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AUTOMORPHISMS ON K3 SURFACES

D. -Q. Zhang

Math. Dept. National Univ. of Singapore

e-mail : matzdq@nus.sg

This is an expository note on our recent works with K. Oguiso. In the present note, we shall often use the following notations and assumptions :

(*) X is a projective K3 surface, σ an automorphism on X of order m ($m \geq 2$), $\zeta_m := exp(2\pi\sqrt{-1}/m)$, ω a non-zero holomorphic 2-form on X and $T_X = (PicX)^{\perp} \subseteq H^2(X, \mathbf{Z})$ the transcendental lattice of X [BPV].

Theorem 1 [Ni 1]. *With the notations and assumptions in* (*), *suppose further that* σ^* *acts trivially on the 1-dimensional space* $H^0(X, \mathcal{O}_X(K_X))$ *of holomorphic 2-forms, i.e.,* $\sigma^*\omega = \omega$. *Then* $m \leq 8$.

In view of Theorem 1, we consider the following hypothesis:

(**) With the notations and assumptions in (*), assume further that $\sigma^*\omega = \zeta_m\omega$.

Under the hypothesis (**), the Euler number $\varphi(m) | rkT_X$, and hence one has [Ni 1]:

$m \leq 66$ *and* $p \leq 19$ *for every prime factor* p *of* m.

The following result can be obtained by the Hodge index theorem and by considering the diagonalization of σ^* at its fixed-points.

Lemma 2. *Assume that the pair (X,σ) satisfies the hypothesis* (**).

(1) *The set $X^{<\sigma>} = \{x \in X | \sigma^i(x) = x$ for some $\sigma^i \neq id\}$ of points with non-trivial stabilizer, is a disjoint union of smooth curves and isolated points.*

(2) *If $X^{<\sigma>}$ contains a curve C of genus ≥ 2, then X^σ is a disjoint union of C, smooth rational curves and isolated points.*

Theorem 3 (see [Z3, Theorems 3 and 3']). *Let X be a projective K3 surface with an involution σ such that $\sigma^*\omega = -\omega$.*

(1) *The fixed locus X^σ is a disjoint union of r smooth curves for some $r \leq 10$.*

If $r = 10$ then X^σ is a union of 9 smooth rational curves and a smooth curve C of genus 0, 1 or 2.

(2) *All pairs (X,σ), modulo isomorphisms, in (1) with $r = 10$ and $g(C) = 1$ (resp. $g(C) = 2$) are parametrized by a subset of $\mathbf{P}^1$ (resp. of $\mathbf{P}^3$).*

(3) *There is a unique (modulo isomorphisms) pair (X,σ) such that X^σ is a union of 10 rational curves. Such X has Picard number 20 and discriminant 4 and hence has infinite automorphism group AutX.*

Remark 4. (1) For X in Theorem 3(3) "$\#AutX = \infty$" was first proved by T. Shioda–H. Inose [SI]; such X is called one of the two most algebraic K3 surfaces by E. B. Vinberg [V] who also determined $AutX$ (see Remark 15, and Example 6, also for the construction of X).

(2) Nikulin claimed that some results in Theorem 3 has been proved in [Ni 2], though

the author has not found any clear statements similar to Theorem 3 above and will try to read [Ni 2] again later.

Theorem 5 [OZ1, Theorems 3 and 4]. *Let X_m be a projective K3 surface with an automorphism σ of order m where $m = 3$ (resp. 2) such that*

(i) $\sigma^*\omega = \zeta_m \omega$,

(ii) *there is no any σ-fixed curve (point-wise) of genus ≥ 2, and*

(iii) *there are at least 6 (resp. 10) σ-fixed rational curves.*

Then such a pair is unique upto isomorphisms, and isomorphic to Shioda-Inose's pair $(S_m, < g_m >)$ to be defined below.

Example 6 (see [OZ1, Examples 1 and 2] for details). Let $\zeta := exp(2\pi\sqrt{-1}/3)$ and let $E_\zeta := \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\zeta)$ be the elliptic curve of period ζ. Let $S_3 \to \overline{S}_3 := {E_\zeta}^2 / < diag(\zeta, \zeta^2) >$ be the minimal resolution of the quotient surface $\overline{S}_3$ [SI, Lemma 5.1].

Then S_3 is the unique projective K3 surface of Picard number 20 and discriminant 3. Let g_3 be the automorphism of S_3 induced by the action diag$(\zeta, 1)$ on ${E_\zeta}^2$. Then this Shioda-Inose pair (S_3, g_3) satisfies all conditions in Theorem 5 with $m = 3$.

