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ON THE CONE OF DIVISORS OF CALABI-YAU FIBER SPACES

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INTRODUCTION

Let $f : X \to S$ be a projective surjective morphism of normal varieties with geometrically connected fibers. We call it a *Calabi-Yau fiber space* if X has only $\mathbb{Q}$-factorial terminal singularities and the canonical divisor K_X is relatively numerically trivial over S. This concept is a natural generalization of that of Calabi-Yau manifolds. Such fiber spaces appear as the output of the minimal model program (MMP). We shall investigate divisors on them by using the log minimal model program (log MMP). We refer the reader to [KMM] for the generalities of the minimal model theory.

We shall consider the following generalizations of conjectures of D. Morrison ([M1, M2]) concerning the finiteness properties of the cones which are generated by nef divisors or movable divisors (cf. Definition 1.1):

Now we consider a generalization of the Morrison conjecture:

Conjecture. *(cf. [M1, M2]). Let $f : X \to S$ be a Calabi-Yau fiber space. Then the following hold:*

(1) The number of the $Aut(X/S)$-equivalence classes of faces of the effective nef cone $\mathcal{A}^e(X/S)$ corresponding to birational contractions or fiber space structures is finite. Moreover, there exists a finite rational polyhedral cone Π which is a fundamental domain for the action of $Aut(X/S)$ on $\mathcal{A}^e(X/S)$ in the sense that

(a) $\mathcal{A}^e(X/S) = \bigcup_{\theta \in \mathrm{Aut}(X/S)} \theta_* \Pi$,

(b) *Int* $\Pi \cap \theta_*$*Int* $\Pi = \emptyset$ *unless* $\theta_* = id$.

(2) The number of the $Bir(X/S)$-equivalence classes of chambers $\mathcal{A}^e(X'/S, \alpha)$ in the effective movable cone $\mathcal{M}^e(X/S)$ for the marked minimal models $f' : X' \to S$ of $f : X \to S$ with markings $\alpha : X' - \to X$ is finite. In other words, the number of isomorphism classes of the minimal models of $f : X \to S$ is finite. Moreover, there exists a finite rational polyhedral cone Π' which is a fundamental domain for the action of $Bir(X/S)$ on $\mathcal{M}^e(X/S)$. □

A *marked minimal model* is a pair consisting of a minimal model and a marking birational map to a fixed model (Definition 1.4).

The *nef cone* $\bar{\mathcal{A}}(X/S)$ is known to be locally rational polyhedral inside the *big cone* $\mathcal{B}(X/S)$ ([K2], Theorem 1.9). In the case dim X = 3, we shall prove a similar statement for the *movable cone* $\bar{\mathcal{M}}(X/S)$: the decomposition of the movable cone into nef cones is locally finite inside the big cone (Theorem 2.6), and the *pseudo-effective cone* $\bar{\mathcal{B}}(X/S)$ itself is locally rational polyhedral away from $\bar{\mathcal{M}}(X/S)$ (Theorem 2.9).

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

It is already known that the above conjectures are true if $\dim X = \dim S = 3$ ([KM], Theorem 2.5). The main result of this paper is the proof of the first parts of the conjectures (1) and (2) in the case where $0 < \dim S < \dim X = 3$ (Main Theorem in §3). In particular, the number of minimal models in a fixed birational class of 3-folds is finite up to isomorphisms if the Kodaira dimension is positive (Corollary).

In the course of the proof, we shall use the $\mathbb{R}$-divisors in an essential way. In fact, $\mathbb{R}$-divisors are more suitable for the analysis of the infinity than the $\mathbb{Q}$-divisors.

It is necessary to consider the birational version (2) of the conjecture in order to carry out our proof for the biregular version (1) (cf. Lemmas 1.15 and 1.16).

The relative setting over the base space S is also essential in our inductive argument on $\dim S$ with fixed $\dim X$. This relative setting seems to correspond to the geometric situation where the size of the metric of the base space S goes to infinity.

