代数幾何学シンポジウム記録
1996年度 pp.79-102

COMPACTIFICATION OF THE MODULI OF ABELIAN VARIETIES

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1996 December 13

0. Introduction.

0.1. We aim at compactifying canonically the moduli of abelian varieties in a way similar to the Mumford-Deligne compactification by stable curves of the moduli of curves, against general belief since Mumford that there exists no canonical (unique in some sense) compactification of the moduli of abelian varieties.

Our idea dates back to over twenty years ago, when the works of [Namikawa76], [Nakamura75] and Ueno [unpublished] pursued the same idea as now through construction of certain kinds of degenerating families of abelian varieties. We may be allowed to mention or emphasize that before knowing Mumford' idea we (Namikawa and the author) had started our consideration and had obtained the primitive idea of stable quasi-abelian varieties through analytic Néron models, and canonical embedding of abelian varieties by the theta functions, though the final formulation of the construction followed Mumford's method.

Very recently Alexeev and the author [AN96] retook up the problem, defined "stable quasi-abelian varieties" over any field and proved a stable reduction theorem as a first step towards compactification of the moduli over $\mathbf{Z}$. In this collaboration [AN96], first, we discussed the problem over any discrete valuation ring possibly in mixed characteristics, and second, we solved some problems in arbitrary dimension which have been left unsolved in dimension greater than four because of the difficulty arising from certain polyhedral decomposition called Delaunay decomposition.

1991 *Mathematics Subject Classification.* Algebraic Geometry, Primary 14J45; Secondary 14E35, 14J30.

Key words and phrases. Abelian variety, Compactification of moduli, Delaunay decomposition, Heisenberg group, Moduli, Stable quasi-abelian variety, Stability, Torus embedding.

Research was supported in part by the Grant-in-aid (No. 08211201, No. 08404001) for Scientific Research, the Ministry of Education

Later [Alexeev96] proved existence of the coarse moduli scheme projective over $\mathbf{Z}$ through a rather complicated definition of the functor.

In the present article we report the Hilbert-stability (Theorem 0.5) of stable quasi abelian varieties in some limited cases. This approach also will lead us to a proof of existence of coarse or fine moduli scheme of stable quasi-abelian varieties (Theorem 0.6).

Let us recall the stable reduction theorem. For comparison we start with the classical stable reduction theorem of (Néron and) Grothendieck in the semi abelian version, hence not in the projective but in the quasi-projective version, which has been proved in [SGA7]. See a résumé by Deligne [SGA7, Expóse I.pp. 1-24] for a short proof of it.

Theorem 0.2. *Let R be a complete discrete valuation ring with the fraction field K and $(G_K, \mathcal{L}_K)$ be a polarised abelian variety over K. Then after a suitable finite ramified cover* Spec $R' \to$ Spec R *it can be extended to a semi abelian group scheme G over R'. Namely there exists a polarised group scheme $(G, \mathcal{L})$ such that $(G, \mathcal{L}) \otimes K' = (G_K, \mathcal{L}_K) \otimes K'$ and the special fiber G_0 is connected and an extension of an abelian scheme by a (split) torus over the residue field R'/I', where I' is the maximal ideal of R'.*

We avoided the notion of cubical invertible sheaves in the above theorem for simplifying the statement. See for instance [MB85, p. 40, 1.1. (ii)].

We now recall the stable reduction theorem [AN96, Theorem 0.1] of abelian varieties in the projective version, not in the semi-abelian version.

Theorem 0.3. *Let R be a complete discrete valuation ring with the fraction field K and $(G_K, \mathcal{L}_K)$ be a polarised abelian variety over K. Then after a suitable finite ramified cover* Spec $R' \to$ Spec R *it can be completed in a canonical way to a flat projective scheme $(P, \mathcal{L})$ over R' with an ample invertible sheaf $\mathcal{L}$ extending $\mathcal{L}_K \otimes K'$.*

Let $(P_0, \mathcal{L}_0)$ be a special fiber of the family $(P, \mathcal{L})$. We call the polarised variety $(P_0, \mathcal{L}_0)$ *a polarised stable quasi-abelian variety* (abbr. SQAV) over the residue field k of R. Although the statement of Theorem 0.3 is somewhat vague at this moment, the object $(P_0, \mathcal{L}_0)$ we obtained is very concrete. This is a "very" canonical limit of a polarised abelian variety. The reason why we call it "very" canonical is intuitively that it is a geometric realisation of limits of canonically chosen theta functions degenerating moderately, or I would say that they are singular varieties which are the closest to a nonsingular abelian variety among degenerate abelian varieties [Namikawa76],[Nakamura75].

Theorem 0.4. [AN96] *Any stable quasi-abelian variety over a field k is*

(1) *a connected, reduced, Gorenstein,*

(2) *(possibly) reducible singular projective variety*
(3) *with trivial dualising sheaf,*
(4) *whose structures of irreducible components and geometric configuration of irreducible components are given by a so-called Delaunay decomposition.*
(5) *The invertible sheaf $\mathcal{L}_0$ is ample, indeed $\mathcal{L}_0^N$ is very ample for $N \geq g+2$.*
(6) *For $N > 0$, $h^0(P_0, \mathcal{L}_0^N) = h^0(P_K, \mathcal{L}_K^N) = \deg(\mathcal{L}_K)N^g$.*
(7) *For $N > 0$, $h^i(P_0, \mathcal{L}_0^N) = 0$ $(i > 0)$.*

[Nakamura96] proved, in some limited cases, in particular, in all cases of dimension $g \leq 4$ that the Hilbert points of the stable quasi-abelian variety $(P_0, \mathcal{L}_0)$ are stable (but not necessarily properly-stable) in the sense of Mumford [MFK94]. *In fact, we need only to apply the result of* [Kempf78]. Though the result of the theorem is unfortunately partial, the absolute majority of stable quasi abelian varieties (over ninety percent of the population, I guess) satisfy the condition of the theorem by taking an étale cover. See Theorem 7.2 for the precise statement.

Theorem 0.5. *Let $(P_0, \mathcal{L}_0)$ be a polarised stable quasi-abelian variety over an algebraically closed field k. Then the Hilbert points of $(P_0, \mathcal{L}_0)$ are stable if the characteristic of k and* $\deg \mathcal{L}_0$ *are coprime and if $\mathcal{L}_0$ is very ample.*

As a consequence of stability we prove

Theorem 0.6. *Let k be an algebraically closed field of any characteristic. Let K be a finite abelian group, any of whose elementary divisors is at least three in the strong sense*[1] *and whose order is coprime to the characteristic of k. Then the functor*[2] *of stable quasi-abelian varieties of dimension $g \leq 4$ with level structure K is coarsely represented by a projective scheme over k.*

We should mention that the above coarse moduli scheme paramatrises the isomorphism classes of stable quasi-abelian varieties with level structures forgotten. In order to pamametrise the isomorphism classes with level structures we will need $Sp(K)$-cover of the moduli in Theorem 0.6 or Mori-Keel or some other versions of existence of quotients. The cases $g \geq 5$ seems to require a somewhat more difficult treatment. Though our result is still very immature, the above form of the representability as well as a simple form of the functor would be a desirable goal of the theory.

[1] This means that $K = \mathbf{Z}/e_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/e_g\mathbf{Z}$, $n|e_1|e_2|\cdots|e_g$, $3 \leq n$
[2] See section 9.

Acknowledgment. We would like to thank many participants of Kinosaki symposium for their cooperation, among others A. Fujiki, T. Saito and T. Kajiwara who helped our understanding of the problem very much through discussions. We also thank V. Alexeev and T. Katsura respectively for collaboration in [AN96] and proof of Theorem 7.1 respectively. The present note is a revised version of our report in the proceedings of Tohoku symposium 1996 July.

1. Examples—Elliptic curves

1.1. Let us start with an elementary example. Let us look at the following degeneration of nonsingular elliptic curves—a variant of the Tate curve.

In what follows we assume that R is a complete discrete valuation ring, I the maximal ideal of R, s a generator of I and $S := \text{Spec } R$. The residue field R/I of R is a (not necessarily algebraically closed) field k.

For $k = 0, 1, 2$ we define

$$\begin{aligned}\theta_k(s, w) &= \sum_{m\in\mathbf{Z}} \mathrm{e}((3m+k)^2\tau/6 + (3m+k)z)\\ &= \sum_{m\in\mathbf{Z}} s^{(3m+k)^2} w^{3m+k}\end{aligned}$$

where $\mathrm{e}(?) = \exp(2\pi i ?)$, and $s = \mathrm{e}(\tau/6)$, $w = \mathrm{e}(z)$. We consider θ_k as (a lifting to the semi-universal covering of) a function converging in the I-adic topology. This is a canonical choice. In the analytic category they are analytic sections of an (relatively ample) invertible sheaf $\mathcal{L}^{\otimes 3}$ of elliptic curves $E(s)$ over a punctured disc. However the argument below is justified in the algebraic category as well.

