fibre $\widetilde{X}_s := (f \circ \tau)^{-1}(s)$ $(s \in S)$ has only terminal singularity.

(3) $K_{\widetilde{X}_s}$ is τ-nef.

Recently K. Altmann constructs in [1, Definition 3.1] affine toric varieties which are called "homogeneous toric deformation". This toric varieties can describe many flat families of toric singularities such as versal deformation space of an isolated Gorenstein toric singularity [2, Theorem 8.1]. In this note, using toric Minimal Model theory, we investigate the existence of simultaneous terminalization of Gorenstein homogeneous toric deformation which is a homogeneous toric deformation whose central fibre is an affine Gorenstein toric variety. Our main results are as follows:

THEOREM 1 *There exists a Gorenstein homogeneous toric deformation whose fibre is n-dimensional* $(n \geq 3)$ *and which admits no simultaneous terminalization even after finite base change.*

We consider the condition of an existence of simultaneous terminalization, and obtain the following results:

THEOREM 2 *A Gorenstein homogeneous toric deformation* $f : X \to \mathbb{C}^m$ *admits a simultaneous terminalization if* X *has a crepant desingularization.*

We introduce the precise definition of K. Altmann's homogeneous toric deformation and Gorenstein homogeneous toric deformation in section 2. Theorem 1 and 2 will be proved in section 3 and 4, respectively. Some examples of 4-dimensional flops are obtained through its application.

2 Homogeneous toric deformation

First we introduce the definition of K .Altmann-n's homogeneous toric deformation.
DEFINITION. A flat morphism $f : X \to \mathbb{C}^m$ is called a homogeneous toric deformation if the following conditions are satisfied:

(1) $X := Spec\mathbb{C}[\sigma^{\vee} \cap M]$ is an affine toric variety.

(2) f is defined by m equations such that $x^{r_i} - x^{r_0}$ where $r_i \in \sigma^{\vee} \cap M$, $(0 \leq i \leq m)$.

(3) Let $L := \oplus_{i=1}^{i=m} \mathbb{Z}(r_i - r_0)$ be the sublattice of N. The central fibre $Y := f^{-1}(0, \cdots, 0)$ is isomorphic to $Spec\mathbb{C}[\bar{\sigma}^{\vee} \cap \bar{M}]$ where $\bar{\sigma} = \sigma \cap L^{\perp}$ and $\bar{M} := M/L$.

(4) $i : Y \to X$ sends the closed orbit in Y isomorphically onto the closed orbit in X.

In this note, we consider a homogeneous toric deformation with some additional conditions:

DEFINITION. We call homogeneous toric deformation $f : X \to \mathbb{C}^n$ as Gorenstein homogeneous toric deformation if it satisfies the following two conditions:

(1) Y is an affine Gorenstein singularity.

(2) The restriction $-r_i$, $(0 \leq i \leq m)$ to $\bar{M}$ determines K_Y.

EXAMPLES.

(1) Most plain example is $f : \mathbb{C}^2(x,y) \to \mathbb{C}(t)$. In this case, f is defined by $x - y = t$.

(2) Let $g : \mathcal{X} \to S$ be a versal deformation space of A_n singularity. It can be described as follows:

$$\mathcal{X} = xy + z^{n+1} + t_1 z^{n-1} + \cdots + t_{n-1}z + t_n \subset \mathbb{C}^{n+3} \to \mathbb{C}^n(t_1, \cdots t_n).$$

Let α_i, $(0 \leq i \leq n)$ be elementary symmetric polynomials on $\mathbb{C}^{n+1}(s_0, \cdots s_n)$ and H a hyperplane such that $\sum_{i=0}^n s_i = 0$. We take a base change by $\alpha_i(s_0, \cdots, s_n) = t_i : H \to \mathbb{C}^n$,

$$\begin{array}{ccc} \mathcal{X} \times_{\mathbb{C}^n} H & \longrightarrow & \mathcal{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ H & \longrightarrow & \mathbb{C}^n. \end{array}$$

