AUTOMORPHISMS ON K3 SURFACES

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This is an expository note on our recent works with K. Oguiso. In the present note, we shall often use the following notations and assumptions:

(*) $X$ is a projective K3 surface, $\sigma$ an automorphism on $X$ of order $m$ ($m \geq 2$), $\zeta_m := \exp(2\pi \sqrt{-1}/m)$, $\omega$ a non-zero holomorphic 2-form on $X$ and $T_X = (\text{Pic}X)^{\perp} \subseteq H^2(X, \mathbb{Z})$ the transcendental lattice of $X$ [BPV].

Theorem 1 [Ni 1]. With the notations and assumptions in (*), suppose further that $\sigma^* \omega = \omega$. Then $m \leq 8$.

In view of Theorem 1, we consider the following hypothesis:

(**) With the notations and assumptions in (*), assume further that $\sigma^* \omega = \zeta_m \omega$.

Under the hypothesis (**), the Euler number $\varphi(m) | r_k T_X$, and hence one has [Ni 1]:

$m \leq 66$ and $p \leq 19$ for every prime factor $p$ of $m$. 
The following result can be obtained by the Hodge index theorem and by considering
the diagonalization of $\sigma^*$ at its fixed-points.

**Lemma 2.** Assume that the pair $(X, \sigma)$ satisfies the hypothesis (**).

1. The set $X^{\sigma} = \{ x \in X | \sigma^i(x) = x \text{ for some } \sigma^i \neq \text{id} \}$ of points with non-trivial
   stabilizer, is a disjoint union of smooth curves and isolated points.

2. If $X^{\sigma}$ contains a curve $C$ of genus $\geq 2$, then $X^\sigma$ is a disjoint union of $C$,
   smooth rational curves and isolated points.

**Theorem 3** (see [Z3, Theorems 3 and 3']). Let $X$ be a projective $K3$ surface with an
involution $\sigma$ such that $\sigma^* \omega = -\omega$.

1. The fixed locus $X^\sigma$ is a disjoint union of $r$ smooth curves for some $r \leq 10$.
   If $r = 10$ then $X^\sigma$ is a union of $9$ smooth rational curves and a smooth curve $C$ of
   genus $0$, $1$ or $2$.

2. All pairs $(X, \sigma)$, modulo isomorphisms, in (1) with $r = 10$ and $g(C) = 1$ (resp.
   $g(C) = 2$) are parametrized by a subset of $\mathbb{P}^1$ (resp. of $\mathbb{P}^3$).

3. There is a unique (modulo isomorphisms) pair $(X, \sigma)$ such that $X^\sigma$ is a union
   of $10$ rational curves. Such $X$ has Picard number $20$ and discriminant $4$ and hence has
   infinite automorphism group $\text{Aut}X$.

**Remark 4.** (1) For $X$ in Theorem 3(3) "$\# \text{Aut}X = \infty$" was first proved by T.
Shioda–H. Inose [SI]; such $X$ is called one of the two most algebraic $K3$ surfaces by E.
B. Vinberg [V] who also determined $\text{Aut}X$ (see Remark 15, and Example 6, also for the
construction of $X$).

(2) Nikulin claimed that some results in Theorem 3 has been proved in [Ni 2], though
the author has not found any clear statements similar to Theorem 3 above and will try to read [Ni 2] again later.

**Theorem 5** [OZ1, Theorems 3 and 4]. Let $X_m$ be a projective K3 surface with an automorphism $\sigma$ of order $m$ where $m = 3$ (resp. 2) such that

(i) $\sigma^*\omega = \zeta_m\omega$,

(ii) there is no any $\sigma$-fixed curve (point-wise) of genus $\geq 2$, and

(iii) there are at least 6 (resp. 10) $\sigma$-fixed rational curves.

Then such a pair is unique up to isomorphisms, and isomorphic to Shioda-Inose's pair $(S_m, g_m)$ to be defined below.

**Example 6** (see [OZ1, Examples 1 and 2] for details). Let $\zeta := \exp(2\pi\sqrt{-1}/3)$ and let $E_\zeta := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\zeta)$ be the elliptic curve of period $\zeta$. Let $S_3 \rightarrow \overline{S}_3 := E_\zeta^2/\langle \text{diag}(\zeta, \zeta^2) \rangle$ be the minimal resolution of the quotient surface $S_3$ [SI, Lemma 5.1].

Then $S_3$ is the unique projective K3 surface of Picard number 20 and discriminant 3. Let $g_3$ be the automorphism of $S_3$ induced by the action $\text{diag}(\zeta, 1)$ on $E_\zeta^2$. Then this Shioda-Inose pair $(S_3, g_3)$ satisfies all conditions in Theorem 5 with $m = 3$.

Let $E_{\sqrt{-1}} := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\sqrt{-1})$ be the elliptic curve of period $\sqrt{-1}$. Let $S_2 \rightarrow \overline{S}_2 := E_{\sqrt{-1}}^2/\langle \text{diag}(-\sqrt{-1}, \sqrt{-1}) \rangle$ be the minimal resolution of the quotient surface $\overline{S}_2$ [SI, Lemma 5.2].

