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Yukawa couplings of Certain Calabi-Yau 3-folds

by

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1 Mirror Symmetry Conjecture for Calabi-Yau 3-folds

Mirror symmetry appeared in the supersymmetric string theory where a Calabi-Yau 3-fold M plays a role as a background for string propagation. It said that a "mirror pair" of Calabi-Yau 3-folds (X, Y) apparently produce isomorphic physical theories. (See [G-P], [Mo2]).
Mathematically, mirror symmetry related certain geometric invariants of a Calabi-Yau 3-fold to a completely different set of geometric invariants of the mirror partner. Mathematician had not found such a symmetry before physisits' predections and calculations. (For more historical backgrounds of mirror symmetry, please consult [Mo2] and references therein). The two models are called A-model and B-model respectively. A-model on a Calabi-Yau 3-fold X has a correlation function related to the so-called Gromov-Witten invariants of a Calabi-Yau 3-folds, which is essentially related to the number of holomorphic rational curves on X. On the other hand, B-model correlation functions on a Calai-Yau 3-fold Y is given by period integrals of holomorphic forms.
A 3-dimensional complex projective manifold X is called a Calabi-Yau 3-fold if $K_X \simeq \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$. For a compact Kähler manifold Y, we set $H^{pq}(Y) = H^q(Y, \Omega^p_Y)$ and $h^{pq}(Y) = \dim_{\mathbf{C}} H^{pq}(Y)$.
Let X be a Calabi-Yau 3-fold and consider the following classical cubic form in A-model:

$$I^{1,1} : H^{1,1}(X) \times H^{1,1}(X) \times H^{1,1}(X) \longrightarrow \mathbf{C}$$

defined by usual cup product

$$I^{1,1}(L_1, L_2, L_3) = \int_X L_1 \wedge L_2 \wedge L_3.$$

Let Y be another Calabi-Yau 3-fold and fix a nowhere vanishng holomorphic 3-form ω and define a classical cubic form in B-model

$$I_\omega^{2,1} : H^{2,1}(Y) \times H^{2,1}(Y) \times H^{2,1}(Y) \longrightarrow \mathbf{C}$$

by

$$I_\omega^{2,1}(\theta_1, \theta_2, \theta_3) = \int_Y \omega \wedge \nabla_{\theta_1} \nabla_{\theta_2} \nabla_{\theta_3} \omega.$$

Here ∇ is the Gauss-Manin connection associated to the Hodge structure on $H^3(Y, \mathbf{C})$.

In A-model side, the quantum deformation of the classical cubic form can be given by

$$I_Q^{1,1} : H^{1,1}(X) \times H^{1,1}(X) \times H^{1,1}(X) \longrightarrow \mathbf{C},$$

$$\begin{aligned} I_Q^{1,1}(L_1, L_2, L_3) &= \textstyle\int_Y L_1 \wedge L_2 \wedge L_3 \\ &\quad + \textstyle\sum_{0 \neq \eta \in H_2(Y,\mathbf{Z})} \Phi_\eta(L_1, L_2, L_3) \frac{q^\eta}{1-q^\eta} \end{aligned}$$

This cubic form is called A-model correlation functions in [Mo2]. Here $\Phi_\eta(L_1, L_2, L_3)$ denote Gromov-Witten invariants. The definitions of Gromov-Witten invariants and the symbol q^η are given in lecture 3 and 4 in [Mo2]. An axiomatic treatment of Gromov-Witten invariants are in [K-M] and the construction of Gromov-Witten invariants using symplectic geometry and psuedo holomrphic curve is given by Ruan [R]. The notion of stable maps due to Kontsevich ([K-M], [Kon]) are used for another definition of Gromov-Witten invariants. For detail, readers may refer to [Kon], [Fu-Pa] and [B-M]. Givental recentely has proved that for Calabi-Yau complete intersections in toric varieties, the predicted enumerative formulas which one calculates by using Batyrev-Borisov candidate mirror partner are in fact correct evaluations of the Gromov-Witten invariants. ([Giv1], [Giv2], [Giv3])). His proof involves stationary phase integrals, equivariant Gromov-Witten invariant and quantum Toda lattices.