Then $\mathcal{X} \times_{\mathbb{C}^n} H$ can be described

$$(xy + \prod_{i=0}^{n}(z + s_i) = 0) \wedge (\sum_{i=0}^{n} s_i = 0) \subset \mathbb{C}^{n+4}(x, y, z, s_0, \cdots, s_n).$$

Let $u_0 := \sum_{i=0}^n s_i$, $u_i := s_i - s_0$, $(1 \leq i \leq n)$ and $z_i := z + s_i$. By using this coordinate, we can describe

$$\mathcal{X} \times_{\mathbb{C}^n} H = (xy + \prod_{i=0}^{n} z_i = 0) \subset \mathbb{C}^{n+3}(x, y, z_0, \cdots, z_n)$$

and $f = (z_1 - z_0, \cdots, z_n - z_0)$. Thus $f : \mathcal{X} \times_{\mathbb{C}^n} H \to H$ is a Gorenstein homogeneous toric deformation.

(3) Let $g : \mathcal{X} \to \mathcal{M}$ be a versal deformation space of n-dimensional $(n \geq 3)$ isolated Gorenstein toric singularity. K. Altmann shows in [2] that for every irreducible component S of $\mathcal{M}$,

$$\begin{array}{ccc} X = \mathcal{X} \times_{S_{\text{red}}} \mathcal{M} & \longrightarrow & \mathcal{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ S_{\text{red}} & \longrightarrow & \mathcal{M} \end{array}$$

the pull back $f : X \to \mathcal{M}_{\text{red}}$ is a Gorenstein homogeneous toric deformation.

3 Simultaneous terminalization for Gorenstein homogeneous toric deformation

In this section, we give a proof of Theorem 1. Let Y be a hypersurface in a quotient space such that

$$Y = (h := x_1 \cdots x_p - x_{p+1} \cdots x_{n+1} = 0) \subset \mathbb{C}^{n+1}(x_1, \cdots, x_{n+1})/G$$

where G is an abelian group acting on $\mathbb{C}^{n+1}$ diagonary. We assume that $\mathbb{C}^{n+1}/G$ has only Gorenstein terminal singularity. Moreover we assume that for every $g \in G$,

$$\sum_{i=1}^{n+1} a_i \leq 1 + \max\{\sum_{i=1}^{p} a_i, \sum_{i=p+1}^{n+1} a_i\}$$

where a_i are characters of the action of g. Then Y has canonical singularity by [6, Theorem 4.6]. Let $X := (h = t) \subset \mathbb{C}^{n+1}/G \times \mathbb{C}(t)$ and $f : X \to \mathbb{C}(t)$ projection. Then a general fibre of f has only $\mathbb{Q}$-factorial terminal singularities because total space $\mathbb{C}^{n+1}/G \times \mathbb{C}$ has only $\mathbb{Q}$-factorial terminal singularities. Suppose that there exists a simultaneous terminalization after some finite base change. Let $(h = t^m) \subset \mathbb{C}^{n+1}/G \times \mathbb{C}$ be finite base change of X and $\tau : \mathcal{X} \to (h = t^m)$ a simultaneous terminalization. We consider the following diagram:

$$\begin{array}{ccccc} (h = t) \subset \mathbb{C}^{n+2} & \leftarrow & (h = t^m) \subset \mathbb{C}^{n+2} & \stackrel{\tau'}{\leftarrow} & \mathcal{X}' \\ \downarrow & & \downarrow & & \downarrow \\ X = (h = t) \subset \mathbb{C}^{n+1}/G \times \mathbb{C} & \leftarrow & (h = t^m) \subset \mathbb{C}^{n+1}/G \times \mathbb{C} & \stackrel{\tau}{\leftarrow} & \mathcal{X} \end{array},$$

where $\mathcal{X}' = \mathcal{X} \times_{\mathbb{C}^{n+1}/G \times \mathbb{C}} \mathbb{C}^{n+2}$. Because general fibres of $f : X \to \mathbb{C}$ has only $\mathbb{Q}$-factorial terminal singularities, the codimension of exceptional sets of τ is greater than two. Thus τ and τ' are small birational morphisms. But $(h = t^m) \subset \mathbb{C}^{n+2}$ is hypersurface singularity whose singular locus has codimension four and thus it is $\mathbb{Q}$-factorial, that is a contradiction. □