Since $\mathcal{L}^{\otimes 3}$ is very ample, the image by θ_k is an elliptic curve over K, whose equation is known as *a Hesse cubic* in $\mathbf{P}^2$. By the representation theory of Heisenberg group [Mumford66-67, I,p.350] it is well known that

$$E(s) : \theta_0^3 + \theta_1^3 + \theta_2^3 = 3\mu(s)\theta_0\theta_1\theta_2$$

where $\mu(s)$ is a so-called theta constant (theta-zerovalue) given explicitly (and clearly) by

$$\mu(s) = \frac{\theta_0^3(s,1) + \theta_1^3(s,1) + \theta_2^3(s,1)}{3\theta_0(s,1)\theta_1(s,1)\theta_2(s,1)}$$

In fact, the Heisenberg group transforms θ_k in essentially two different manners

$$\begin{aligned}&\theta_0 \mapsto \theta_0, \theta_1 \mapsto \zeta_3\theta_1, \theta_2 \mapsto \zeta_3^2\theta_2\\ &\theta_0 \mapsto \theta_1 \mapsto \theta_2 \mapsto \theta_0\end{aligned}$$

where ζ_3 is a primitive cube root of unity. As is easily seen the above equation is the (almost!) unique possibility of the equation invariant under these actions. This elliptic curve is a universal elliptic curve with level three structure[3]. It has nine 3-division points given by $(1,\zeta,0)$, $(0,1,\zeta)$ and $(\zeta,0,1)$ (ζ: cube roots of -1) if $\mu(s)\neq\infty$ or $\mu(s)^3\neq 1$. If $\mu(s)=\infty$ and $\mu(s)^3=1$, then the curve $E(s)$ is a union of three lines with three ordinary double points, say a 3-gon of rational curves. The curve carries a natural very ample invertible sheaf $\mathcal{L}_0\simeq O(1)$. We note that $h^0(E(0),\mathcal{L}_0)=3=\sharp(\mathbf{Z}/3\mathbf{Z})$ by Theorem 0.4.

We also see that there exists a unit u in R such that $1/\mu(s)=3us^2$, $u=1 \bmod I$. In this sense $E(s)$ is *a Tate curve* with multiplicative period $q=s^6$.

It might be instructive to compute the limit $E(0)$, as $\mu(s)$ tends to infinity, when the parameter s approaches zero, from the view point of Néron model —a geometric realization of theta functions in this case.

The Néron model over S of the relative elliptic curve E (= a one-dimensional abelian scheme over K) in this case has a special fiber isomorphic to $\mathbf{G}_m\times(\mathbf{Z}/3\mathbf{Z})$, which is a Zariski open subset of the 3-gon $x_0x_1x_2=0$. The last fact is checked by setting $w=sa$, s^3a and s^5a for nonzero $a\in R\setminus I$, where we also dare to consider $a\in k:=R/I$ for brevity. Let us set $w=sa$. Then we first see that

$$\begin{aligned}
\theta_0(s,sa)&=\sum_{m\in\mathbf{Z}} s^{9m^2+3m}a^{3m}\\
&=1+s^6a^{-3}+s^{12}a^3+s^{30}a^{-6}+\cdots\\
\theta_1(s,sa)&=\sum_{m\in\mathbf{Z}} s^{(3m+1)^2+3m+1}a^{3m+1}\\
&=s^2a+s^2a^{-2}+s^{20}a^4+s^{20}a^{-5}+\cdots\\
\theta_2(s,sa)&=\sum_{m\in\mathbf{Z}} s^{(3m+2)^2+3m+2}a^{3m+2}\\
&=a^{-1}+s^6a^2+s^{12}a^{-4}+\cdots
\end{aligned}$$

Therefore we have in $\mathbf{P}^2$

$$\begin{aligned}
\lim_{s\to 0}\,[\theta_k(s,sa)]&=\lim_{s\to 0}\,[1+o(s),0+o(s),1/a+o(s)]\\
&=[1,0,1/a]
\end{aligned}$$

Similarly we see

$$\begin{aligned}
\lim_{s\to 0}\,[\theta_k(s,s^3a)]&=[0,1/a,1]\\
\lim_{s\to 0}\,[\theta_k(s,s^5a)]&=[1/a,1,0]
\end{aligned}$$

[3]See 1.3 for level structures.

TABLE 1. Stability of reduced cubic curves

curves (sing.)	stability	Stab. gr.
smooth elliptic	properly stable	finite
3-gon	stable not properly stable	2-dim
irred. a node	semi-stable not stable	$\mathbf{Z}/2\mathbf{Z}$
a triple point	unstable	2-dim

In addition, if we put $w = a$, $w = s^2a$ or $w = s^4a$, then we see that

$$\lim_{s\to 0}\,[\theta_k(s, s^0a)] = [1, 0, 0]$$
$$\lim_{s\to 0}\,[\theta_k(s, s^2a)] = [0, 0, 1]$$
$$\lim_{s\to 0}\,[\theta_k(s, s^4a)] = [0, 1, 0]$$

In the geometric invariant theory the cubic $x_0x_1x_2 = 0$ is stable but not properly stable [MFK, p.80]. In fact, the 3-gon has a two dimensional stabilizer group $\simeq G_m^2$, while proper-stability was by definition stability with finite stabilizer group [MFK, p.37].

The stability of the cubic is also proved by using Kempf's criterion [Kempf78] as well as by Gieseker's method [Gieseker82]. The purpose of the present article is to generalize this fact—to prove (or simply to report) Theorem 0.5.

1.2. Now we look at another example, which shows in fact that the very ampleness condition of $\mathcal{L}_0$ in Theorem 0.5 is necessary for stability. Let us define

$$\vartheta_k = \sum_{m\in\mathbf{Z}} s^{9m(m-1)+6mk}w^{3m+k} \quad (k = 0, 1, 2)$$

These theta functions on the elliptic curve $C(s)$ with multiplicative period s^6 are a canonical choice in the present case. We see easily

$$\vartheta_0(0, w) = 1 + w^3$$
$$\vartheta_1(0, w) = w$$
$$\vartheta_2(0, w) = w^2$$

Hence the limit curve is a rational curve with an ordinary double point $[1, 0, 0]$ [4]

$$C(0) : x_1^3 + x_2^3 = x_0x_1x_2.$$

The functions $\vartheta_k(0, w)$ are sections of $\mathcal{L}_0^3$ ($\simeq O(1)$), where $\mathcal{L}_0$, an invertible sheaf on $C(0)$ with $\deg \mathcal{L}_0 = 1$, is ample but not very ample, while $\mathcal{L}_0^3$ is very ample. We

[2]We do not know the equation of $C(s)$.

note that $h^0(C(0), \mathcal{L}_0^3) = 3$ by Theorem 0.4. The cubic curve $C(0)$ is semi-stable but not stable. The elliptic curve $C(s)$ is also a Tate curve with multiplicative period $q = s^6$.

We note that the stability in [MFK,p.80] of a cubic curve is just stability (Hilbert stability) of the third Hilbert point of the cubic curve. Therefore Theorem 0.5 seems to be the best possible.

1.3. Here we would like to remind the readers of the classical analytic theory. Let

$$\Gamma(3) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}); \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 3 \right\}$$

Let $\mathbf{H}$ be the upper half plane $\{\tau \in \mathrm{C}; \mathrm{Im}(\tau) > 0\}$, on which $\Gamma(3)$ acts by

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

Then $\mathbf{H}$ has four cusps ∞, 0, 1 and 2 on the rational boundary $\{\infty\} \cup \mathbf{Q}$ of $\mathbf{H}$, which are inequivalent under $\Gamma(3)$. These cusps correspond in the paragraph 1.1 to the points $\mu = \infty$ or $\mu^3 = 1$, or in geometric terms, the four 3-gons of rational curves.

Let $F(\tau)$ be an elliptic curve with periods 1 and τ. The level three structure on $F(\tau)$ is by definition a choice of basis of 3-division points of $F(\tau)$, where a natural choice is $e_1 := \{z = 1/3\}$ and $e_2 = \{z = \tau/3\}$. With an identification $F(\tau(s)) = E(s)$, they will be $e_1 := [1, \zeta_3, 0]$ and $e_2 = [0, 1, -1]$ on $E(s)$ where the zero $z = 0$ of the elliptic curve is chosen to be $e_0 = [1, -1, 0]$, while ζ_3 is a primitive cubic root of unity.

The quotient curve $M_3^0 := \mathbf{H}/\Gamma(3)$ is a rational curve with four points deleted, which can be compactified into a smooth rational curve M_3 by adding four cusps mentioned above. The curve M_3 admits over it a universal generalized elliptic curve S_3 with level three structure, which is just a minimal compactification of the Néron model over M_3 of the universal elliptic curve $S_3 \times_{M_3} M_3^0$. The complex surface S_3 is perhaps more familiar as Shioda elliptic modular surface of level three.