Let $E_{\sqrt{-1}} := \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\sqrt{-1})$ be the elliptic curve of period $\sqrt{-1}$. Let $S_2 \to \overline{S}_2 := {E_{\sqrt{-1}}}^2 / < diag(-\sqrt{-1}, \sqrt{-1}) >$ be the minimal resolution of the quotient surface $\overline{S}_2$ [SI, Lemma 5.2].

Then S_2 is the unique projective K3 surface of Picard number 20 and discriminant 4. Let g_2 be the automorphism of S_2 induced by the action diag$(-1, 1)$ on ${E_{\sqrt{-1}}}^2$. Then this Shioda-Inose pair (S_2, g_2) satisfies all conditions in Theorem 5 with $m = 2$.

Corollary 7 [OZ1, Theorems 1 and 2]. *There is only one isomorphism class of rational* *log Enriques surface of Type D_{19}, and only one of Type A_{19} (see Definitions below).*

Let Z be a rational normal projective surface with at worst isolated quotient singular points. Z is a *(rational) log Enriques surface* if a positive multiple mK_Z of the canonical Weil divisor K_Z is linearly equivalent to zero.

$m := min\{n \in \mathbf{Z}_{>0} | nK_Z \sim 0\}$ is called the *index* of Z. Let

$$\pi : Y := Spec_{\mathcal{O}_Z} \oplus_{i=0}^{m-1} \mathcal{O}_Z(-iK_Z) \to Z$$

be the *canonical* Galois $\mathbf{Z}/m\mathbf{Z}$-covering. By the definition, we have:

Lemma 8. (1) *π is unramified over the smooth part $Z - SingZ$.*

(2) *Y is a projective K3 surface with at worst Du Val (= rational double) singular points. Let $g : X \to Y$ be a minimal resolution.*

(3) *Let σ be an order-m automorphism on X (or on Y) coming from the map π so that $Gal(Y/Z) = <\sigma>$. Then $\sigma^*\omega = \zeta_m\omega$, after replacing σ by a new generator of $Gal(Y/Z)$.*

By Lemma 8(3), we can apply Lemma 2 (here $m \geq 2$ because $Z = Y/\sigma$ is rational).

Remark 9. The following two things are essentially equivalent:

(A) A pair (X, σ), where X is a projective K3 surface and σ an order m ($m \geq 2$) automorphism on X such that $\sigma^*\omega = \zeta_m\omega$ and that $X^{<\sigma>}$ is non-empty but consists of only rational curves and isolated points. By Lemma 2, $X^{<\sigma>}$ is now a disjoint union of smooth rational curves and isolated points.

(B) A (rational) log Enriques surface Z of index m.

In fact, for (B) $\Rightarrow$ (A), we define X, σ as in Lemma 8.

For (A) $\Rightarrow$ (B), we let $X \to Y$ be a contraction of a σ-stable divisor D containing all curves in $X^{<\sigma>}$, into Du Val singular points. Now Define $Z := Y/\sigma$.

Question 10. Let Z be a rational log Enriques surface of index m with $\pi : Y \to Z$ as its canonical covering.

We know that Y is a projective K3 surface with at worst singular points of Dynkin types A_r ($r \geq 1$), D_s ($s \geq 4$) and E_t ($t = 6, 7, 8$).

What is the possible combination of Dynkin types of singular points on Y ?

Definition 11. A log Enriques surface Z is of *Type* $A_r + D_s + E_t + \cdots$ if the canonical covering Y of Z satisfies Sing $Y = A_r + D_s + E_t + \cdots$.

Remark 12. The sum of "weights" $r + s + t + \cdots$ in Definition 11 has an upper bound 19, because the Picard number of a K3 surface has an upper bounded 20.

Z is an *extremal log Enriques surface* if this sum $r + s + t + \cdots$ equals 19.

Theorem 13 [OZ3, Main Theorem]. *There are exactly 7 isomorphism classes of extremal (rational) log Enriques surfaces. Their Types are as follows:*

$$D_{19}, D_{16} + A_3, D_{13} + A_6,$$

$$D_7 + A_{12}, D_7 + D_{12}, D_4 + A_{15}, A_{19}.$$

Example 14. Let (S_m, g_m) ($m = 2, 3$) be Shioda-Inose's pairs in Example 6. On S_m where $m = 3$ (resp. 2), there are 24 normal crossing (g_m-stable) smooth rational curves shown in [OZ1, Figures 1 and 2] or [SI, Figures 2 and 3]; among these 24, there are divisors Δ_i of the first six Dynkin types (resp. divisor Δ_7 of Dynkin type A_{19}) in Theorem 13.

Let $S_m \to \overline{S}_m$ be the contraction of Δ_i and let $Z(i) := \overline{S}_m/g_m$. Then $Z(i)$'s are nothing but 7 extremal rational log Enriques surfaces in Theorem 13.