In B-model side, the quantum deformation of $I_\omega^{2,1}$ is defined to be

$$I_{Q,\omega}^{2,1} = I_\omega^{2,1},$$

that is, we keep $I_\omega^{2,1}$ not deformed. From the view point of Mirror symmetry, the asymptotic behavior of the B-model correlation function $I_\omega^{2,1}$ is very important when the complex structure of Y tends to the large complex structure limit. Readers may consult lecture 6 and lecture 7 in [Mo2].

2 A review on Mordell-Weil groups

In this section, we recall the Mordell-Weil group of abelian scheme over rational function field of a complex projective curve. Let C be a smooth projective curve defined over $\mathbf{C}$, and $K = \mathbf{C}(C)$ the field of rational functions on C. Let A/K denote an abelian variety defined over K. Then the theory of Néron model says that there exist group scheme

$$\pi : \mathcal{A}^0 \longrightarrow C$$

whose generic fiber $\mathcal{A}^0_\eta$ is isomorphic to A over K. *The Mordell-Weil group* of A is defined to be
the group of K-rational points of A

$$MW(A/K) = A(K).$$

The Néron universal property ensures that

$$MW(A/K) = \{\sigma : C \longrightarrow \mathcal{A}^0, \quad \text{regular section of } \pi\}$$

In general, the group $MW(A/K)$ is not finitely generated, for A may have nontrivial $K/\mathbf{C}$-trace B. However, a theorem of Lang implies that the Mordell-Weil group of the quotient abelian variety A/B_K is finitely generated.
In [Man1], Manin construct a height pairing on the Mordell-Weil group $MW(A/K)$. In order to construct the height, he needed to suppose that $\mathcal{A}^0 \pi C$ has a smooth relative compactification

$$\begin{array}{ccc} \mathcal{A}^0 & \hookrightarrow & \mathcal{A} \\ \pi \searrow & & \swarrow h \\ & C & \end{array}$$

such that $\mathcal{A}$ is smooth projective.
The group $D_{\mathbf{C}}(\mathcal{A})$ of all divisors defined over $\mathbf{C}$ on $\mathcal{A}$ splits into the direct sum of the two subgroups

$$D_{\mathbf{C}}(\mathcal{A}) = D^C \oplus D^A$$

where D^C is generated by the irreducible components of the fibers of h and D^A generated by irreducible divisors which maps onto C.
Moreover we suppose that:

1. all translation automorphisms of $\mathcal{A}^0/C$ extend to biregular automorphisms of $\mathcal{A}/C$, and

2. the map $\gamma : D_{\mathbf{C}}(\mathcal{A}) \rightarrow D_K(A)$ which maps each $\mathbf{C}$-divisors of $\mathcal{A}$ into the divisors induced by it on the genereic fiber A is null on D^A and defines an A-isomorphism beween D^A and $D_K(A)$. Further for a rational function $f \in K(A)$ we have

$$\gamma((f)_{\mathcal{A}}) = (f)_A,$$

where $(f)_{\mathcal{A}}$ (resp. $(f)_A$) is a principal divisor on $\mathcal{A}$ (resp. on A).

Let $X \in D_K(A)$, then the Tate height on $MW(A/K)$ relative to a divisor X is defined in [Man1]

$$\hat{h}_X : MW(A/K) \longrightarrow \mathbf{R},$$

and moreover if X is symmetric, that is, $X^- = X$, where X^- is the image of the inversion map on A, $\hat{h}_X$ is really quadratic. The following theorem is due to Manin ([Man1], Theorem 4.).

Theorem 2.1 *Let $D = D^C$/principal divisors on C. On the subgroup $MW^1(A/K)$ which acts trivially on D, the Tate height may be computed by the formula*

$$\hat{h}_X(\sigma) = (\gamma^{-1}(X), \sigma(C)) - (\gamma^{-1}(X), 0(C)).$$

Here the pairings (,) in the right hand side denote the intersection pairing of divisors and curves and $\sigma \in MW(A/K)$ is considered as a section $\sigma : C \to \mathcal{A}$. (We also denote by 0 the zero element of $MW(A/K)$.)

Corollary 2.1 *If $h : \mathcal{A} \longrightarrow C$ has only irreducible fibers, and $X \in D_K(A)$ is symmetric and $(\gamma^{-1}(X), 0(X)) = 0$, the Tate height is quadratic integral-valued and given by*

$$\hat{h}_X(\sigma) = (\gamma^{-1}(X), \sigma(C)) = (L, C).$$

where L is a line bundle on $\mathcal{A}$ algebraically eqivalent to $\gamma^{-1}(X)$.