For a smooth elliptic curve with $\tau \neq i, \zeta_3$ there are exactly 12 choices of level three structures, which are in fact classified by $PSL(2, \mathbf{F}_3) := SL(2, \mathbf{F}_3)/\{\pm 1\} (\simeq A_4)$.

The level three structures on a 3-gon are classified by the coset of $PSL(2, \mathbf{F}_3)$ by the image of the stabilizer subgroup $\mathrm{Stab}(\infty)/\pm 1 \simeq \mathbf{Z}/3\mathbf{Z}$ in $PSL(2, \mathbf{Z})$. This is because the effect of the different choice of e_2 is cancelled out by nontrivial automorphisms of lines in the 3-gon. Therefore for the 3-gon of rational curves there are exactly four inequivalent choices of level three structures. This explains existence of four cusps in M_3, or equivalently four 3-gons of the form $E(s)$, in other words, four singular fibers of S_3.

We remark that there are six or four choices of level three structures on the elliptic curve with $\tau = i$ or $\tau = \zeta_3$. This shows that rationality of $M_1 := \mathbf{H}/SL(2, \mathbf{Z}) \cup \{\infty\}$

and M_3 is consistent with the Hurwitz formula

$$2 \cdot 0 - 2 = 12(2 \cdot 0 - 2) + 4(3 - 1) + 4(3 - 1) + 6(2 - 1).$$

2. Cubical structures

Let us give a very brief summary of cubical structures here.

2.1. Let A be an abelian scheme over an algebraically closed field k, L an invertible sheaf on A. Then the theorem of the square [Mumford74, p.59, Corollary 4] says

$$T^*_{x+y}L \otimes L \simeq T^*_x(L) \otimes T^*_y(L)$$

for any point $x, y \in A$. Let $\Lambda(L) := m^*(L) \otimes p_1^*(L)^{-1} \otimes p_2^*(L)^{-1}$ on $A^2 := A \times A$ where $m : (x, y) \mapsto x + y$. Then we see by using the theorem of the cube [Mumford74, p.89] that the invertible sheaves on A^3

$$\Theta_{12}(L) = (m \times 1)^*\Lambda(L) \otimes p_{13}^*\Lambda(L)^{-1} \otimes p_{23}^*\Lambda(L)^{-1}$$
$$\Theta_{23}(L) = (1 \times m)^*\Lambda(L) \otimes p_{12}^*\Lambda(L)^{-1} \otimes p_{13}^*\Lambda(L)^{-1}$$

are trivial. The pair of the above two sheaves $\Theta_{12}(L)$ and $\Theta_{23}(L)$ together with their trivialisations fixed is a cubical structure on L. See [Breen83, Introduction and §1].

One can rephrase the above fact as follows. Let $N := T^*_x(L) \otimes L^{-1}$. Then $N \in \mathrm{Pic}^0(A) = A^t \simeq \mathrm{Ext}(A, \mathbf{G}_m)$. Therefore $N \setminus \{0\}$ is an extension of A by a split torus $\mathbf{G}_m$, which admits an abelian group scheme structure. Moreover $T^*_y(N) \simeq N$ for any $y \in A(k)$.

It seems that the cubical structure of L is an intrinsic manifestation of this fact without referring to translation by k-points of A.

2.2. Let G be a semi-abelian scheme over a (complete) discrete valuation ring. Assume for simplicity that G is a group S-scheme over an abelian S-scheme A with any fiber T_s split torus. Namely we have an exact sequence of group schemes

$$1 \to T \xrightarrow{i} G \xrightarrow{\pi} A \to 0$$

where T is a split S-torus.

Cubical structures on G are defined in a way similar to the above. However we have to assume triviality of the sheaves $\Theta_{12}(L)$ and $\Theta_{23}(L)$ in general contrary to the case of abelian schemes.

There is an equivalence between the category of cubical $\mathbf{G}_m$-torsors and the category of rigidified invertible sheaves (rigidified along the unit of G) where a $\mathbf{G}_m$-torsor is a line bundle (associated with an invertible sheaf) minus zero section. This means that there is one to one correspondence between a cubical invertible sheaf and a rigidified invertible sheaf on a semi-abelian group scheme. Any rigidified invertible sheaf on G has a unique cubical structure [Breen83, Proposition 2.4].

Moreover by [Breen83, p.38, Proposition 3.10], [MB85, p.37, 7.2.2], the category of cubical $\mathbf{G}_m$-torsors on G with restriction to T trivial is equivalent to that of cubical $\mathbf{G}_m$-torsors on A. We note that this fact is proved essentially by using Rosenlicht's lemma [SGA7, p.265, VIII, Lemme 4.1].

However by our assumption that T is a split torus, the restriction to T of any invertible sheaf of G is always trivial. Therefore the categories of cubical $\mathbf{G}_m$-torsors on G and of cubical $\mathbf{G}_m$-torsors on A are equivalent. This means that for any cubical invertible sheaf L on G there exists a unique cubical invertible sheaf M such that $\pi^*(M) = L$. If L is ample[5], then M is ample and vice versa[6].

3. Degeneration data

The purpose of this section is to sketch the description of degenerations of abelian varieties given by Faltings-Chai[FC90, II.4.1,5.1]. See also [AN96, section two].

Notation 3.1. a) R is a Noetherian normal integral domain complete with respect to an ideal $I = \sqrt{I}$, $S = \operatorname{Spec} R$, $S_0 = \operatorname{Spec} R/I$, K is the fraction field and $\eta = \operatorname{Spec} K$ is the generic point.

We will assume that R is a complete discrete valuation ring complete with respect to the maximal ideal I-adic topology. We will denote by $k = R/I$ the residue field.

b) G/S is a semiabelian scheme of relative dimension g with abelian generic fibre G_η (with a chosen unit section). The special fibre G_0 is a semiabelian scheme over k, namely an extension of an abelian scheme A_0 of relative dimension g' by a torus T_0 of relative dimension g'', $g' + g'' = g$. We assume T_0 to be split, and this always holds after a finite base change of S.

c) $\mathcal{L}$ is a rigidified ample invertible sheaf on G [7].

d) Associated to G/S and $\mathcal{L}$ are the formal scheme $G_{\text{for}} = \lim G \otimes R/I^n$ and an invertible sheaf $\mathcal{L}_{\text{for}} = \lim \mathcal{L} \otimes R/I^n$. The scheme G_{for} fits into an exact sequence

$$0 \to T_{\text{for}} \to G_{\text{for}} \xrightarrow{\pi_{\text{for}}} A_{\text{for}} \to 0$$

By the theory of cubical structures [Breen83] [MB85, p.40, Theorem 1.1 (ii)] there exists a unique cubical structure on $\mathcal{L}$ (viewed as a $\mathbf{G}_m$-torsor), which induces a cubical structure of the sheaf $\mathcal{L}_{\text{for}}$.

[5] For global sections $f \in H^0(G, L^n)$ G_f is affine and forms a base of Zariski topology of G for $n > 0$.

[6] Note that π is affine.

[7] See Remark 3.2

Then $\mathcal{L}_{\mathrm{for}}$ is descended to a unique cubical ample invertible sheaf $\mathcal{M}_{\mathrm{for}}$ on A_{for}, that is, $\mathcal{L}_{\mathrm{for}} = \pi^*_{\mathrm{for}}(\mathcal{M}_{\mathrm{for}})$ [8]. Since there exists an ample sheaf on A_{for}, A_{for} is algebraisable. Namely by the algebraisation theorem of Grothendieck there exists an abelian S-scheme A with an ample invertible sheaf $\mathcal{M}$ such that the formal completion $(\hat{A}, \hat{\mathcal{M}})$ of $(A, \mathcal{M})$ is $(A_{\mathrm{for}}, \mathcal{M}_{\mathrm{for}})$.

By our assumption that T_0 is a k-split torus, T_{for} is a formal S-split torus by [SGA3, IX, Théorèm 3.6], [FC90, 2.2]. Let X be the character group of T_{for}. Then by setting $T := \mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{G}_m)$, T algebraises T_{for}.