4 Simultaneous minimal model of Gorenstein homogeneous toric deformation

THEOREM **3** *Let $f : X := Spec\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}^m$ be a Gorenstein homogeneous toric deformation and $\tau : \widetilde{X} \to X$ a toric Minimal Model of X. Assume that* $\dim X = n + m$. *Then*

(1) *$f \circ \tau : \widetilde{X} \to \mathbb{C}^m$ is a flat morphism.*

(2) $K_{\widetilde{X}_t}$ *is* τ*-nef*

(3) $\widetilde{X}_t$ *has only canonical complete intersection singularities in quotient space such that*

$$(F_i - F_0 = 0) \subset \mathbb{C}^{n+m}(x_1, \cdots, x_{n+m})/G, \quad (1 \leq i \leq m)$$

where G *is an abelian group acting on* $\mathbb{C}^{n+m}$ *diagonary.* $\mathbb{C}^{n+m}/G$ *has only Gorenstein terminal singularity and* F_i *are invariant monomials of the action of* G*. The monomials* F_i *can be written as*

$$\begin{aligned} F_0 &= \prod_{i=1}^{p_1} x_i \\ F_j &= \prod_{i=p_j+1}^{p_{j+1}} x_i \quad 1 \leq j \leq m-1 \\ F_m &= \prod_{i=p_m+1}^{n+m} x_i \end{aligned}$$

where $1 \leq p_1 < p_2 < \cdots < p_m \leq n+m$.

(4) *If* $\widetilde{X}$ *is smooth,* $\widetilde{X}_t$ *has only terminal singularities.*

REMARK. If $\dim X = 2+m$ (i.e. f is 2-dimensional fibration), then we can write F_i, $(1 \leq i \leq m)$ as

$$F_i = x_{i+1} \quad (0 \leq i \leq m-1), \quad F_m = x_m x_{m+1}$$

by changing indices if necessary. Because F_i are invariant monomials under the action of G, the action of each element of $g \in G$ is nontrivial only x_m, x_{m+1} axises. But $\mathbb{C}^{2+m}/G$ has only Gorenstein terminal singularity, the action of G must be trivial. Thus each fibre of $f \circ \tau$ is smooth, and τ gives simultaneous resolution.

PROOF OF THEOREM 3. By K. Altmann's [1, Theorem 3.5 , Remark 3.6], we can state the construction of σ as follows:

(1) σ can be written as $\sigma = \mathbb{R}_{\geq 0}P$ where P is a $(n+m-1)$-dimensional polygone such that

$$\begin{aligned} P \;:=\; & Conv(\cup_{i=0}^{m} R_i \times e_i) \\ & \text{where } R_i \times e_i = \{(x_1, \cdots, x_{n-1}, 0, \cdots, 1, \cdots, 0) \in \mathbb{R}^{n+m} | (x_1, \cdots, x_{n-1}) \in R_i\} \end{aligned}$$

and R_i, $(0 \leq i \leq m)$ are integral polytopes in $\mathbb{Z}^{n-1}$.

(2) f can be written as $(x^{r_i} - x^{r_0})$, $(1 \le i \le m)$ where $r_i : N_{\mathbb{R}} = \mathbb{R}^{n+m} \to \mathbb{R}$ is $n+i$-th projection.

Thus, all primitive one dimensional generators of σ are contained in the hyperplane defined by $r_0 + \cdots + r_m = 1$. Let $\tau : \widetilde{X} \to X$ be a toric minimal model of X and $\sigma = \cup \sigma_\lambda$ is a corresponding cone decomposition. Then these cones are satisfies the following three conditions:

(1) σ_λ is simplex.