1. Ample cone and movable cone

Definition 1.1. In this paper, $f : X \to S$ will always be a projective surjective morphism of normal varieties defined over $\mathbb{C}$ with geometrically connected fibers unless stated otherwise. A Cartier divisor D on X is said to be *f-nef* (resp. *f-movable*, *f-effective*, *f-big*) if $(D \cdot C) \geq 0$ holds for any curve C on X which is mapped to a point on S (resp. if $\dim \operatorname{Supp} \operatorname{Coker}(f^*f_*\mathcal{O}_X(D) \to \mathcal{O}_X(D)) \geq 2$, if $f_*\mathcal{O}_X(D) \neq 0$, if $\kappa(X_\eta, D_\eta) = \dim X - \dim S$ for the generic point $\eta \in S$). A linear combination of Cartier divisors with coefficients in $\mathbb{R}$ is called an *$\mathbb{R}$-Cartier divisor*. The real vector space

$$\begin{aligned} N^1(X/S) &= \{\text{Cartier divisor on } X\}/(\text{numerical equivalence over } S) \otimes_{\mathbb{Z}} \mathbb{R} \\ &= \{\mathbb{R}\text{-Cartier divisor on } X\}/(\text{numerical equivalence over } S) \end{aligned}$$

is finite dimensional. We set $\rho(X/S) = \dim N^1(X/S)$. The class of an $\mathbb{R}$-Cartier divisor D in $N^1(X/S)$ is denoted by $[D]$.

The *f-nef cone* $\bar{\mathcal{A}}(X/S)$ (resp. the *closed f-movable cone* $\bar{\mathcal{M}}(X/S)$, the *f-pseudo-effective cone* $\bar{\mathcal{B}}(X/S)$) is the closed convex cone in $N^1(X/S)$ generated by the numerical classes of f-nef divisors (resp. f-movable divisors, f-effective divisors). We have the following inclusions:

$$\bar{\mathcal{A}}(X/S) \subset \bar{\mathcal{M}}(X/S) \subset \bar{\mathcal{B}}(X/S) \subset N^1(X/S)$$

The interior $\mathcal{A}(X/S) \subset \bar{\mathcal{A}}(X/S)$ (resp. $\mathcal{B}(X/S) \subset \bar{\mathcal{B}}(X/S)$) is the open convex cone generated by the numerical classes of f-ample divisors (f-big divisors) and called an *f-ample cone* (resp. *f-big cone*). We do not know such a characterization for the interior of $\bar{\mathcal{M}}(X/S)$. We denote by $\mathcal{B}^e(X/S)$ the *f-effective cone*, the convex cone generated by f-effective Cartier divisors. We call $\mathcal{A}^e(X/S) = \bar{\mathcal{A}}(X/S) \cap \mathcal{B}^e(X/S)$ and $\mathcal{M}^e(X/S) = \bar{\mathcal{M}}(X/S) \cap \mathcal{B}^e(X/S)$ the *f-effective f-nef cone* and *f-effective f-movable cone*, respectively. By definition, we have $\mathcal{B}(X/S) \subset \mathcal{B}^e(X/S)$.

Remark 1.2. (1) The base space S can be a complex analytic space if we make a suitable modification. In this case, one needs an additional assumption which guarantees the finiteness of $\rho(X/S)$. For example, we consider S as a germ of a neighborhood of a compact subset $\mathcal{K} \subset S$ as in [K2].

(2) If the log abundance theorem for $\mathbb{R}$-divisors holds, e.g., if $\dim X = 3$ ([KeMM] and [Sho]), then $\mathcal{A}^e(X/S)$ and $\mathcal{M}^e(X/S)$ are generated by the classes of $\mathbb{Q}$-divisors as convex cones for a Calabi-Yau fiber space $f : X \to S$ (Proposition 2.4). But there may exist rational points in $\bar{\mathcal{M}}(X/S)$ which do not belong to $\mathcal{M}^e(X/S)$. (cf. Example 3.8 (2)).

(3) Even if $[D]$ and $[D']$ are f-effective and non-zero, we might have $[D+D'] = 0$. Unlike the case of $\bar{\mathcal{A}}(X/S)$, $\bar{\mathcal{M}}(X/S)$ may contain a linear subspace of $N^1(X/S)$.

Example 1.3. (1) If $f : X \to S$ is a birational morphism, then $\mathcal{B}(X/S) = N^1(X/S)$.