The sequence $0 \to T_{\mathrm{for}} \to G_{\mathrm{for}} \to A_{\mathrm{for}} \to 0$ is also algebraisable because the extension class of it is given by an element of $\mathrm{Ext}(A_{\mathrm{for}}, T_{\mathrm{for}}) \simeq \mathrm{Ext}(A, T)$ [FC90, p.34]. The dual abelian variety G^t_η is also extended to a semiabelian S-scheme G^t by taking the connected Néron model [9] after taking finite ramified cover of S if necessary. See [SGA7, I, p.20 Appendice]. Then similarly we see that the dual G^t_{for} is algebraisable. Namely there exists a semiabelian scheme $\tilde{G}^t$ such that the formal completion of $\tilde{G}^t$ is isomorphic to G_{for}. Thus we obtain the so called Raynaud extensions for G_{for} and G^t_{for}

$$0 \to T \to \tilde{G} \xrightarrow{\pi} A \to 0$$

$$0 \to T^t \to \tilde{G}^t \xrightarrow{\pi^t} A^t \to 0$$

plus the homomorphisms $c : X \to A^t$, $c^t : Y \to A$ decoding them. In other words, $c \in \mathrm{Hom}(X, A^t) \simeq \mathrm{Ext}(A, T)$ and $c^t \in \mathrm{Hom}(Y, A) \simeq \mathrm{Ext}(A^t, T^t)$ describe the extension classes of semiabelian schemes $\tilde{G}$ and $\tilde{G}^t$ respectively.

e) Finally, the polarisation $\lambda(\mathcal{L}_\eta) : G_\eta \to G^t_\eta$ induces a morphism $\lambda : G \to G^t$ by the universal property of Néron model of G^t_η. It induces also a formal morphism $\lambda_{\mathrm{for}} : G_{\mathrm{for}} \to G^t_{\mathrm{for}}$, which defines two polarisations $\phi : Y \to X$ and $\lambda_A = \lambda(\mathcal{M}) : A \to A^t$. Since we are given the formal morphism λ_{for}, the extension classes of G_{for} and G^t_{for} are compatible by $\lambda_{A_{\mathrm{for}}} = \lambda(\mathcal{M})_{\mathrm{for}}$ so that $c_{\mathrm{for}}\phi = \lambda_{A_{\mathrm{for}}} c^t_{\mathrm{for}}$. After algebraisation it follows that $c\phi = \lambda_A c^t$. From this it follows that the formal morphism λ_{for} is algebraised into a morphism from $\tilde{G}$ onto $\tilde{G}^t$.

Remark 3.2. Note that if R is a discrete valuation ring with the quotient field K then according to the semistable reduction theorem any abelian variety G_K over K can be extended to a semiabelian scheme G over R as the connected Néron model of G_K, so the condition b) above is no restriction. Moreover since by taking a finite extension of K if necessary there exists an invertible sheaf $H \in \mathrm{Pic}^0(G_K)$ such that the invertible sheaf $\mathcal{L}_K \otimes H$ is symmetric, namely $i^*(\mathcal{L}_K \otimes H) = \mathcal{L}_K \otimes H$ for the

[8] This is true because T_{for} is a split torus. Otherwise we need to take a symmetric invertible $\mathcal{L}_{\mathrm{for}} \otimes [-1]^*\mathcal{L}_{\mathrm{for}}$ for descent.

[9] We mean by the connected Néron model the Néron model with closed fiber irreducible.

involution $i = [-1]_{G_K}$ of G_K [10]. Therefore *we can assume from the start that $\mathcal{L}_K$ is symmetric, ample and rigidified along the zero section.* $\mathcal{L}_K$ associates to some Cartier divisor, which extends uniquely to a smooth scheme G. Therefore $\mathcal{L}_K$ extends to a symmetric invertible sheaf on G uniquely because G_0 is irreducible. On the other hand by [Raynaud70, p.158 XI, 1.2 and p.170 XI 1.13] $\mathcal{L}_K^{\otimes n}$ extends to G as an ample invertible sheaf for some $n > 0$ if $\mathcal{L}_K$ is symmetric and ample. Since $\mathcal{L}_K$ satisfies the condition in this case, the extension $\mathcal{L}$ is ample.

3.3. The space of theta functions $\Gamma(G_\eta, \mathcal{L}_\eta)$ on the generic fibre is embedded into $\Gamma(G_{\text{for}}, \mathcal{L}_{\text{for}}) \underset{A}{\otimes} K$. The latter has the torus action. Therefore, every theta function $\theta \in \Gamma(G_\eta, \mathcal{L}_\eta)$ can be written as a Fourier series of eigenfunctions, and this series should converge in the I-adic topology. The theorem of Faltings and Chai says that the coefficients of these Fourier series satisfy the same functional equations as in the classical complex analytic case.

We restrict ourselves to the totally (or maximally) degenerate case, that is the case when A_0 (and hence A) is trivial. Then $G_{\text{for}} = T_{\text{for}}$ and $\widetilde{G} = T$. The invertible sheaf $\mathcal{L}_{\text{for}}$ is trivial on T_{for}, and therefore

$$\Gamma(G_\eta, \mathcal{L}_\eta) = \Gamma(G, \mathcal{L}) \underset{R}{\otimes} K \hookrightarrow \Gamma(G_{\text{for}}, \mathcal{L}_{\text{for}}) \underset{R}{\otimes} K =$$
$$\prod_{x \in X} \Gamma(S, O_S) \underset{R}{\otimes} K \cdot w^x = \prod_{x \in X} K \cdot w^x$$

Therefore, every theta function $\theta \in \Gamma(G_\eta, \mathcal{L}_\eta)$ can be written as a formal Fourier power series

$$\theta = \sum_{x \in X} \sigma_x(\theta) w^x$$

with $\sigma_x(\theta) \in K$.

Theorem 3.4. [Faltings–Chai90] *There exists a function $a : Y \to K^*$ and a bilinear function $b : Y \times X \to K^*$ with the following properties:*

(1) $a(y+z+w)a(y)a(z)a(w) = a(y+z)a(z+w)a(w+y) \quad \forall y, z, w \in Y$ *(in particular, $a(0) = 1$).*
(2) $b(y, z) = b(z, y) = a(y+z)a(y)^{-1}a(z)^{-1} \quad \forall y, z \in Y$
(3) $b(y, y) \in I \quad \forall y \neq 0$, *equivalently, for every $n \geq 0$ $a(y) \in I^n$ for almost all $y \in Y$*

[10] See the proof of Theorem 7.1.

(4) *The K-vector space $\Gamma(G_\eta, \mathcal{L}_\eta)$ is identified with the vector space of Fourier series θ that satisfy $\sigma_{x+\phi(y)}(\theta) = a(y)b(y,x)\sigma_x(\theta)$.*

Definition 3.5. We define the functions $A : Y \to \mathbf{Z}$, $B : Y \times X \to \mathbf{Z}$ and elements $\bar{b}(y,x) \in R^*$, $\bar{a}(y) \in R^*$ by

$$B(y,x) = val_s(b(y,x)), \quad A(y) = val_s(a(y)) = \frac{1}{2}(B(y,\phi(y)) + r(\phi(y)))$$
$$b(y,x) = \bar{b}(y,x)s^{B(y,x)}, \quad a(y) = \bar{a}(y)s^{(B(y,\phi(y))+r(\phi(y)))/2}$$

for some $r \in \mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{Z})$. It is easy to see that B is bilinear.

We note that B is *positive definite* by Theorem 3.4 (3).

4. Construction of $(P, \mathcal{L})$

4.1. From [AN96], [Namikawa76] we recall that

$$\begin{aligned}
\mathrm{Alg} : &= R[a(x)w^x\vartheta; x \in X] \\
&\simeq R[\xi_x\vartheta; x \in X], \quad \xi_x := s^{B(x,x)/2+r(x)/2}w^x \\
\zeta_{x,c} : &= \xi_{x+c}/\xi_c = s^{B(\alpha(\sigma),x)+r(x)/2}w^x \quad (x + c \in C(c,\sigma)) \\
\widetilde{P} : &= \text{normalization of } \mathrm{Proj}(\mathrm{Alg}) \\
S_y^*(a(x)w^x\vartheta) &= a(x+y)w^{x+\phi(y)}\vartheta
\end{aligned}$$

where Alg is the graded algebra with $\deg(a(x)w^x\vartheta) = 1$ and $\deg a = 0$ for $a \in R$.

The endomorphism S_y^* induces a natural action of $\widetilde{P}$, which we denote by the same letter S_y. Let $\widetilde{\mathcal{L}}$ be the pull back of $O_{\mathrm{Proj}}(1)$ to $\widetilde{P}$.

Then our construction of $(P, \mathcal{L})$ [AN96] can be stated in the following

Theorem 4.2.

(1) *Let $(\widetilde{P}_0, \widetilde{\mathcal{L}}_0)$ be the closed fibre of $(\widetilde{P}, \widetilde{\mathcal{L}})$. The restriction of $\widetilde{\mathcal{L}}_0$ to any irreducible component of $\widetilde{P}_0$ is ample* [11].

[11] $\widetilde{P}$ is the normalization of Proj so that $\widetilde{\mathcal{L}}_0$ may not be very ample.

