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Resonance in Hypergeometric Systems related to Mirror Symmetry *

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In the late 1980's physicists discovered a fascinating phenomenon in Conformal Field Theory – they called it Mirror Symmetry – and pointed out that this had far reaching consequences in the enumerative geometry of Calabi-Yau threefolds; see [9] for some of the early articles about mirror symmetry and [7] for a recent survey. It is a technique mathematicians had never dreamed of: the number of rational curves of a given degree on one Calabi-Yau threefold is computed from the variation of Hodge structure on the cohomology in a family of different Calabi-Yau threefolds. One is therefore interested in an efficient computation of the variation of Hodge structure in families of Calabi-Yau varieties.

In [1] Batyrev made the observation that behind many examples of mirror symmetry one can see a simple combinatorial duality: the CY threefolds are hypersurfaces (more precisely, members of the anti-canonical linear system) in two toric varieties, constructed from a pair of dual lattice polytopes in $\mathbb{R}^4$. In [2] he analyzed the Hodge structure of Calabi-Yau hypersurfaces in toric varieties and showed that the periods of a (suitably normalized) holomorphic d-form on a d-dimensional CY hypersurface in a toric variety satisfy a system of Gel'fand-Kapranov-Zelevinskii hypergeometric differential equations with appropriate parameters ([2] thm 14.2). However, the rank of this GKZ system is larger than the rank of the period lattice. So, even if one would have all solutions for this system, one would still need a method to decide which solutions are periods. In [6] Hosono, Lian and Yau gave a method for determining the complete system of differential equations for the periods and applied this method in some examples. Their resulting system looks complicated. Fortunately, what we need for mirror symmetry are the periods, i.e. the solutions, not the differential equations!

My approach is based on two observations: firstly, implicit in [2] is a variation of mixed Hodge structure which is an extension of the variation of Hodge structure for the family of CY hypersurfaces and for which the GKZ system is the complete system of differential equations; secondly, [2] does in fact tell precisely where the holomorphic d-form of the Calabi-Yau hypersurface lies in this extended VMHS. In this note I present a simple explicit formula for the solutions of the GKZ system for the extended VMHS. By differentiating these

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A GKZ hypergeometric system ([4] def.1) is a system of partial differential equations for functions Φ of N variables $v_1, \ldots, v_N$. It depends on parameters A and b: parameter $\mathsf{A} = (a_{ij})$ is a $\nu \times N$-matrix of rank ν with entries in $\mathbb{Z}$ and $a_{11} = a_{12} = \ldots = a_{1N} = 1$; parameter $\mathsf{b} = (b_1, \ldots, b_\nu)$ is a vector in $\mathbb{C}^\nu$. Let $\mathbb{L} \subset \mathbb{Z}^N$ be the kernel of the matrix A. The GKZ hypergeometric system with parameters A and b is:

$$\left(-b_i + \sum_{j=1}^{N} a_{ij}\, v_j \frac{\partial}{\partial v_j}\right) \Phi = 0 \quad \text{for } i = 1, \ldots, \nu \tag{1}$$

$$\left(\prod_{j\,:\,\ell_j > 0} \left[\frac{\partial}{\partial v_j}\right]^{\ell_j} - \prod_{j\,:\,\ell_j < 0} \left[\frac{\partial}{\partial v_j}\right]^{-\ell_j}\right) \Phi = 0 \quad \text{for } (\ell_1, \ldots, \ell_N) \in \mathbb{L} \tag{2}$$

In the situation of [2] thm 14.2 matrix A is such that when we delete its first row the columns of the resulting $(\nu - 1) \times N$-matrix are the integral lattice points contained in the Newton polytope Δ of a Laurent polynomial equation for the $(\nu-2)$-dimensional hypersurface in a $(\nu-1)$-dimensional torus. The CY variety is the closure of this affine hypersurface in the toric variety associated with Δ. Parameter b for the case of an appropriately normalized holomorphic $(\nu - 2)$-form is $(-1, 0, \ldots, 0)$. For the GKZ system of the extended VMHS we have the same parameter A, but $\mathsf{b} = (0, 0, \ldots, 0)$.