Then $S_2$ is the unique projective K3 surface of Picard number 20 and discriminant 4. Let $g_2$ be the automorphism of $S_2$ induced by the action $\text{diag}(-1, 1)$ on $E_{\sqrt{-1}}^2$. Then this Shioda-Inose pair $(S_2, g_2)$ satisfies all conditions in Theorem 5 with $m = 2$.

**Corollary 7** [OZ1, Theorems 1 and 2]. There is only one isomorphism class of rational
Let $Z$ be a rational normal projective surface with at worst isolated quotient singular points. $Z$ is a \textit{(rational) log Enriques surface} if a positive multiple $mK_Z$ of the canonical Weil divisor $K_Z$ is linearly equivalent to zero.

$m := \min\{n \in \mathbb{Z}_{>0} | nK_Z \sim 0\}$ is called the \textit{index} of $Z$. Let

$$\pi : Y := \text{Spec}_{\mathbb{Z}} \mathbb{Z}_m \oplus \mathbb{O}_\mathbb{Z}(-iK_Z) \rightarrow Z$$

be the \textit{canonical Galois} $\mathbb{Z}/m\mathbb{Z}$-covering. By the definition, we have:

**Lemma 8.** (1) $\pi$ is unramified over the smooth part $Z - \text{Sing}Z$.

(2) $Y$ is a projective $K3$ surface with at worst Du Val (rational double) singular points. Let $g : X \rightarrow Y$ be a minimal resolution.

(3) Let $\sigma$ be an order-$m$ automorphism on $X$ (or on $Y$) coming from the map $\pi$ so that $\text{Gal}(Y/Z) = <\sigma>$. Then $\sigma^*\omega = \zeta_m\omega$, after replacing $\sigma$ by a new generator of $\text{Gal}(Y/Z)$.

By Lemma 8(3), we can apply Lemma 2 (here $m \geq 2$ because $Z = Y/\sigma$ is rational).

**Remark 9.** The following two things are essentially equivalent:

(A) A pair $(X, \sigma)$, where $X$ is a projective $K3$ surface and $\sigma$ an order $m$ ($m \geq 2$) automorphism on $X$ such that $\sigma^*\omega = \zeta_m\omega$ and that $X^{<\sigma>}$ is non-empty but consists of only rational curves and isolated points. By Lemma 2, $X^{<\sigma>}$ is now a disjoint union of smooth rational curves and isolated points.

(B) A (rational) log Enriques surface $Z$ of index $m$.

In fact, for (B) $\Rightarrow$ (A), we define $X, \sigma$ as in Lemma 8.
For (A) \(\Rightarrow (B)\), we let \(X \rightarrow Y\) be a contraction of a \(\sigma\)-stable divisor \(D\) containing all curves in \(X^{<\sigma>}\), into Du Val singular points. Now Define \(Z := Y/\sigma\).

**Question 10.** Let \(Z\) be a rational log Enriques surface of index \(m\) with \(\pi : Y \rightarrow Z\) as its canonical covering.

We know that \(Y\) is a projective K3 surface with at worst singular points of Dynkin types \(A_r (r \geq 1), D_s (s \geq 4)\) and \(E_t (t = 6, 7, 8)\).

What is the possible combination of Dynkin types of singular points on \(Y\) ?

**Definition 11.** A log Enriques surface \(Z\) is of Type \(A_r + D_s + E_t + \cdots\) if the canonical covering \(Y\) of \(Z\) satisfies \(\text{Sing } Y = A_r + D_s + E_t + \cdots\).

**Remark 12.** The sum of "weights" \(r + s + t + \cdots\) in Definition 11 has an upper bound 19, because the Picard number of a K3 surface has an upper bounded 20.

\(Z\) is an extremal log Enriques surface if this sum \(r + s + t + \cdots\) equals 19.

**Theorem 13** [OZ3, Main Theorem]. There are exactly 7 isomorphism classes of extremal (rational) log Enriques surfaces. Their Types are as follows:

\[
D_{19}, D_{16} + A_3, D_{13} + A_6, \\
D_7 + A_{12}, D_7 + D_{12}, D_4 + A_{15}, A_{19}.
\]

**Example 14.** Let \((S_m, g_m) (m = 2, 3)\) be Shioda-Inose's pairs in Example 6. On \(S_m\) where \(m = 3\) (resp. 2), there are 24 normal crossing \((g_m\)-stable\) smooth rational curves shown in [OZ1, Figures 1 and 2] or [SI, Figures 2 and 3]; among these 24, there are divisors \(\Delta_i\) of the first six Dynkin types (resp. divisor \(\Delta_7\) of Dynkin type \(A_{19}\)) in Theorem 13.
Let $S_m \to \overline{S}_m$ be the contraction of $\Delta_i$ and let $Z(i) := \overline{S}_m/g_m$. Then $Z(i)$’s are nothing but 7 extremal rational log Enriques surfaces in Theorem 13.