(2) *For n large enough, nY acts on $\widetilde{P}_0$ freely so that the quotient $\widetilde{P}_0/nY$ is a scheme of finite type over k covered with (the isomorphic images of) affine open subsets of $\widetilde{P}_0$. The invertible sheaf $\widetilde{\mathcal{L}}_0$ is descended to an invertible sheaf $\widetilde{\mathcal{L}}_0/nY$ on the quotient $\widetilde{P}_0/nY$. A union of only finitely many irreducible components of $\widetilde{P}_0$ dominates $\widetilde{P}_0/nY$ so that $\widetilde{\mathcal{L}}_0/nY$ is ample and $(\widetilde{P}_0, \widetilde{\mathcal{L}}_0)/nY$ is a projective scheme over k.*

(3) *$(\widetilde{P}_0, \widetilde{\mathcal{L}}_0)/Y$ is a quotient of a projective k-scheme of $(\widetilde{P}_0, \widetilde{\mathcal{L}}_0)/nY$ by a finite group Y/nY so that it is a projective scheme over k* [12].

(4) *$(\widetilde{P}_{\mathrm{for}}, \widetilde{\mathcal{L}}_{\mathrm{for}})/Y$ is a flat projective formal S-scheme.*

(5) *There exists a flat projective S-scheme $(P, \mathcal{L})$ such that the formal completion $(P_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$ of it along the closed fibre is isomorphic to $(\widetilde{P}_{\mathrm{for}}, \widetilde{\mathcal{L}}_{\mathrm{for}})/Y$.*

Proof. (5) follows from the algebraisation theorem of Grothendieck. The rest is clear from the statements. $\square$

Remark 4.3. The above construction is still insufficient because the closed fibre P_0 *can be nonreduced.* We need to take a certain finite ramified cover to obtain a reduced closed fibre. The modification is not difficult but only technical, so we omit it here. See [AN96]. By the modification we obtain Theorems 0.3-0.4.

5. Delaunay decomposition

Definition 5.1. Let B be a symmetric positive definite integral $g \times g$-matrix, which determines a distance $\| \ \|_B$ on the Euclidean space $X_{\mathbf{R}}$. For an arbitrary $\alpha \in X_{\mathbf{R}}$ we say that a lattice element $a \in X$ is α-nearest if

$$\|a - \alpha\|_B = \min\{\|b - \alpha\|_B; b \in X\}$$

We define a (closed) *B-Delaunay cell σ* (or simply a Delaunay cell if B is understood) to be the closed convex hull of all lattice elements which are α-nearest for some $\alpha \in X_{\mathbf{R}}$. Note that for a given Delaunay cell σ the element α is uniquely defined only if σ has the maximal possible dimension, equal to g. In this case we sometimes call α *the centre* or *the hole of σ.*

Together all the B-Delaunay cells constitute a locally finite decomposition of $X_{\mathbf{R}}$ into infinitely many bounded convex polyhedra which we call the *B-Delaunay decomposition* Del_B.

[12] $\widetilde{\mathcal{L}}_0/nY$ is descended to $\widetilde{P}_0/Y$.

Remark 5.2. It is clear from the definition that the Delaunay decomposition is invariant under translation by the lattice X and that the 0-dimensional cells are precisely the elements of X.

Definition 5.3. For a given B-Delaunay cell σ consider all $\alpha \in X_{\mathbf{R}}$ that define σ. They themselves form a locally closed bounded convex polyhedron which we denote $\mathrm{Vor}^0(\sigma)$ and call an *open B-Voronoi cell* (even though it is only locally closed). We denote the closure of this cell by $\mathrm{Vor}(\sigma)$. All the closed B-Voronoi cells make a (closed) polyhedral *Voronoi decomposition* Vor_B of $X_{\mathbf{R}}$.

5.4. As we vary the bilinear form B, the corresponding decompositions Del_B and Vor_B themselves change. Since the vertices of Del_B are all in X, it is clear that Del_B have a discrete set of values. The Voronoi decompositions, however, have some continuous moduli. The Delaunay and Voronoi decompositions are dual, with respect to B, in the following sense:

Lemma 5.5.

(1) *The natural maps $\sigma \mapsto V(\sigma)$, $\hat{\sigma} \mapsto D(\hat{\sigma})$ define a one-to-one correspondence between closed Delaunay and Voronoi cells;*
(2) $\dim \sigma + \dim V(\sigma) = \dim \hat{\sigma} + \dim D(\hat{\sigma}) = g$;
(3) *$\tau \subset \sigma$ iff $V(\sigma) \subset V(\tau)$, $\hat{\tau} \subset \hat{\sigma}$ iff $D(\hat{\sigma}) \subset D(\hat{\tau})$;*
(4) *for $c \in Y$ the cell $V(c)$ is a g-dimensional polyhedron with vertices at centers $\alpha(\sigma)$, where σ goes over all maximal-dimensional cells containing c;*
(5) *for an arbitrary Delaunay cell σ the cell $V(\sigma)$ is a polyhedron with vertices at centers $\alpha(\sigma)$, where σ goes over all maximal-dimensional cells containing all of the vertices of σ.*

Proof. Straightforward. See also [Namikawa76]. □

Delaunay decompositions enter the theory through the following

Theorem 5.6. *Let P_0 be the central fibre of the flat family $(P, \mathcal{L})$. Let σ,τ be Delaunay cells in the Delaunay decomposition* $\mathrm{Del}(P)$ *corresponding to $(P, \mathcal{L})$.*

(1) *For each σ there exists a T-invariant subscheme $O(\sigma)$ of the central fibre P_0 which is a torus of dimension* $\dim \sigma$ *over k,*
(2) *$\sigma \subset \tau$ iff $O(\sigma)$ is contained in $O(\tau)$,*

(3) *For each σ there exists a unique closed subscheme $V(\sigma)$ which is the closure of $O(\sigma)$ and has dimension* $\dim\sigma$ *over k,*
(4) $\sigma \subset \tau$ *iff $V(\sigma)$ is contained in $V(\tau)$,*
(5) $P_0 = \bigcup_{\sigma\in \mathrm{Del}(P) \bmod Y} V(\sigma)$

See [Namikawa76], [AN96].

6. Heisenberg group of $(P, \mathcal{L})$

We will prove

Lemma 6.1. *By choosing a suitable finite extension K' of K if necessary, the finite group X/Y operates freely on $(P,\mathcal{L})\otimes R'$ via S_z $(z \in X)$ defined below. The quotient $(P_{quot}, \mathcal{L}_{quot}) := (P,\mathcal{L})\otimes R'/(X/Y)$ is a proper flat family of principally polarised* [13] *stable quasi-abelian varieties over* Spec R' *where R' is the integral closure of R in K'.*

Proof. We define an algebra homomorphism $S^*_{z,n}$ $(z \in X)$ of $\mathrm{Alg}\otimes R'/I'^{n+1}$ by the same formula as [AN96];

$$S^*_{z,n}(\bar{a}(x)\xi_x\vartheta) = \bar{a}(x+z)\xi_{x+z}\vartheta \bmod I'^{n+1}$$
$$S^*_{z,n}(\zeta_{x,c}) = \bar{b}(z,x)\zeta_{x,c+z} \bmod I'^{n+1}.$$

It is clear that each $S^*_{z,n}$ is an algebra homomorphism. Since $S^*_{z,n}$ is degree-preserving, $S^*_{z,n}(\tilde{\mathcal{L}}\otimes R'/I'^{n+1}) \simeq \tilde{\mathcal{L}}\otimes R'/I'^{n+1}$. Since $S^*_{z,n}$ $(n \geq 0)$ are compatible and commutes with S^*_y $(y \in Y)$, we have an R'/I'^{n+1}-isomorphism of $(P,\mathcal{L})\otimes R'/I'^{n+1}$, which we denote by $S_{z,n}$. Therefore we have an R'-isomorphism of $(P_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})\otimes R'$. Hence by the algebraisation theorem of Grothendieck, it is algebraised by an R'-isomorphism S_z of $(P,\mathcal{L})\otimes R'$. It is also clear that the finite group X/Y operates on $(P,\mathcal{L})\otimes R'$ freely via the algebraised actions S_z and these give a descent data for $(P,\mathcal{L})\otimes R'$. The rest follows from [AN96] and the theory of descent. □

Remark 6.2. If G_0 is a k-split torus, (in fact, this is the case for a suitable extension K' of K), $\tilde{G}$ is an algebraic R'-split torus, and it is an open subscheme of $\tilde{P}\otimes R'$. [AN96, 3.13]. Hence we may assume for simplicity that there exists a g-dimensional Delaunay cell σ_0 such that $0 \in \sigma_0$ and rel. int. $O(\sigma_0) = \tilde{G}\otimes k'$. In other words,

$$\tilde{G} = \mathrm{Spec}\ R'[\zeta^{\pm 1}_{x,0}; x \in C(0,\sigma_0)\cap X]$$

[13] The term principally polarised means that $h^0(P_{quot}, \mathcal{L}_{quot}) = 1$.

where $C(0,\sigma_0)$ is the cone spanned by Delaunay vectors of σ_0 starting from the vertex 0 of σ_0. For simplicity we assume (or introduce the notation) $w^x := \zeta_{x,0}$,

$$\zeta_{x,0} = s^{B(\alpha(\sigma_0),x)+r(x)/2}w^x = w^x \quad (\forall x \in C(0,\sigma_0)\cap X),$$

which is equivalent to the following

$$B(\alpha(\sigma_0),x) + r(x)/2 = 0 \quad (\forall x \in X).$$

In what follows we also assume the equation.