(2) Let $\{k_1, \cdots, k_{n+m}\}$ be one dimensional primitive generators of σ_λ. Then all k_i are contained in the hypersurface defined by $r_0 + \cdots + r_m = 1$.

(3) The polytope

$$\Delta_\lambda := \sum_{i=0}^{n+m} \alpha_i k_i, \quad \sum_{i=0}^{n+m} \alpha_i \le 1, \quad \alpha_i \ge 0$$

contains no lattice points except for its vertices.

Let $X_\lambda := Spec\mathbb{C}[\sigma_\lambda^\vee \cap M]$ and $k_i^\vee$, $(1 \le i \le n+m)$ dual vectors of k_i. Then X_λ can be written as follows:

$$X_\lambda \cong \mathbb{C}^{n+m}(x_1, \cdots, x_{n+m})/G$$

where $x_i = x^{k_i^\vee}$, $G := N/ \oplus_{i=1}^{n+m} \mathbb{Z}k_i$ and the action of G is diagonary. Because k_j are contained in the hypersurface defined by $r_0 + \cdots + r_m = 1$ and $\langle r_i, k_j \rangle \ge 0$ $(r_i \in \sigma^\vee)$,

$$\begin{cases} \langle r_i, k_j \rangle = 1 & \text{for } p_i < j \le p_{i+1} \\ \langle r_i, k_j \rangle = 0 & \text{other } j \end{cases}$$

where $0 = p_0 < p_1 < p_2 < \cdots < p_m < p_{m+1} = n+m$. Thus we can write

$$x^{r_i} = \prod_{j=p_i+1}^{p_{i+1}} x_j.$$

The monomials x^{r_i} are invariant under the action of G, because $r_i \in \sigma_\lambda^\vee \cap M$. Thus if we denote $F_i = x^{r_i}$, a proof of theorem (1), (2) and (3) are completed. Finally, we show (4). Since X_λ is smooth, general fibre of $f \circ \tau$ is smooth. We study a singularity of central fibre. Because X_λ is smooth we may assume $k_i = e^i$ $(1 \le i \le n+m)$ where e^i are the standard basis of $\mathbb{Z}^{n+m}$. Then r_i can be written

$$r_i = (e^{p_i+1})^\vee + \cdots + (e^{p_{i+1}})^\vee.$$

Thus a generator of the cone of central fibre $f \circ \tau$ can be written

$$e^{s_1} + \cdots + e^{s_m}, \quad p_i < s_i \le p_{i+1},$$

by the definition of homogeneous toric deformation. So all generators of this cone contains hypercube of $\mathbb{Z}^{n+m}$, hence central fibre has only terminal singularities. □

A toric minimal model of X is not unique, and which can be joined by flop each other. We can obtain some examples of 4-dimensional flop through its application.

EXAMPLE. There is a 4-dimensional flop which satisfy the following diagram:

$$\begin{array}{ccc} \mathbb{P}(1,a,b) \subset Z & \dashrightarrow & \mathbb{P}(1,a,b) \subset Z^+ \\ \phi \searrow & & \swarrow \phi^+ \\ & W & \end{array}$$

where $\phi(\mathbb{P}(1,a,b)) = \phi^+(\mathbb{P}(1,a,b)) = pt$.

REMARK. In the case $a = b = 1$, it is known as "Mukai transformation".

CONSTRUCTION OF EXAMPLE. Let $\bar{\sigma}$ be 3-dimensional cone whose primitive generators are

$$\bar{\sigma} = \langle(1,0,1),(0,1,1),(-a,-b,1),(1,1,1),(-a,1-b,1),(1-a,-b,1)\rangle \subset \mathbb{R}^3.$$

We consider a deformation of toric singularity $Y := Spec\mathbb{C}[\bar{\sigma} \cap \bar{M}]$. The polytope $Q := \bar{\sigma} \cap \{(x,y,z) \in \mathbb{R}^3 | z = 1\}$ has Minkowski decomposition such that $R_0 + R_1 + R_2$ where

$$R_0 = \langle(1,0),(0,0)\rangle, \quad R_0 = \langle(0,1),(0,0)\rangle, \quad R_0 = \langle(-a,-b),(0,0)\rangle.$$