(2) If a generic fiber X_η of $f : X \to S$ is a curve, then the *degree* of a divisor D is defined by $\deg D = \deg D_\eta = (D \cdot F)$ for a general fiber F, and

$$\bar{\mathcal{B}}(X/S) = \{z \in N^1(X/S); \deg z \geq 0\}.$$

Definition 1.4. A *minimal model* of $f : X \to S$ (or of X over S) is a projective morphism $f' : X' \to S$ which satisfies the following conditions (cf. [KMM]):

(1) There exists a birational map $\alpha : X' - \to X$ such that $f' = f \circ \alpha$.
(2) X' has only $\mathbb{Q}$-factorial terminal singularities.
(3) $K_{X'}$ is f'-nef.

The pair (X', α) is called a *marked minimal model* with a marking α. If $f : X \to S$ is also minimal, then α is an isomorphism in codimension 1, and we obtain an isomorphism $\alpha_* : N^1(X'/S) \to N^1(X/S)$ such that $\alpha_*(\bar{\mathcal{M}}(X'/S)) = \bar{\mathcal{M}}(X/S)$ and $\alpha_*(\bar{\mathcal{B}}(X'/S)) = \bar{\mathcal{B}}(X/S)$. We denote $\alpha_*(\bar{\mathcal{A}}(X'/S)) = \bar{\mathcal{A}}(X'/S, \alpha)$. We sometimes write $\bar{\mathcal{A}}(X'/S)$ instead of $\bar{\mathcal{A}}(X'/S, \alpha)$ if there is no danger of confusion. Two marked minimal models (X_i, α_i) $(i = 1, 2)$ are said to be *isomorphic* if there exists an isomorphism $\beta : X_1 \to X_2$ such that $\alpha_1 = \alpha_2 \circ \beta$.

Let $f : X \to S$ be a minimal model. We denote by $\mathrm{Aut}(X/S)$ (resp. $\mathrm{Bir}(X/S)$) the group of biregular (resp. birational) automorphisms of X over S. Any $\theta \in \mathrm{Bir}(X/S)$ doen not contract any divisor on X, since K_X is f-nef and X has only terminal singularities. Thus there is a linear representation

$$\sigma : \mathrm{Bir}(X/S) \to GL(N^1(X/S), \mathbb{Z})$$

given by $\sigma(\theta)([D]) = \theta_*([D])$.

Lemma 1.5. *Let (X_i, α_i) $(i = 1, 2)$ be marked minimal models of a minimal $f : X_0 \to S$. Then the following conditions are equivalent:*

(1) (X_1, α_1) and (X_2, α_2) are isomorphic.
(2) $\mathcal{A}(X_1/S, \alpha_1) = \mathcal{A}(X_2/S, \alpha_2)$ in $N^1(X/S)$.
(3) $\mathcal{A}(X_1/S, \alpha_1) \cap \mathcal{A}(X_2/S, \alpha_2) \neq \emptyset$ in $N^1(X/S)$. □

Corollary 1.6. *There is a 1-1 correspondence between the orbit space*

$$Bir(X_0/S)/Aut(X_0/S)$$

and the set of isomorphism classes of the marked minimal models (X, α) of X_0 over S such that X is isomorphic to X_0 over S. □

Definition 1.7. $f : X \to S$ is said to be a *Calabi-Yau fiber space* if X has only $\mathbb{Q}$-factorial terminal singularities and $[K_X] = 0$ in $N^1(X/S)$. This concept is more general than the usual Calabi-Yau manifold in the following points: (1) there is no assumption on the fundamental group nor the irregularity of the generic fiber, (2) X may be mildly singular, (3) we consider relatively over the base space S. For example, if dim X = dim S (resp. = dim $S + 1$), then f is a *crepant resolution* of singularities (resp. an elliptic fibration). We note that $h^1(\mathcal{O}_{X_\eta})$ may be non-zero even if $h^1(\mathcal{O}_X) = 0$. Any minimal model which satisfies the abundance theorem yields a Calabi-Yau fiber space (cf. [KMM]). The point is that we can treat these cases in a unified way.