Definition 6.3. It is clear that the lattice X is spanned by $C(0,\sigma_0)\cap X$. Hence $\widetilde{G}$ is an R'-torus Spec $R'[w^x; x \in X]$. We denote $\widetilde{G}$ by $\widetilde{G}(\sigma_0)$. By [Mumford72] the quotient of $\widetilde{G}(\sigma_0)_{\mathrm{for}}$ by the periods Y is algebraised into a semi-abelian group scheme over R', which we denote by $G(\sigma_0)$. By [Mumford72] $G(\sigma_0)_\eta$ $(= P_\eta)$ is independent of the choice of σ_0.

Lemma 6.4. *Assume that $B : Y\times X \to \mathbf{Z}$ extends to a bilinear form $B : X\times X \to \mathbf{Z}$. We define a R'-valued point $\tilde{s}(z)$ of $\widetilde{P}$ by $\tilde{s}(z)^*(w^x) = \bar{b}(z,x)s^{B(z,x)}(=: b(z,x))$. Then $\tilde{s}(z)$ is algebraised by an R'-valued point $s(z)$ of P and S_z is induced from translation of $G_\eta \otimes K'$ by $s(z)\otimes K'$.*

Proof. By Remark 6.2 we are given a R'-valued point e of P, which gives rise to a formal R'-valued point e_{for} of P_{for}. By the Remark 6.2 we may assume that a lifting $\tilde{e}_{\mathrm{for}} : \mathrm{Spec}\, R' \to \widetilde{P}\otimes R'$ of e_{for} is given by $\tilde{e}^*_{\mathrm{for}}(w^x) = 1$ $(\forall x \in X)$.

It is easy to see from the definition that $\tilde{s}(z)$ is an R'-valued point of $\widetilde{P}$. In fact, $\tilde{s}(z)^*(\zeta_{x,-z}) = \tilde{s}(z)^*(s^{B(-z,x)}w^x) = \bar{b}(z,x)$. Then we have $s(z)_n := \tilde{s}(z) \mod I'^{n+1} \in \widetilde{P}\otimes R'/I'^{n+1}$, hence a formal R'-valued point $s(z)_{\mathrm{for}}$ of P_{for}. Since it is proper over R' we have an algebraisation $s(z)$, an R'-valued point of P. Next we define translation $T_{s(z)}$ by $s(z)$ as follows. First we define on $\mathrm{Alg}\otimes R'$

$$T^*_{s(z),n}(\zeta_{x,c}) = \bar{b}(z,x)\zeta_{x,c+z} \mod I'^{n+1}$$

because by definition $\zeta_{x,c} = s^{B(\alpha(\sigma),x)+r(x)/2}w^x$ if $x+c\in\sigma$. Hence $T_{\tilde{s}(z),n}$ is defined on $(\widetilde{P},\widetilde{\mathcal{L}})\otimes R'/I'^{n+1}$ for any $n\geq 0$ in a compatible way, and $T_{\tilde{s}(z),n}$ commutes with $S_{y,n}$ $(\forall y \in Y)$. Moreover since $T^*_{\tilde{s}(z),n}$ on $\mathrm{Alg}\otimes R'$ is degree-preserving, it descends to an action $T_{s(z),n}$ of $(P,\mathcal{L})\otimes R'/I'^{n+1}$, hence an action $T_{s(z),\mathrm{for}}$ on $(P_{\mathrm{for}},\mathcal{L}_{\mathrm{for}})\otimes R'$. By algebraisation theorem $T_{s(z),\mathrm{for}}$ is algebraised into an action $T_{s(z)}$ on $(P,\mathcal{L})\otimes R'$. Since $S^*_z(\zeta_{x,0}) = \bar{b}(z,x)\zeta_{x,z}$, $T^*_{s(z),\mathrm{for}} = S^*_{z,\mathrm{for}}$, hence $T_{s(z)} = S_z$. □

Remark 6.5. $s(z)$ is a R'-valued point of $G(\sigma_0 - z)$. This confusing fact comes from our notational convention $S_z^*(\zeta_{x,0}) = \bar{b}(z,x)\zeta_{x,c+z}$, in other words, $S_z(U(c)) \subset U(c-z)$.

Definition 6.6. For a R'-valued point α of $G(\sigma_0)$ we define translation action T_a upon P by α as follows. Suppose that α is given by a R'-valued point $\tilde{\alpha}$ of $\tilde{G}(\sigma_0)$ $\tilde{\alpha}^*(w^x) = \alpha(x) \in R'^* := R' \setminus I'$ where $\alpha \in \mathrm{Hom}\,(X, R'^*) = \tilde{G}(\sigma_0)(R')$. Then we define

$$T_{\alpha,n}^*(\zeta_{x,c}) = \alpha(x)\zeta_{x,c} \mod I'^{n+1}.$$

It is clear that $S_{y,n}^* T_{\alpha,n}^* = T_{\alpha,n}^* S_{y,n}^*$ $(\forall y \in Y)$. Hence $T_{\alpha,n}^*$ descends to a morphism from $P \otimes R'/I'^{n+1}$ to itself. By algebraisation theorem, we have an algebraisation $T_\alpha : P \otimes R' \to P \otimes R'$, which we call translation by α.

Definition 6.7. By modifying slightly [Mumford66-67] we define a functor from the category $(Schemes/K)$ to the category $(Sets)$

$$H(G_\eta, \mathcal{L}_\eta)(T) = \{T_x \in \mathrm{Aut}_T(G_\eta(T)); x \in G_\eta(T), T_x^*(\mathcal{L}_T) \simeq \mathcal{L}_T\}$$

for a scheme T over K. Since $\mathcal{L}$ is ample, this functor is representable by a finite group subscheme of $\mathrm{Aut}(G_\eta)$, which we denote by $H(G_\eta, \mathcal{L}_\eta)$. By [ibid.] if K is algebraically closed and if characteristic K and $d := \deg \mathcal{L}_\eta$ are coprime, $H(G_\eta, \mathcal{L}_\eta)$ is a finite abelian reduced group scheme of order d^2 where $\deg \mathcal{L}_\eta := (\mathcal{L}_\eta^g)/g! = [X : Y]$ [Mumford66-67, I, p.289]. In fact, it is isomorphic to $\ker \Lambda(\mathcal{L}_\eta)$ where $\Lambda(\mathcal{L}_\eta) : G_\eta \to G_\eta$ is the polarisation morphism by $\mathcal{L}_\eta$. It follows that there exists a suitable finite extension K' of K such that $H(G_\eta, \mathcal{L}_\eta)(K')$ is of order d^2. Therefore by choosing such an extension K' we may assume that $H(G_\eta, \mathcal{L}_\eta)(K'')$ is of order d^2 for an arbitrary finite extension K'' of K'.

Lemma 6.8. *By choosing a suitable finite extension K' of K if necessary, with the notation in Lemma 6.4 and Definition 6.6 we define*

$$\begin{aligned} K((P,\mathcal{L}) \otimes R') &:= \{T_{s(z)}; z \in X/Y\} \\ \hat{K}((P,\mathcal{L}) \otimes R') &:= \{T_\alpha; \alpha \in G(R'), \alpha(y) = 1 \ (\forall y \in Y)\} \\ H((P,\mathcal{L}) \otimes R') &:= K((P,\mathcal{L}) \otimes R') \oplus \hat{K}((P,\mathcal{L}) \otimes R'). \end{aligned}$$

Then

$$H((P,\mathcal{L}) \otimes R') \otimes K' = H(G_\eta, \mathcal{L}_\eta)(K').$$

Moreover the Weil pairing on $H((P,\mathcal{L}) \otimes R')$ is given by

$$e_{\mathcal{L}\otimes R'}((T_{s(z)}, T_\alpha), (T_{s(w)}, T_\beta)) = \alpha(w)\beta(z)^{-1}.$$