Assume that $MW(A/K)$ is finitely genrated and torsion-free. Under the assumption of corollary, the associated symmetric bilinear form

$$<,>: MW(A/C) \times MW(A/K) \longrightarrow \mathbf{Z}$$

gives a natural lattice sturucture on $MW(A/K)$. In case when A/K is an elliptic curve, one has a good minimal model $h : \mathcal{A} \longrightarrow C$ by using minimal model theory of projective surfaces.[1].

Shioda [Sh1], [Sh3] actually showed that this lattice structure can be calculated by using the intersection theory of the surface $\mathcal{A}$, and he called the lattice *Mordell-Weil lattice.*

Later, he extended his results to the case where A is the Jacobian variety of a higher genus curve T over K ([Sh2]).

For example, if $h : \mathcal{A} \longrightarrow \mathbf{P}^1$ is a minimal rational elliptic surface with only irreducible fibers, one has isometry

$$MW(A/K) \simeq E_8$$

where E_8 is the famous even unimodular lattice of rank 8.

[1]Actually, in this case, one does not need the assumption on the irreduciblity of fibers and other condition in the corollary

3 Calabi-Yau 3-folds with abelian fibration

In this section, we shall give two examples of Calabi-Yau 3-folds with fibrations of abelian surfaces.
First of all, we make the following

Definition 3.1 *A fibration of curves $f : S \longrightarrow C$ from a smooth projective surface S to a curve C is called a Lefschetz fibration if all cloed fibers has at most one node as its singularity.*

Example I (Example of C. Schoen [Sch].)
Let $f_i : S_i \longrightarrow \mathbf{P}^1$ $(i = 1, 2)$ is minimal rational elliptic surfaces with sections and assume that both of f_i are Lefschetz fibrations. Moreover assume that the sets of critical values of f_i has no common elements. Then consider the fiber product $W = S_1 \times_{\mathbf{P}^1} S_2$.

$$\begin{array}{ccccc} & & W & & \\ & p_1 \swarrow & & \searrow p_2 & \\ S_1 & & & & S_2 \\ & f_1 \searrow & & \swarrow f_2 & \\ & & \mathbf{P}^1 & & \end{array} .$$

Then it is easy to see that W is a nonsingular Calabi-Yau threefold and induced fibration $h : W \longrightarrow BP^1$ is a fibration of the product of two elliptic curves. Moreover we fix zero sections 0_1 and 0_2 of S_1 and S_2 respectively and let F_i denote the class of general fiber of f_i, then set

$$L_i = 0_i(\mathbf{P}^1) + F_i.$$

It is easy to see that the Tate height of each section $\sigma \in MW(S_i/\mathbf{P}^1)$ is given by

$$< \sigma, \sigma >= (L_i, \sigma(\mathbf{P}^1)),$$

and the Mordell-Weil lattices are isometric to E_8. Then one can easily prove

Proposition 3.1 *The Mordell-Weil lattice structure of $MW(W/\mathbf{P}^1)$ with respect to line bundle $L = p_1^*(L_1) + p_2^*(L_2)$ is isometric to*

$$(MW(W/\mathbf{P}^1), <, >) \simeq E_8 \oplus E_8$$

The Hodge diamond of W is given by

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 19 & & 0 & \\ 1 & & 19 & & 19 & & 1 \\ & 0 & & 19 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Example II([Sa-Sak].)
We recall a construction of genus g fibrations in [Sa-Sak]. For $\Sigma_0 = \mathbf{P}^1 \times \mathbf{P}^1$, we let $p_i : \Sigma_0 \longrightarrow \mathbf{P}^1$ $(i = 1,2)$ be the natural projections onto i-th factor and $F_i = p_i^*(apoint)$ the divisor class of a fiber of p_i. A curve B in Σ_0 is said to be of bidegree (a, b) if B is linear equivalent to $aF_1 + bF_2$.
Let B be a smooth curve in Σ_0 of bidegree $(2, 2g+2)$ and let $\pi : X \longrightarrow \Sigma_0$ be the double covering whose branch locus is B. Set $f = \pi p_1$ and $\varphi = \pi p_2$. Then we have the following two fibrations:

$$\begin{array}{ccc} & X & \\ f \swarrow & & \searrow \varphi \\ \mathbf{P}^1 & & \mathbf{P}^1 \end{array} .$$