In [4] Gel'fand-Kapranov-Zelevinskii gave solutions for the GKZ system in the form of so-called Γ-series

$$\sum_{\ell \in \mathbb{L}} \prod_{j=1}^{N} \frac{v_j^{c_j + \ell_j}}{\Gamma(c_j + \ell_j + 1)} \tag{3}$$

where Γ is the usual gamma-function, $\ell = (\ell_1, \ldots, \ell_N) \in \mathbb{L} \subset \mathbb{Z}^N$. The series depends on additional parameters $c_1, \ldots, c_N \in \mathbb{C}$ which must satisfy

$$a_{i1} c_1 + \ldots + a_{iN} c_N = b_i \quad \text{for } i = 1, \ldots, \nu. \tag{4}$$

In order to be able to interpret (3) as a function one also needs a triangulation of the polytope $\Delta := \operatorname{conv}\{\mathsf{a}_1, \ldots, \mathsf{a}_N\}$; here $\mathsf{a}_1, \ldots, \mathsf{a}_N$ are the columns of matrix A viewed as points in $\mathbb{R}^\nu$. The triangulation is used to formulate additional conditions on $c_1, \ldots, c_N \in \mathbb{C}$ which ensure that in (3) the coefficient in the term for ℓ is zero if ℓ is not in a certain pointed cone.

However, the parameter $\mathsf{b} = 0$ is resonant for triangulations with more than one maximal simplex and the Γ-series (3) do not provide enough solutions; cf. [4]. The classical trick for obtaining enough solutions for resonant hypergeometric systems is to differentiate the power series solutions with respect to the parameters of the hypergeometric system. This is what Hosono, Lian and Yau do for the present GKZ hypergeometric system: [6] formula (3.28).

In this note we take a different approach to find solutions for (1)-(2) in case $\mathsf{b} = 0$. First multiply the Γ-series (3) with $\prod_{j=1}^{N} \Gamma(c_j + 1)$. The result can be written as

$$\sum_{\ell \in \mathbb{L}} \frac{\prod_{j:\,\ell_j<0} \prod_{k=0}^{-\ell_j-1} (c_j - k)}{\prod_{j:\,\ell_j>0} \prod_{k=1}^{\ell_j} (c_j + k)} \prod_{j=1}^{N} v_j^{\ell_j} \cdot \prod_{j=1}^{N} v_j^{c_j} \tag{5}$$

or more elegantly, using the notation

$$(t)_0 := 1, \quad (t)_r := t \cdot (t+1) \cdot \ldots \cdot (t+r-1) \quad \text{for } r \in \mathbb{Z},\ r > 0 \tag{6}$$

for Pochhammer symbols,

$$\sum_{\ell \in \mathbb{L}} \frac{\prod\limits_{j:\,\ell_j<0} (-c_j)_{-\ell_j}}{\prod\limits_{j:\,\ell_j>0} (1+c_j)_{\ell_j}} \prod_{j:\,\ell_j<0} (-1)^{\ell_j} \prod_{j=1}^{N} v_j^{\ell_j} \cdot \prod_{j=1}^{N} v_j^{c_j} \tag{7}$$

The key observation in our method is that for (7) to make sense it is not necessary that $c_1, \ldots, c_N$ be complex numbers. It also works if $c_1, \ldots, c_N$ are taken from a $\mathbb{Q}$-algebra in which they are nilpotent and satisfy the linear relations (4) for $\mathsf{b} = 0$, i.e.

$$a_{i1}c_1 + \ldots + a_{iN}c_N = 0 \quad \text{for } i = 1, \ldots, \nu. \tag{8}$$