Proof. Recall that $S_z = T_{s(z)}$ $(z \in X)$ by Lemma 6.4. It is clear that S_y $(y \in Y$ induces identity transformation of $P \otimes R'$ so that $K((P,\mathcal{L}) \otimes R') \simeq X/Y$. Let $\alpha \in G(R')$. Then we have

$$T_\alpha^* S_z^*(a(x)w^x\vartheta) = \alpha(x+z)a(x+z)w^{x+z}\vartheta$$
$$S_z^* T_\alpha^*(a(x)w^x\vartheta) = \alpha(x)a(x+z)w^{x+z}\vartheta.$$

Hence $T_\alpha^* S_z^* = S_z^* T_\alpha^*$ if and only if $\alpha(z) = 1$. Moreover $T_\alpha^* S_z^* = S_z^* T_\alpha^*$ $(\forall z \in Y)$ if and only if T_α^* descends to $(P_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}) \otimes R'$ and is algebraised by algebraisation theorem into an isomorphism of $(P,\mathcal{L}) \otimes R'$, equivalently $(P, T_\alpha^*(\mathcal{L})) \otimes R' \simeq (P,\mathcal{L}) \otimes R'$. It is also easy to see that $T_\alpha^* S_z^* = \alpha(z) S_z^* T_\alpha^*$ on $\mathcal{L}$. Therefore we have

$$\begin{aligned} e_{\mathcal{L}\otimes R'}((T_{s(z)}, T_\alpha),(T_{s(w)}, T_\beta)) :&= S_z^* T_\alpha^* S_w^* T_\beta^* (S_z^* T_\alpha^*)^{-1} (S_w^* T_\beta^*)^{-1} \\ &= \alpha(w)\beta(z)^{-1}. \end{aligned}$$

Lemma follows from $|H(G_\eta, \mathcal{L}_\eta)(K')| = |X/Y|^2$ and [Mumford66-67, I, p.310]. □

Definition 6.9. We call $H((P,\mathcal{L}) \otimes R')$ the reduced Heisenberg group of of $(P,\mathcal{L})$. The total Heisenberg group $\mathcal{G}((P,\mathcal{L}) \otimes R')$ of $(P,\mathcal{L})$ is a central extension of $H((P,\mathcal{L}) \otimes R')$ by R'^*,

$$1 \to R'^* \to \mathcal{G}((P,\mathcal{L}) \otimes R') \to H((P,\mathcal{L}) \otimes R') \to 1$$

The group structure of $\mathcal{G}((P,\mathcal{L}) \otimes R')$ is defined by

$$(a, T_{s(z)}, T_\alpha) \cdot (b, T_{s(w)}, T_\beta)) = (ab\beta(z), T_{s(z+w)}, T_{\alpha\beta})$$

Usually the Weil pairing $e_{\mathcal{L}\otimes R'}$, a skew symmetric bimultiplicative form on $H((P,\mathcal{L}) \otimes R')$ is defined [Mumford66-67, I,p.293] by

$$\begin{aligned} e_{\mathcal{L}\otimes R'}((T_{s(z)}, T_\alpha),(T_{s(w)}, T_\beta)) &= [(1, T_{s(z)}, T_\alpha),(1, T_{s(w)}, T_\beta))] \\ &= \alpha(w)\beta(z)^{-1}. \end{aligned}$$

where $[u,v] := uvu^{-1}v^{-1}$ is the commutator for $u, v \in \mathcal{G}((P,\mathcal{L}) \otimes R')$. This coincides with the above Lemma 6.8. The Weil pairing is clearly nondegenerate.

A subgroup K of $H((P,\mathcal{L}) \otimes R')$ is called isotropic (resp. maximally isotropic) if $e_{\mathcal{L}\otimes R'} = 1$ on $K \times K$ (resp. if it is isotropic and maximal among isotropic subgroups). A subgroup $\widetilde{K}$ of $\mathcal{G}((P,\mathcal{L}) \otimes R')$ is called a level subgroup if $\widetilde{K} \cap (R'^*) = \{1\}$ and if the image K of $\widetilde{K}$ is maximally isotropic. Any level subgroup $\widetilde{K}$ is of order $\deg(\mathcal{L}) = |X/Y|$. The image K of a level subgroup $\widetilde{K}$ is called a reduced level subgroup of $H((P,\mathcal{L}) \otimes R')$.

As an abstract group $H((P,\mathcal{L})\otimes k')\simeq (X/Y)^{\oplus 2}$ and $\mathcal{G}((P,\mathcal{L})\otimes k')$ is a central extension of it by k'^*, whose group structure is uniquely determined by X/Y.

$\mathcal{G}((P,\mathcal{L})\otimes k')$ has a unique irreducible representation of weight one [Mumford66-67, I,Theorem 2] where we say that the representation has weight n if the centre k'^* operates by the scalar multiplication of n-th power.

In the above definitions, we omit $\otimes k'$ from the notation if $(P_0,\mathcal{L}_0)$ is defined over an algebraically closed field k.

Lemma 6.10. *Assume that k is algebraically closed. Then $H^0(P,\mathcal{L})$ is an irreducible $\mathcal{G}(P,\mathcal{L})$-module of weight one.*

Proof. We recall

$$\theta_x := \sum_{y\in Y} a(x+y)w^{x+y}$$
$$S_z^*(\theta_x\vartheta)=\theta_{x+z}\vartheta,\quad S_\alpha^*(\theta_x\vartheta)=\alpha(x)\theta_x\vartheta$$

where $x,z\in X$, $\alpha\in\hat{K}(P,\mathcal{L})$. In particular, $S_y^*(\theta_x\vartheta)=\theta_x\vartheta$ $(\forall y\in Y)$. By taking mod I we obtain a representation of $\mathcal{G}((P,\mathcal{L})\otimes k)$. This is a standard realisation of a representation of $\mathcal{G}((P,\mathcal{L})\otimes k)$, which is known to be irreducible if k is algebraically closed [Mumford66-67, I,Theorem 2]. □

See [Nakamura96] for the definition of $\mathcal{G}((P,\mathcal{L})\otimes k')$ and a similar lemma in the general case.

7. Embedding theorem

The following theorem in the nonsingular case might be known to specialists, which was communicated to the author by T. Katsura with a complete proof.

Theorem 7.1. *Let A be an abelian variety over an algebraically closed field k, L an ample invertible sheaf on A. Suppose that the reduced Heisenberg group $H(L)$ contains the group A_n of all n-torsion points of A for some $n\,(\geq 3)$[14] prime to the characteristic of k. Then L is very ample.*

[14]This is the same as the condition that any elementary divisor of $H(L)$ is at least in the strong sense.

Proof. Let $\Lambda(L)$ be the polarisation homomorphism $\Lambda(L)(x) = T_x^*L \otimes L^{-1}$ $(x \in A)$, $H(L) := \operatorname{Ker}\phi_L$ the reduced Heisenberg group and e_L the Weil pairing. By definition it is clear that $H(L)$ is a subgroup of $H(L^n)$. Let A_n be the group of all n-torsions of A. By the assumption $A_n \subset H(L)$. Since $e_{L^n}(x,y) = e_L(x,y)^n = 1$ $(\forall x, y \in A_n)$, L^n descends to $A/A_n \simeq A$ by [Mumford66-67, I, p.291]. That is, there exists an ample invertible sheaf M on A such that $L^n = (n_A)^*(M)$ where n_A is the multiplication morphism of A by n. By [Mumford74, p.59],

$$L^n = (n_A)^*M \simeq M^{(n^2+n)/2} \otimes i^*(M)^{(n^2-n)/2}$$

where i is the inversion of A.

Next we will prove that $L = (M')^n$ for some ample invertible sheaf M' on A. For this we will prove $i^*(M) \otimes M^{-1} \in \operatorname{Pic}^0(A)$. First note that $i^*N \otimes N = (i + id_A)^*N = \mathbf{1}_A$ (trivial bundle on A) for any $N \in \operatorname{Pic}^0(A)$, hence $i^*N = N^{-1}$. Then for any $x \in A$, we have

$$\Lambda(i^*M)(x) = i^*(\Lambda(M)(i(x))) = -\Lambda(M)(-x) = \Lambda(M)(x).$$

Hence $i^*M \otimes M^{-1} \in \operatorname{Pic}^0(A)$. Since $\operatorname{Pic}^0(A)$ is a divisible group, there exists an invertible sheaf $G' \in \operatorname{Pic}^0(A)$ such that $L = (M \otimes G')^n$. By Lefschetz's theorem, L is very ample because $n \geq 3$. □

We can also prove

Theorem 7.2. *Let $(P_0, \mathcal{L}_0)$ be a polarised stable quasi-abelian variety over an algebraically closed field k, $H(P_0, \mathcal{L}_0)$ the reduced Heisenberg group. Suppose that any elementary divisor of the finite abelian group $H(P_0, \mathcal{L}_0)$ is at least three in the strong sense, then the complete linear system $|\mathcal{L}_0|$ is base-point free. If the Delaunay decomposition* $\operatorname{Del}(P_0)$ *is simplicially generating,*[15] *hence in particular if the dimension of the toric part of P_0 is less than 5, then $\mathcal{L}_0$ is very ample.*

8. Stability

8.1. Let $(P_0, \mathcal{L}_0)$ be a stable quasi-abelian variety over k. Suppose that $\mathcal{L}_0$ is very ample. Then for any positive integer n we have an epimorphism

$$\phi_n : S^n H^0(P_0, \mathcal{L}_0) \to H^0(P_0, \mathcal{L}_0^n)$$

which determines a point of the Grassmannian variety. Let $n(g) := n^g \deg(\mathcal{L}_0)$.