Remark 15. (1) Every K3 surface of (maximum possible) Picard number 20 satisfies discr.$X \geq 3$ [SI]. This might be the reason why Vinberg call *the* two K3 surfaces X with Picard number 20 and discr.$X = 3, 4$, the most algebraic K3 surfaces.

(2) The same rational log Enriques surfaces of Type D_{19} and A_{19} were constructed by "bottom up" (rather than "top down" here) in [Z1]. I. Naruki and M. Reid then asked about the uniqueness of these two surfaces. See Reid [R] for his result towards a kind of uniqueness theorem.

We know that there is a unique rational log Enriques surface of Type D_{19} and one of Type A_{19}. One may ask the same uniqueness question for D_n, A_n with smaller n. The following are some of the answers, where $D_{17} + *$, etc. means $D_{17}+$ something.

Theorem 16 [OZ2, Theorems 1 and 2]. *There is exactly one (resp. two) isomorphism class(es) of rational log Enriques surface(s) of Type $D_{18} + *$ ($= D_{18}$ as a matter of fact) (resp. $A_{18} + *$ ($= A_{18}$ as a matter of fact)).*

Theorem 17. [Z5, Theorem 4]. *There is no any rational log Enriques surface of Type $D_{17} + *$.*

Theorem 18 [OZ4, Z4, Z5]).

(1) *Any rational log Enriques surface of Type $A_{17} + *$ has index $2, 3, 4$, or 5.*

(2) *There are exactly two isomorphism classes of rational log Enriques surfaces of*

Type $A_{17} + *$ *(*$= A_{17}$ *as a matter of fact) and index* 5 *(cf. Theorem 19 and Remark 20).*

(3) *There are exactly three isomorphism classes of rational log Enriques surfaces of Type* $A_{17} + *$ *(*$= A_{17} + A_1$ *as a matter of fact) and index* 4.

(4) *There is at least one and at most three isomorphism classes of rational log Enriques surfaces of Type* $A_{17} + *$ *(*$= A_{17}$ *as a matter of fact) and index* 3.

(5) *There are exactly three isomorphism classes of rational log Enriques surfaces of Type* $A_{17} + A_1$ *and index* 2.

Theorem 19 (cf. [OZ5, Theorem 4] and [OZ5]). *Let* Z *be a rational log Enriques surface of Type* A_{17} *and index 5. Let* $Y \to Z$ *be the canonical* $\mathbf{Z}/5\mathbf{Z}$*-covering,* $X \to Y$ *a minimal resolution, and* σ *an order-5 automorphism on* X *(or on* Y*) such that* $Gal(Y/Z) = <\sigma>$*. Then we have:*

$(***)$ *discr.*$X = 5$, $<\sigma> = Ker(AutX \to Aut(PicX))$, *and the Euler number* $\varphi(5) = rkT_X$.

Remark 20. According to the result (announced in a 3-page paper by S. P. Vorontsov but without detailed proof), there is only one isomorphism class of X with a σ satisfying (***) above. We have a detailed proof of the same result [OZ5]. Kondo [Ko] has constructed such a pair $(X, <\sigma>)$.

For general m, we have the following results:

Theorem 21 [OZ5]. *Let* X *be a projective K3 surface with an automorphism* σ *of order* m *where* $m = 2$ *(resp.* $3, 5, 7, 11, 13, 17,$ *or* 19*) such that*

(i) $\sigma^*|PicX = id$,

(ii) $\sigma^* \omega = \zeta_m \omega$, *and*

(iii) *there is no any* σ*-fixed curve of genus* ≥ 2, *and there are at least* 10 *(resp.* $6, 3, 2, 1, 1, 0$, *or 0)* σ*-fixed rational curves.*

Then such a pair (X, σ) *is unique upto isomorphisms. Moreover, discr.*$X = m$.

Let X be a projective K3 surface and let $H_X := Ker(AutX \to Aut(PicX))$. Then H_X is a finite cyclic group of order m_X say [Ni 1]. By [Ni 1], $\varphi(m_X)|rkT_X$. Kondo [Ko] determined all possible values of m_X; in particular, if T_X is non-unimodular, then either m_X is prime with $2 \leq m \leq 19$, or $m_X = 2^r$ $(r = 0, 2, 3, 4)$, 3^s $(s = 2, 3)$, or 25.

Corollary 22 [OZ5]. *There is a unique projective K3 surface such that* m_X *is prime and* $\varphi(m_X) = rkT_X$. *Moreover, such* X *satisfies discr.*$X = m_X$.

When $m = 13, 17$ or 19, we can prove that all conditions (i), (ii) and (iii) in Theorem 21 will be satisfied automatically. That is, we have:

Corollary 23 [OZ5]. *For each of* $m = 13, 17$ *and* 19, *there is exactly one isomorphism class of projective K3 surface with an automorphism of order* m.

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