In order to ensure that in (7) the coefficient in the term for ℓ is zero if ℓ is not in a certain pointed cone we need additional conditions on $c_1, \ldots, c_N$. Very convenient for this purpose are the relations in the definition of the Stanley-Reisner ring of the triangulation $\mathfrak{T}$ of Δ (viewed as a simplicial complex):

$$\begin{array}{lcl} c_{i_1} \cdot \ldots \cdot c_{i_s} \; = \; 0 & \text{if} & \\ \operatorname{conv}\{\mathsf{a}_{i_1}, \ldots, \mathsf{a}_{i_s}\} & \text{is not} & \text{a simplex in the triangulation } \mathfrak{T}. \end{array} \tag{9}$$

The sum (7) will then only involve terms with ℓ satisfying

$$\operatorname{conv}\{\mathsf{a}_j \mid \ell_j < 0\} \quad \text{is a simplex in triangulation } \mathfrak{T} \tag{10}$$

Thus we are lead to introduce the ring $\mathcal{S}_{\mathbb{L},\mathfrak{T}}$ which is the quotient of the polynomial ring $\mathbb{Q}[C_1, \ldots, C_N]$ by the ideal corresponding to relations (8) and (9). It turns out that this ring is finite dimensional as a $\mathbb{Q}$-vector space. This implies that $c_1, \ldots, c_N$ are nilpotent. The expression $v_j^{c_j}$ in (7) should be interpreted as $\exp(c_j \log v_j)$. Thus (7) does contain powers of logarithms.

The expression (7) satisfies the GKZ system (1)-(2) with $\mathsf{b} = 0$[1] . Expanding this expression in terms of a vector space basis of $\mathcal{S}_{\mathbb{L},\mathfrak{T}}$ one finds as coefficients functions of $v_1, \ldots, v_N$ which are solutions of the GKZ system. Expanding (7) by

[1]The same resonant GKZ-system, the same form of its solutions and the same interpretation of the Artinian ring was found by Givental; see [5] thm 3. However, Givental starts from S^1-equivariant Floer cohomology of the space of contractible loops on the toric variety associated with the dual polytope; i.e. on the mirror side from our starting point!

monomials in the nilpotent c's is in fact Taylor expansion, hence differentiation, with respect to the c's. Thus in some sense our formula (7) is a systematized version of the classical trick.

By looking at the logarithms appearing in these solutions of this GKZ system one can easily conclude that they are linearly independent over $\mathbb{C}$. The dimension of the vector space $\mathcal{S}_{\mathsf{L},\mathfrak{T}}$ equals the number of maximal simplices in the triangulation $\mathfrak{T}$. In particular, if all maximal simplices have volume 1, this dimension equals the volume of Δ. Since according to [4] the rank of this GKZ system is $\operatorname{vol}\Delta$, we conclude that our method gives a basis for the solution space of (1)-(2) with $\mathsf{b} = 0$ precisely if all maximal simplices have volume 1.

Thus we have completely determined the extended VMHS. For CY hypersurfaces in toric varieties the next step is to apply $v_1\frac{\partial}{\partial v_1}$ to (7); for this the indices are chosen such that a_1 is the unique lattice point in the interior of Δ. Something similar works for CY complete intersections in toric varieties.

Details of the general theory will be published elsewhere. I finish this report with an example.

An example

Consider the Laurent polynomial $f :=$

$$v_1 + v_2x_1 + v_3x_2 + v_4x_3 + v_5x_1^{-3}x_2^{-1}x_3^{-1} + v_6\,x_1^{-2}x_4^{-1} + v_7x_4 + v_8x_1^{-1} \quad (11)$$

as a polynomial in the variables x_1, x_2, x_3, x_4. The equation $f = 0$ defines for generic values of the coefficients $v_1, \ldots, v_8$ a smooth hypersurface in the 4-dimensional torus $(\mathbb{C}^*)^4$. Matrix A for this Laurent polynomial is

$$\mathsf{A} := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -3 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix} \quad (12)$$