**Remark 15.** (1) Every K3 surface of (maximum possible) Picard number 20 satisfies $\text{discr.} X \geq 3 [SI]$. This might be the reason why Vinberg call the two K3 surfaces $X$ with Picard number 20 and $\text{discr.} X = 3, 4$, the most algebraic K3 surfaces.

(2) The same rational log Enriques surfaces of Type $D_{19}$ and $A_{19}$ were constructed by "bottom up" (rather than "top down" here) in [Z1]. I. Naruki and M. Reid then asked about the uniqueness of these two surfaces. See Reid [R] for his result towards a kind of uniqueness theorem.

We know that there is a unique rational log Enriques surface of Type $D_{19}$ and one of Type $A_{19}$. One may ask the same uniqueness question for $D_n, A_n$ with smaller $n$. The following are some of the answers, where $D_{17} + *$, etc. means $D_{17} +$ something.

**Theorem 16** [OZ2, Theorems 1 and 2]. *There is exactly one (resp. two) isomorphism class(es) of rational log Enriques surface(s) of Type $D_{18} + * (= D_{18}$ as a matter of fact) (resp. $A_{18} + * (= A_{18}$ as a matter of fact)).

**Theorem 17.** [Z5, Theorem 4]. *There is no any rational log Enriques surface of Type $D_{17} + *$.

**Theorem 18** [OZ4, Z4, Z5]).

(1) *Any rational log Enriques surface of Type $A_{17} + *$ has index 2, 3, 4, or 5.*

(2) *There are exactly two isomorphism classes of rational log Enriques surfaces of*
Type $A_{17} + *$ (= $A_{17}$ as a matter of fact) and index 5 (cf. Theorem 19 and Remark 20).

(3) There are exactly three isomorphism classes of rational log Enriques surfaces of Type $A_{17} + *$ (= $A_{17} + A_1$ as a matter of fact) and index 4.

(4) There is at least one and at most three isomorphism classes of rational log Enriques surfaces of Type $A_{17} + *$ (= $A_{17}$ as a matter of fact) and index 3.

(5) There are exactly three isomorphism classes of rational log Enriques surfaces of Type $A_{17} + A_1$ and index 2.

**Theorem 19** (cf. [OZ5, Theorem 4] and [OZ5]). Let $Z$ be a rational log Enriques surface of Type $A_{17}$ and index 5. Let $Y \rightarrow Z$ be the canonical $\mathbb{Z}/5\mathbb{Z}$-covering, $X \rightarrow Y$ a minimal resolution, and $\sigma$ an order-5 automorphism on $X$ (or on $Y$) such that $\text{Gal}(Y/Z) = \langle \sigma \rangle$. Then we have:

$$(* * *) \text{discr.} X = 5, \quad \langle \sigma \rangle = \text{Ker}(\text{Aut} X \rightarrow \text{Aut}(\text{Pic} X)), \quad \text{and the Euler number} \quad \varphi(5) = \text{rk} T_X.$$ 

**Remark 20.** According to the result (announced in a 3-page paper by S. P. Vorontsov but without detailed proof), there is only one isomorphism class of $X$ with a $\sigma$ satisfying (***) above. We have a detailed proof of the same result [OZ5]. Kondo [Ko] has constructed such a pair ($X, \langle \sigma \rangle$).

For general $m$, we have the following results:

**Theorem 21** [OZ5]. Let $X$ be a projective $K3$ surface with an automorphism $\sigma$ of order $m$ where $m = 2$ (resp. $3, 5, 7, 11, 13, 17, \text{ or } 19$) such that

(i) $\sigma^*|\text{Pic} X = \text{id},$
(ii) $\sigma^* \omega = \zeta_m \omega$, and

(iii) there is no any $\sigma$-fixed curve of genus $\geq 2$, and there are at least 10 (resp. 6, 3, 2, 1, 1, 0, or 0) $\sigma$-fixed rational curves.

Then such a pair $(X, \sigma)$ is unique up to isomorphisms. Moreover, $\text{discr}.X = m$.

Let $X$ be a projective K3 surface and let $H_X := \text{Ker}(\text{Aut} X \to \text{Aut}(\text{Pic} X))$. Then $H_X$ is a finite cyclic group of order $m_X$ say $[N_1]$. By $[N_1]$, $\varphi(m_X) \mid \text{rk} T_X$. Kondo [Ko] determined all possible values of $m_X$; in particular, if $T_X$ is non-unimodular, then either $m_X$ is prime with $2 \leq m \leq 19$, or $m_X = 2^r (r = 0, 2, 3, 4)$, $3^s (s = 2, 3)$, or 25.

**Corollary 22 [OZ5].** There is a unique projective K3 surface such that $m_X$ is prime and $\varphi(m_X) = \text{rk} T_X$. Moreover, such $X$ satisfies $\text{discr}.X = m_X$.

When $m = 13, 17$ or 19, we can prove that all conditions (i), (ii) and (iii) in Theorem 21 will be satisfied automatically. That is, we have:

**Corollary 23 [OZ5].** For each $m = 13, 17$ and 19, there is exactly one isomorphism class of projective K3 surface with an automorphism of order $m$.

References


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