Note that f is a fibration of (hyperelliptic) curves of genus g and φ is a fibration of conics, and hence X is a rational surface. Let $K = \mathbf{C}(\mathbf{P}^1)$ be the rational function field and considering the generic fiber X_η of f as a curve of genus g over K, we define the Jacobian variety of X_η

$$J = Jac(X_\eta).$$

¿From now on we restrict our attention to the case of $g = 2$. Then the Néron model of J exists

$$\mathcal{J}^0 \longrightarrow \mathbf{P}^1.$$

Theorem 3.1 *([Sa3], [Sa4]) Assume that $g = 2$ and $f : X \longrightarrow \mathbf{P}^1$ is a Lefschetz fibration. Then we have the following assertions:*

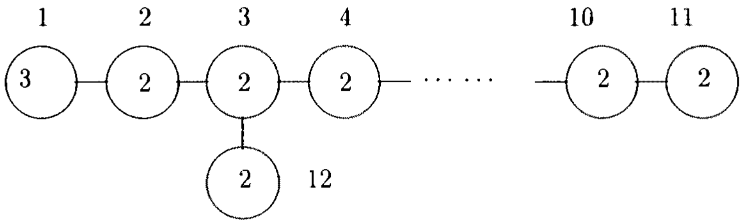
1. *$\mathcal{J}^0 \longrightarrow \mathbf{P}^1$has a good smooth compactification $h : \mathcal{J} \longrightarrow \mathbf{P}^1$ whose total space is a Calabi-Yau 3-fold.*

2. *There exists a natural embedding of $X \hookrightarrow \mathcal{J}$.*

3. *With respect the line bundle $L = X + X^- + 2F$ where F is a class of fiber of h, the Mordell-Weil lattic structure on $MW(J/K)$ is isometric to*

$$(MW(J/K), <, >_L) = D_{12}^+$$

Here D_{12}^+ is a unimodular lattice whose dynkin diagram is given by



The existence of good smooth compactification in the Lefschetz fibration case is due to Nakamura [N]. The Hodge diamond of $\mathcal{J}$ is given by the following:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 14 & & 0 & \\ 1 & & 14 & & 14 & & 1 \\ & 0 & & 14 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

4 Theta functions of lattices and Yukawa coupling

In examples in §3, we know the structure of Mordell-Weil lattices. By virtue of result in Cor. 2.1, this let us know the number of rational curves *arising from section* whose intersection number with respect to special line bundles are fixed by the theta function of lattices.

Let us start with lattice theta functions. For any positive integral lattice L, we let

$$\theta_L(z) = \sum_{\eta\in L} q^{<\eta,\eta>}$$

be the theta function of the lattice L where $q = \exp(\pi i z)$. We have

$$\theta_L(z) = \sum_{m=1}^{\infty} N_L(m) q^m.$$

where

$$N_L(m) = \#\{\eta \in L| < \eta, \eta >= m\}.$$

Let us recall the Jacobi theta functions:

$$\theta_2(z) = 2q^{1/4}\prod_{m=1}^{\infty}(1-q^{2m})(1+q^{2m})^2,$$

$$\theta_3(z) = \prod_{m=1}^{\infty}(1-q^{2m})(1+q^{2m-1})^2,$$

$$\theta_4(z) = \prod_{m=1}^{\infty}(1-q^{2m})(1-q^{2m-1})^2.$$

Then the theta function of D_{12}^{+} is given by (see 7.3, Ch. 4 in [C-S]):

$$\theta_{D_{12}^{+}}(z) = 1/2(\theta_2^{12}(z) + \theta_3^{12}(z) + \theta_4^{12}(z))$$

On the other hand, the theta function of E_8 is Eisenstein series $E_2(z)$, while

$$\theta_{E_8 \oplus E_8}(z) = (E_2(z))^2 = E_4(z).$$

By using the expansion of these Theta function, we know the number of rational curves arising from section of fixed degree (= height) with respet to the specific line bundle.
The further calculation of Gromov-Witten invariants needs more effort, however the author expects that there should be good differential equation which was satisfied by the correlation functions associated the Calabi-Yau 3-folds in §3.

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