Thus the corresponding toric homogeneous deformation space of Y is a toric variety $X := Spec\mathbb{C}[\sigma \cap M]$ where

$$\sigma := \mathbb{R}_{\geq 0}\langle e_1, e_2, e_3, e_4, e_5, e_6\rangle,$$

$$\begin{array}{lll} e_1 := (1,0,1,0,0), & e_2 := (0,0,1,0,0), & e_3 := (0,1,0,1,0) \\ e_4 := (0,0,0,1,0), & e_5 := (-a,-b,0,0,1), & e_6 := (0,0,0,0,1). \end{array}$$

We construct two different crepant resolutions of X. Because $ae_1 - ae_2 + be_3 - be_4 + e_5 - e_6 = 0$, by Reid[5, Lemma 3.2], there are two cone decompositions of σ such that

$$\sigma = \langle e_2, e_4, e_6, e_1, e_3\rangle \cup \langle e_2, e_4, e_6, e_3, e_5\rangle \cup \langle e_2, e_4, e_6, e_5, e_1\rangle$$

and

$$\sigma = \langle e_1, e_3, e_5, e_2, e_4\rangle \cup \langle e_1, e_3, e_5, e_4, e_6\rangle \cup \langle e_1, e_3, e_5, e_6, e_2\rangle.$$

Let $\tilde{Z}$ and $\tilde{Z^+}$ be the toric varieties corresponding to above cone decompositions. Then the exceptional sets of $\phi : \tilde{Z} \to X$ and $\phi^+ : \tilde{Z^+} \to X$ are isomorphic to $\mathbb{P}(1,a,b)$ and $\phi(\mathbb{P}(1,a,b)) = \phi^+(\mathbb{P}(1,a,b)) = pt$. There is a diagram

$$\begin{array}{ccccc} \tilde{Z} & \xrightarrow{\phi} & X & \xleftarrow{\phi^+} & \tilde{Z^+} \\ f\downarrow & & \downarrow & & \downarrow f^+ \\ \mathbb{C}^2 & = & \mathbb{C}^2 & = & \mathbb{C}^2. \end{array}$$

Then the exceptional set of ϕ and ϕ^+ are contained in a fibre of f and f^+ respectively, since $\phi(\mathbb{P}(1,a,b)) = \phi^+(\mathbb{P}(1,a,b)) = pt$. Let $\imath : \mathbb{C} \to \mathbb{C}^2$ be the diagonal map. We define Z, Z^+ and W the pull back by $\imath$ of $\tilde{Z} \to \mathbb{C}^2$, $\tilde{Z^+} \to \mathbb{C}^2$ and $X \to \mathbb{C}^2$ respectively. □

References

[1] K. Altmann *Minkowski sums and homogeneous deformation of toric variety,* Tôhoku. Math. J., **47** (1995), 151–184.

[2] ———, *The versal deformation of an isolated toric Gorenstein singularity,* alg-geom9405008.

[3] E. Brieskorn, *Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen,* Math. Ann., **166** (1996), 76–102.

[4] ———, *Die Auflösung der rationalen Singularitäten holomorpher abbildungen,* Math. Ann., **178** (1968), 255–270.

[5] M. Reid, *Decomposition of Toric Morphism,* in *Arithmetic and Geometry,* (M. Artin and J. Tate, eds.) vol. II, *Geometry, Progress in Math.* **36**, Birkhäuser, Boston, Basel, Stuttgart, (1983), 539–414.

[6] ———, *Young person's guide to canonical singularities, Proceedings of Symposia in Pure and Mathematics.* **46**, (1987), 345–414.

[7] G. Tyurina, *Resolution of singularities of flat deformations of rational double points,* Functional Anal., **4** (1970), 68–73.