By taking the Plücker coordinates we obtain a point of the projective space

$$\overset{n(g)}{\wedge} \phi_n : \overset{n(g)}{\wedge} S^n H^0(P_0, \mathcal{L}_0) \to \overset{n(g)}{\wedge} H^0(P_0, \mathcal{L}_0^n) \simeq k.$$

[15]Let a_i $(i \in I)$ be Delaunay vectors of σ with $0 \in \sigma$, $C(0,\sigma) = \sum_{i \in I} \mathbf{R}_+ a_i$. "Simplicially generating" means roughly that the semi group $C(0,\sigma) \cap X$ is generated by a_i [AN96, 1.12].

We call $\overset{n(g)}{\wedge}\phi_n$ *the n-th Hilbert point of* $(P_0, \mathcal{L}_0)$. It is not difficult to see that ϕ_n is a $\mathcal{G}(P_0, \mathcal{L}_0)$-equivariant homomorphism so that so is $\overset{n(g)}{\wedge}\phi_n$. Since $\mathcal{G}(P_0, \mathcal{L}_0)$ has no nontrivial one dimensional representation, $\overset{n(g)}{\wedge}\phi_n$ is $\mathcal{G}(P_0, \mathcal{L}_0)$-invariant.

By Lemma 6.10 the following is a corollary to [Kempf78].

Theorem 8.2. *Let* $(P_0, \mathcal{L}_0)$ *be a polarised stable quasi-abelian variety over an algebraically closed field* k. *Then the Hilbert points of* $(P_0, \mathcal{L}_0)$ *are stable if the characteristic of* k *and* $\deg \mathcal{L}_0$ *are coprime and if* $\mathcal{L}_0$ *is very ample.*

Proof. $\mathcal{G}(P_0, \mathcal{L}_0)$ operates linearly on $H^0(P_0, \mathcal{L}_0)$ keeping the Hilbert points of $(P_0, \mathcal{L}_0)$ invariant. Since $H^0(P_0, \mathcal{L}_0)$ is an irreducible $\mathcal{G}(P_0, \mathcal{L}_0)$-module by Lemma 6.10, $\mathcal{G}(P_0, \mathcal{L}_0) \cap SL(\deg(\mathcal{L}_0), k)$ is contained in no parabolic subgroup of $SL(\deg(\mathcal{L}_0), k)$. By [Kempf78, Corollary 5.1], the Hilbert points of $(P_0, \mathcal{L}_0)$ are stable in the sense of Mumford. $\square$

This is enough for constructing the complete moduli of abelian varieties up to dimension 4 in the subsequent formulation. As we remarked in the end of the paragraph 1.1, the Hilbert points of $(P_0, \mathcal{L}_0)$ is not necessarily properly stable.

By [Kempf78] Chow-stability of the image of $(P_0, \mathcal{L}_0)$ follows similarly under the same assumptions.

9. Moduli

The following definition of $\mathcal{SQAV}_{g,K}(= \mathcal{SQAV}_{g,K}^{\text{temporary}})$ will simplify $\mathcal{SQAV}_g$ in [Alexeev96].

Definition 9.1. Let H be a finite abelian group, $e_H : H \times H \to k^*$ a skew symmetric bimultiplicative form. The pair (H, e_H) is called *a symplectic finite abelian group* if e_H is nondegenerate. If (H, e_H) be a symplectic finite abelian group, then there exists a maximal totally isotropic subgroup K of H such that $H \simeq K \oplus \hat{K}$ and $\hat{K} \simeq \mathrm{Hom}_{\mathbf{Z}}(K, k^*)$. Hence in particular $|H| = |K|^2$.

Conversely let K be a finite abelian group. We set $\hat{K} := \mathrm{Hom}_{\mathbf{Z}}(K, k^*)$ and $H := H(K) = K \oplus \hat{K}$. We define $e_H : H \times H \to k^*$ by $e_H(a \oplus \alpha, b \oplus \beta) = \alpha(b)\beta(a)^{-1}$ where $a, b \in K$, $\alpha, \beta \in \hat{K}$. Then it is clear that (H, e_H) is a symplectic finite abelian group. We denote e_H by ℓ_K when it is necessary to emphasize dependence on K.

Let $\mathcal{G}(K)$ be a group defined by the group law

$$(a, z, \alpha) \cdot (b, w, \beta) = (ab\beta(z), zw, \alpha\beta)$$

where $a, b \in k$, $z, w \in K$ and $\alpha, \beta \in \hat{K}$. It is clear that $\mathcal{G}(K)$ contains K as a level subgroup, that is, as a maximal isotropic subgroup.

Suppose that the characteristic of k and the order $|K|$ of K are coprime.

Definition 9.2. A triple $(P, \mathcal{L}, \mathcal{K})$ is called a polarised stable quasi-abelian variety over k with level structure K if the following conditions are satisified

(1) $(P, \mathcal{L})$ is a polarised stable quasi-abelian variety over k
(2) Let $\mathcal{G}(P, \mathcal{L})$ be the total Heisenberg group of $(P, \mathcal{L})$ with Weil pairing $e_{\mathcal{L}}$. Then the triple $(\mathcal{G}(P, \mathcal{L}), \mathcal{K}, e_{\mathcal{L}})$ is isomorphic to $(\mathcal{G}(K), K, \ell_K)$.

The condition (2) implies that $|\mathcal{K}| = \deg L$.

Given a noetherian k-scheme T, $(P, \mathcal{L}, \mathcal{K})$ is called a polarised stable quasi-abelian scheme of relative dimension g over T with level structure K if

(1) $(P, \mathcal{L})$ is a flat proper T-scheme with a relatively ample invertible sheaf $\mathcal{L}$,
(2) $\mathcal{K}$ is a flat finite reduced subgroup scheme of $\mathrm{Aut}_T(P, \mathcal{L})$
(3) for any geometric point s of T, $(P_s, \mathcal{L}_s)$ is a polarised stable quasi-abelian variety of dimension g over $k(s)$ with level structure $\mathcal{K}_s \simeq K$.

The condition (3) implies

$$(\mathcal{G}(P_s, \mathcal{L}_s), \mathcal{K}_s, e_{\mathcal{L}_s}) \simeq (\mathcal{G}(\mathcal{K}_s), \mathcal{K}_s, \ell_{\mathcal{K}_s}) \simeq (\mathcal{G}(K), K, \ell_K)$$

We define a functor $\mathcal{SQAV}_{g,K}$ $(= \mathcal{SQAV}_{g,K}^{\text{temporary}})$ as follows. For any noetherian k-scheme T, we set

$$\begin{aligned} \mathcal{SQAV}_{g,K}(T) = & \text{ the set of polarised stable quasi-abelian } T\text{-schemes} \\ & (P, \mathcal{L}, \mathcal{K}) \text{ of relative dimension } g \text{ with level structure } K \\ & \text{modulo } T\text{-isom with } \mathcal{K} \text{ forgotten}. \end{aligned}$$

As the readers may see, the part *modulo $T-$isom with $\mathcal{K}$ forgotten* is unnatural in the definition of the functor, which should be replaced by *modulo $T-$isom.* Therefore the formulation here might be changed in the near future.

It follows from Theorem 8.2 and Mumford [MFK94].

Theorem 9.3. *Let k be an algebraically closed field. Suppose that the characteristic of k and the order of K is coprime and that any elementary divisor of K is at least 3 in the strong sense* [15]. *Then the functor $\mathcal{SQAV}_{g,K}$ is coarsely represented by a projective scheme over k if $g \leq 4$.*

Projectivity (and properness) follows from [MFK94] and Theorem 0.3 plus the definition by Lemma 6.8.

[15] See Theorem 0.6.

Let ζ_{e_g} be a primitive e_g-th root of unity. By modifying the above functor into the over-$\mathbf{Z}[\zeta_{e_g}, 1/|K|]$-version [16] , and by applying [Seshadri77, Theorems 2,3,4,pp.263-269] we infer from the above theorem

Theorem 9.4. *Suppose that $g \leq 4$ and any elementary divisor of K is at least 3 in the strong sense. Then the functor $SQAV_{g,K}$ is coarsely represented by a projective scheme over $\mathbf{Z}[\zeta_{e_g}, 1/|K|]$.*

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