Let $\mathsf{a}_1, \ldots, \mathsf{a}_8$ denote the columns of A viewed as points in $\mathbb{R}^5$. Let Δ be the convex hull of $\{\mathsf{a}_1, \ldots, \mathsf{a}_8\}$, i.e. the *Newton polytope* of f (for generic values of the coefficients $v_1, \ldots, v_8$). Δ is a 4-dimensional pyramid with apex a_2 and base the double tetrahedron formed by the 3-simplices $\operatorname{conv}\{\mathsf{a}_3, \mathsf{a}_4, \mathsf{a}_5, \mathsf{a}_6\}$ and $\operatorname{conv}\{\mathsf{a}_3, \mathsf{a}_4, \mathsf{a}_5, \mathsf{a}_7\}$. Point a_8 is the centre of this double tetrahedron: $\mathsf{a}_8 = (\mathsf{a}_3 + \mathsf{a}_4 + \mathsf{a}_5)/3 = (\mathsf{a}_6 + \mathsf{a}_7)/2$. Point a_1 is the unique lattice point in the interior of Δ: $\mathsf{a}_1 = (\mathsf{a}_2 + \mathsf{a}_8)/2$.

There are six triangulations of Δ. There is only one for which all maximal simplices have volume 1; namely the following triangulation $\mathfrak{T}$ with 12 maximal simplices

$$\begin{matrix} [1\,2\,3\,4\,6] & [1\,2\,3\,4\,7] & [1\,2\,3\,5\,6] & [1\,2\,3\,5\,7] & [1\,2\,4\,5\,6] & [1\,2\,4\,5\,7] \\ [1\,3\,4\,6\,8] & [1\,3\,4\,7\,8] & [1\,3\,5\,6\,8] & [1\,3\,5\,7\,8] & [1\,4\,5\,6\,8] & [1\,4\,5\,7\,8] \end{matrix} \quad (13)$$

(here [1 2 3 4 6] means conv $\{a_1, a_2, a_3, a_4, a_6\}$, etc.)

From (12) and (13) one easily computes $\mathcal{S}_{\mathbb{L},\mathcal{T}}$. From (12) one gets in particular $c_3 = c_4 = c_5$ and $c_6 = c_7$. From (13) one gets in particular $c_3c_4c_5 = 0$ and $c_6c_7 = 0$. Hence $c_5^3 = c_6^2 = 0$. A vector space basis for $\mathcal{S}_{\mathbb{L},\mathcal{T}}$ is:

$$1\,;\ c_5,\, c_6,\, c_8\,;\ c_5^2,\, c_5c_6,\, c_5c_8,\, c_6c_8\,;\ c_5^2c_6,\, c_5^2c_8,\, c_5c_6c_8\,;\ c_5^2c_6c_8$$

One can substitute all the concrete information into (7). From (10) one can see that for each term in the sum ℓ_1 is ≤ 0 and $\ell_2, \ldots, \ell_7$ are ≥ 0. The sum contains terms with $\ell_8 \geq 0$ as well as terms with $\ell_8 < 0$.

Now apply $v_1\frac{\partial}{\partial v_1}$ to (7). The result takes the form $c_1\Omega$. I will give an explicit formula for Ω. One easily checks that $c_1c_8 = 0$, and hence $c_1\Omega$ contains only terms with $\ell_1 \leq 0$ and $\ell_2, \ldots, \ell_8 \geq 0$. As a basis for $\mathbb{L}$ we take the rows of the matrix

$$L := \begin{pmatrix} -6 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 & 0 & 1 & 1 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{14}$$

Then we have for $\ell = (\ell_1, \ldots, \ell_8) \in \mathbb{L}$

$$(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7, \ell_8) = (\ell_5, \ell_6, \ell_8) \cdot L \tag{15}$$

Similarly the linear relations among the c's can be summarized as

$$(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8) = (c_5, c_6, c_8) \cdot L \tag{16}$$

The chosen basis of $\mathbb{L}$ is also used to introduce new variables:

$$\begin{aligned} z_5 &= v_1^{-6} v_2^3 v_3 v_4 v_5 \\ z_6 &= v_1^{-4} v_2^2 v_6 v_7 \\ z_8 &= v_1^{-2} v_2 v_8 \end{aligned}$$