Definition 1.8. Let $f : X \to S$ be a Calabi-Yau fiber space, and $X \xrightarrow{g} T \xrightarrow{h} S$ a factorization such that g is also a Calabi-Yau fiber space and h is not an isomorphism. Then $g^* : N^1(T/S) \to N^1(X/S)$ is injective, and $g^*\bar{\mathcal{A}}(T/S) = g^*N^1(T/S) \cap \bar{\mathcal{A}}(X/S)$ is a face of $\bar{\mathcal{A}}(X/S)$. There are 2 cases:

(1) g is a birational morphism. In this case, it is called a *birational contraction*. We have $\rho(X/T) + \rho(T/S) = \rho(X/S)$ ([KMM]). If $\rho(X/T) = 1$, then it is called *elementary* or *primitive*.

(2) dim $X >$ dim T. In this case, g is called a *fiber space structure* (cf. [O]).

Let D be an f-effective but not f-nef $\mathbb{R}$-divisor. If ϵ is a sufficiently small positive number, then the pair $(X, \epsilon D)$ is log terminal, and there exists an extremal ray R for this pair ([KMM]). Let $\phi : X \to Y$ be a contraction morphism over S associated to R. Since K_X is f-nef, ϕ is a primitive birational contraction morphism. It is called a *divisorial contraction* or a *small contraction* if the exceptional locus of ϕ is a prime divisor or not, respectively. In the latter case, the log flip of ϕ is called a *D-flop*.

In this paper, a prime divisor E on X is said to be *f-exceptional* if there exists a minimal model $f' : X' \to S$ of f and a divisorial contraction $\phi : X' \to Y$ over S whose exceptional divisor is the strict transform of E.

As a consequence of the cone theorem ([KMM]), we obtain

Theorem 1.9. *([K2, Theorem 5.7]) Let $f : X \to S$ be a Calabi-Yau fiber space. Then the cone*

$$\bar{\mathcal{A}}(X/S) \cap \mathcal{B}(X/S) = \mathcal{A}^e(X/S) \cap \mathcal{B}(X/S)$$

is locally rational polyhedral inside the open cone $\mathcal{B}(X/S)$. Moreover, any face F of this cone corresponds to a birational contraction $\phi : X \to Y$ over S by the equality $F = \phi^*(\bar{\mathcal{A}}(Y/S) \cap \mathcal{B}(Y/S))$. □

The following is an easy generalization of the characterization of nef and big divisors in [K1, Lemma 3] to $\mathbb{R}$-divisors:

Proposition 1.10. *Let $f : X \to S$ be a proper morphism of normal varieties. Then*

$$\bar{\mathcal{A}}(X/S) \cap \mathcal{B}(X/S) = \{z \in \bar{\mathcal{A}}(X/S); z_\eta^n > 0\}$$

where n is the dimension of the generic fiber X_η of f. □

The following is in [W] for the case of Calabi-Yau 3-folds:

Corollary 1.11. *Let $f : X \to S$ be a Calabi-Yau fiber space, and let $\mathcal{W} = \{z \in N^1(X/S); z_\eta^n > 0\}$. Then the cone $\bar{\mathcal{A}}(X/S) \cap \mathcal{W}$ is locally rational polyhedral inside the cone $\mathcal{W}$.* □

Remark 1.13. (1) Conjecture 1.12 were inspired by the mirror symmetry conjecture of Calabi-Yau threefolds. Some positive evidences are given in [B], [GM] and [OP] for (1) and [Nm1] for (2).

(2) With respect to our relative formulation over the base space S, the variety X can be an arbitrary minimal model which satisfies the abundance theorem ([KMM]), if we take S to be the canonical model $\mathrm{Proj}(\bigoplus_{m=0}^{\infty} H^0(X, mK_X))$.

(3) If we replace the ample cone $\mathcal{A}(X)$ by the Kähler cone $\mathcal{K}(X)$, then the conjecture is clearly false.

(4) The finiteness questions such as the finite generation of the canonical ring, the termination of flips, the finiteness of the cones, the boundedness of the moduli space and the Zariski decomposition, seem to be mutually related (cf. [A], [G]).

Example 1.14. (1) Let X be an abelian variety. Then we have

$$\bar{\mathcal{A}}(X) = \bar{\mathcal{B}}(X) = \{z \in N^1(X); z^n \geq 0\}^o$$

where ^{o} denotes an irreducible component of the cone, since X does not contain a rational curve, and there is no divisorial contraction nor flop of X.