Then

$$\Omega = \sum_{m_5, m_6, m_8 \geqslant 0} \gamma_{m_5,m_6,m_8} \cdot z_5^{m_5} z_6^{m_6} z_8^{m_8} \cdot z_5^{c_5} z_6^{c_6} z_8^{c_8}$$

with coefficients γ_{m_5,m_6,m_8} =

$$\frac{(1 + 6c_5 + 4c_6 + 2c_8)_{(6m_5+4m_6+2m_8)}}{(1 + 3c_5 + 2c_6 + c_8)_{(3m_5+2m_6+m_8)}((1 + c_5)_{m_5})^3((1 + c_6)_{m_6})^2(1 + c_8)_{m_8}}$$

In this formula the c's must be interpreted in $\mathcal{S}_{\mathbb{L},\mathcal{T}} / \mathrm{Ann}(c_1)$. In particular $c_8 = 0$. One easily checks

$$\mathcal{S}_{\mathbb{L},\mathcal{T}} / \mathrm{Ann}(c_1) = \mathbb{Q}[C_5, C_6] / (C_5^3\,,\, C_6^2)$$

The expression for Ω can be simplified further by introducing

$$w_5 := \frac{z_5}{(1 - 4z_8)^3} \quad \text{and} \quad w_6 := \frac{z_6}{(1 - 4z_8)^2}$$

This gives

$$\Omega = \frac{1}{\sqrt{1-4z_8}} \sum_{m_5,m_6 \geqslant 0} \frac{(\frac{1}{2}+3c_5+2c_6)_{(3m_5+2m_6)}}{((1+c_5)_{m_5})^3((1+c_6)_{m_6})^2} \cdot (4^3 w_5)^{m_5} (4^2 w_6)^{m_6} \cdot w_5^{c_5} w_6^{c_6}$$

If we now expand Ω in terms of the obvious basis for $\mathcal{S}_{\mathbb{L},\mathfrak{T}} / \mathrm{Ann}(c_1)$:

$$\Omega = g_{00} + g_{10}c_5 + g_{01}c_6 + g_{20}c_5^2 + g_{11}c_5c_6 + g_{21}c_5^2c_6$$

then $g_{00}, \ldots, g_{21}$ form a basis for the period lattice of the (compact) Calabi-Yau threefold given by the Laurent polynomial f; see (11).

With this basis one can compute the Yukawa coupling, and thus (assuming mirror symmetry) count numbers of rational curves, on the mirror CY threefold. Details of this computation and its results will be discussed elsewhere.

I finish this note with a description of the mirror CY threefold [2]. This is the double covering of $\mathbb{P}^2 \times \mathbb{P}^1$ branched along a surface of degree $(6,4)$. If one describes this double covering by a homogeneous equation $z^2 = p(x_1, x_2, x_3; y_1, y_2)$ then the weights of the variables for the action of $\mathbb{C}^* \times \mathbb{C}^*$ are: z has weight $(3,2)$; x_1, x_2, x_3 have weight $(1,0)$ and y_1, y_2 have weight $(0,1)$ (compare this with the basis of $\mathbb{L}$ in (14)). From these weights one gets the polytope Δ with its marked points $\mathsf{a}_1, \ldots, \mathsf{a}_7$. In order to have a triangulation $\mathfrak{T}$ of Δ for which all maximal simplices have volume 1, we must insert the point a_8. The triangulation gives a refinement of the outer normal fan of the dual polytope of Δ. It gives a toric variety $\mathbb{V}$, in which the double covering of $\mathbb{P}^2 \times \mathbb{P}^1$ sits as a hypersurface $\mathbb{X}$. This construction really is Batyrev's version of mirror symmetry!

$\mathcal{S}_{\mathbb{L},\mathfrak{T}}$ is in fact the cohomology ring of $\mathbb{V}$ (see [3] § 5.2) and $\mathcal{S}_{\mathbb{L},\mathfrak{T}} / \mathrm{Ann}(c_1)$ is the image of $H^*(\mathbb{V})$ in $H^*(\mathbb{X})$. The elements c_5 and c_6 can be identified as the pullbacks of the hyperplane classes of $\mathbb{P}^2$ and $\mathbb{P}^1$ respectively.

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