Although the shape of this cone is quite different from a finite rational polyhedral cone, the conjecture seems to be true in this case, too. One checks it by an explicit calculation in the case where $X \cong E \times \cdots \times E$ for an elliptic curve E without complex multiplications (Corollary 2.11). A related result is in [NN].

(2) Let X be a K3 surface with an ample class h, and Σ the set of all the (-2)-curves on X. Then

$$\bar{\mathcal{A}}(X) = \{z \in N^1(X); z^2 \geq 0, z \cdot h \geq 0, z \cdot C \geq 0 \quad \forall C \in \Sigma\}$$

and $\bar{\mathcal{B}}(X)$ is the closed convex cone generated by the cone $\{z \in N^1(X); z^2 \geq 0, z \cdot h \geq 0\}$ and the $C \in \Sigma$. This duality between $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ will be generalized in Theorem 2.9. In this case, the conjecture is verified in [St] (see Theorem 2.1). See also [Kov].

Our strategy is to analyse the birational automorphism group first and try to prove Conjecture (2), and then consider the biregular automorphism group toward Conjecture (1).

Lemma 1.15. *Let $f : X \to S$ be a Calabi-Yau fiber space. Assume that the number of faces of $\mathcal{A}^e(X/S)$ which correspond to primitive birational contractions is finite up to the action of $Bir(X/S)$. Then it is also finite up to the action of $Aut(X/S)$.* □

Lemma 1.16. *Let $f : X \to S$ be a Calabi-Yau fiber space. Assume that the number of faces of $\mathcal{A}^e(X/S)$ which correspond to fiber space structues is finite up to the action of $Bir(X/S)$. In addition, assume that the first part of Conjecture 1.12 (2) is true for any Calabi-Yau fiber space which factors f non-trivially. Then it is also finite up to the action of $Aut(X/S)$.* □

2. General results for dimension 2 or 3.

Theorem 2.1. *Let $f : X \to S$ be a Calabi-Yau fiber space such that dim $X = 2$. Then Conjecture 1.12 is true.* □

Remark 2.2. (1) The global Torelli theorem proved in [PSS] guarantees the existence of sufficiently large automorphism group, and is the key point in the proof.

(2) The above theorem is also valid over any field k of characteristic 0.

Theorem 2.3. *(cf. [K2, p.120]). Let $f_0 : X_0 \to S$ be a Calabi-Yau fiber space with dim $X_0 = 3$, and D an $\mathbb{R}$-divisor such that $[D] \in \mathcal{M}^e(X_0/S)$. Then there exists a sequence of D-flops such that the strict transform of D becomes relatively nef over S. Therefore,*

$$\mathcal{M}^e(X_0/S) = \bigcup_{(X,\alpha)} \mathcal{A}^e(X/S,\alpha)$$

where the union on the right hand side is taken for all the marked minimal models (X,α) of X_0 over S. □

Proposition 2.4. *Let $f : X \to S$ be a Calabi-Yau fiber space such that dim $X = 3$. Then the cones $\mathcal{A}^e(X/S)$ and $\mathcal{M}^e(X/S)$ are generated by the numerical classes of $\mathbb{Q}$-Cartier divisors as convex cones.* □

The following gives a positive answer to Conjecture 1.12 in a special case, where we note that $\mathrm{Bir}(X/S) = \{\mathrm{id}\}$:

Theorem 2.5. *([KM]). Let S be a normal 3-fold, and $f : X \to S$ a minimal resolution. Then $\bar{A}(X/S)$ is a finite polyhedral cone, and there exists only finitely many marked minimal models of f. In other words, Conjecture 1.12 is true if dim X = dim $S = 3$.* □

The following is a generalization of the above theorem:

Theorem 2.6. *Let $f_0 : X_0 \to S$ be a Calabi-Yau fiber space such that dim $X_0 = 3$. Then the decomposition*

$$\mathcal{M}^e(X_0/S) \cap \mathcal{B}(X_0/S) = \bigcup_{(X,\alpha)} \mathcal{A}^e(X/S,\alpha) \cap \mathcal{B}(X_0/S)$$

is locally finite inside the open cone $\mathcal{B}(X_0/S)$ in the following sense: if Σ is a closed convex cone contained in $\mathcal{B}(X_0/S) \cup \{0\}$, then there exist only a finite number of cones $\mathcal{A}^e(X/S,\alpha) \cap \mathcal{B}(X_0/S)$ which intersect Σ. □

The accumulation occurrs only toward the boundary $\partial\bar{\mathcal{B}}(X/S)$:

Corollary 2.7. *Let $f : X \to S$ be a Calabi-Yau fiber space such that dim $X = 3$. Then the cone $\bar{\mathcal{M}}(X/S) \cap \mathcal{B}(X/S)$ is locally rational polyhedral inside the open cone $\mathcal{B}(X/S)$. Moreover, the faces of this cone correspond to divisorial contractions of some marked minimal models.* □

Remark 2.8. One f-exceptional divisor may correspond to several faces of $\bar{\mathcal{M}}(X/S)$.

Theorem 2.9. *Let $f : X \to S$ be a Calabi-Yau fiber space such that dim $X = 3$. Then the cone $\bar{\mathcal{B}}(X/S)$ is locally rational polyhedral inside the open cone*

$$N^1(X/S) \setminus (\bar{\mathcal{M}}(X/S) \cap \partial\bar{\mathcal{B}}(X/S)).$$

Moreover, it is generated by $\bar{\mathcal{M}}(X/S)$ and the numerical classes of the f-exceptional divisors. □

We have the following positive evidence for Conjecture 1.12 in the case where X is a direct product of an elliptic curve without complex multiplications. By the mirror symmetry, its complexified Kähler cone $\mathbb{R}^{\frac{1}{2}n(n+1)} \times \sqrt{-1}\mathcal{A}(X)$ should be isomorphic to the moduli space of marked principally polarized abelian varieties under the mirror map, as is proved in the following proposition. An abelian variety which is isogenous to X may correspond to a non-principal polarization.

Proposition 2.10. *Let $X = E \times \cdots \times E$ (n-times) for an elliptic curve E without complex multiplications. Then $Aut(X) = Bir(X)$, $\rho(X) = \frac{1}{2}n(n+1)$, and*

$$Im(\sigma : Aut(X) \to GL(N^1(X), \mathbb{Z})) \cong GL(n, \mathbb{Z}).$$

Moreover, there is a linear isomorphism $\tau : N^1(X) \to S(n, \mathbb{R})$ to the real vector space of symmetric (n,n)-matrices which sends $\mathcal{A}(X)$ to the cone of positive definite matrices and which is compatible with the natural $GL(n, \mathbb{Z})$-actions. □

Corollary 2.11. *Conjecture 1.12 is true for $X = E \times \cdots \times E$ for an elliptic curve E without complex multiplications.* □

We make a remark on the behaviour of the cones of divisors under deformations extending [W] and [Nm2]. This result is not used in the rest of this paper.

Proposition 2.12. *Let X be a Calabi-Yau fiber space over a point such that dim $X = 3$ an $h^2(\mathcal{O}_X) = 0$. Let $\pi : \mathcal{X} \to B$ be a flat family of deformations of $X = X_0$ over a germ $(B, 0)$. Then there exist at most countably many proper closed analytic subsets C_λ of B, which may contain 0, such that $\bar{\mathcal{A}}(X_t)$, $\bar{\mathcal{M}}(X_t)$ and $\bar{\mathcal{B}}(X_t)$ are constant in $N^1(X) \cong N^1(X_t)$ for $t \in B \setminus \bigcup_\lambda C_\lambda$.* □

3. Main results

Main Theorem. *Let $f : X \to S$ be a Calabi-Yau fiber space such that dim $X = 3$ and dim $S = 2$ or $= 1$. Then there exist only finitely many chambers for the marked minimal models of f and finitely many faces of them up to the action of $Bir(X/S)$, hence the first parts of Conjecture (1) and (2) are true.* □

Corollary. *Let X be an algebraic variety of dimension 3 whose Kodaira dimension $\kappa(X)$ is positive. Then there exist only finitely many minimal models of X up to isomorphisms.* □

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