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# LOW RANK VECTOR BUNDLES FORMED FROM HYPERPLANE CONFIGURATION

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In Kinosaki symposium I talked on a new rank-3 bundle on the four dimensional complex projective space $\mathbb{P}_4(\mathbb{C})$. This will appear as a joint paper with H.Abo and W.Decker:

A rank 3 bundle which gives rise to an elliptic conic bundle

This will be written up soon but is not yet completed. Thus I like to summarize our knowledge on low rank vector bundles on $\mathbb{P}_n(\mathbb{C})(n \geq 4)$. (My version is limitted to algebraic geometrical or complex analytic aspect.)

As is well known, the theory of the vector bundles on $\mathbb{P}_2(\mathbb{C})$ and $\mathbb{P}_3(\mathbb{C})$ is well undersood. (as is founed, for example, in works of W.Barth,K.Hulek,Le Poitier

for $\mathbb{P}_2$ and in those of G.Horrocks, M.Atiyah-E.Ree, R.Hartshorne, M.Maruyama-G.Trautmann for $\mathbb{P}_3$.)

However, the situation is largely different for $\mathbb{P}_n(n \geq 4)$. The following seems to be a common impression for these subjects.

There are not so many low rank bundles, but such bundles, if found, must be of interesting nature.

In this article "vector bundle" is used for "indecomposable one". It is of low rank if the rank < the dimension of the underlying manifold.

## § 1 Low rank bundles.

First I list all low rank vector bundles on $\mathbb{P}_n(n \geq 4)$, which have been obtained hitherto

(1) Null correlation bundle and its generalization, mathematical instanton bundle.(This is of rank $2n$ defined on $\mathbb{P}_{2n+1}$(cf.[OSS] and [S-T])

(2) A rank $(n-1)$-bundle on $\mathbb{P}_n$ found by Tango(cf.[Ta])

(3) A rank 3 bundle on $\mathbb{P}_5$ found by Horrocks(cf/[Ho])

(4) A rank 2 bundle on $\mathbb{P}_4$ found by Horrocks-Mumford(cf.[H-M])

The first example may be known for a long time.(I do not know

the history of this bundle. My impression may be wrong.) It may be pointed out that (2),(3) and (4) are published respectively in 1976, 1978 and 1973.(Thus, for a long time, such bundles have been not found.) Investigating these bundles, a first impression of me and other persons is that it is not difficult to find other ones. But, as a matter of fact, it turns out to realize that it is a hard problem after some trials.

Among many interesting properties of (1), we may point out its relation to quaternion algebra and twister geometry.

I understand that Tango found (2) through a construction of morphism of $\mathbb{P}_n$ to a Grassmannian manifold. Also it is remarked that Ein(cf.[E])found that (2) yields quintic elliptic scroll.

The third one (3) is, I understand, constructed from the null correlation bundle. Since I have not investigated this so closely, an interested reader may read the original version and some related papers

The final one (4), abbreviated as HM-bundle, is a very beautiful and famous subject. Many interesting studies have been done for it. For the relavant informations for these, see [Hu]. A most brilliant feature of this is, usually regarded, the existence of a big symmetry. The fact that the general section is abelian surface(and so, needless to say, an irregular surface.) We should also point out that (3) and (4) are found by means of monad, combined with represntation theory.

On the otherhand, surfaces of low degree in $\mathbb{P}_4$ have been studied systematically by some authors.(See, for example, [DES], [Po], [ADHPR].)They found that irregular surfaces in $\mathbb{P}_4$ are rather rare and, in many cases, they are obtained from interesting sheaves.

## § 2 Hyperplane configuration.

I began my trial to contruct bundles and sheaves, by using ideas in stratification theory at the middle of 1980(cf.[Sa1]). A very naive idea is to finsd good frames and cocycle relations among them(transition matrix). It is seen that HM-bundle has also a beautiful propery and is reconstructed from hyperplane configuration of $\mathbb{P}_4$, which corresponds to the homogeneous coordinates of it. Soon after Kaji found that the null correlation bundle on $\mathbb{P}_3$ is constructed from two hyperplanes in $\mathbb{P}_3$. It is seen that the null corrrelation bundle on $\mathbb{P}_{2n+1}$ is constructed from $(2n-1)$-hyperplanes. We do not think that the relation between sheaf construction and hyperplane configuration is accidental. Also we should take account into the theory of hypergeometic function, which is formed from these objects.

**Case of rank two.** The construction of HM-bundles uses monomials $x_1x_4$ and $x_2x_3$(cf.[HM]). Such monomials appear also in the transition matrix of it. It may be natural to try to find similar data to

construct more bundles and sheaves. After trials I was aware of the fact that

$$\{1,4\} \text{ and } \{2,3\} \text{ coincides with } (\mathbb{F}_p^*)^2 \text{ and } \mathbb{F}_p - (\mathbb{F}_p^*)^2.$$

I noticed this in 1991. Some discussions with Kyoji Saito were useful; see [SEKS]. It is easy to see that a similar rank two sheaf $\mathfrak{E}_p$ on $\mathbb{P}_{p-1}(\mathbb{C})$ is defined for each prime $p \equiv 1(\mathrm{mod}.4)$ I believe that this is a very beautiful subject. The construction of this sheaf and some properties are reported in [AS] in the Sendai symposuim(1996). In giving our main result, the formula $c_4$, we omitted to divide by 4.(This was pointed out my classmate, Mrs Murakami.) The formula is as follows.

$$c_4(\mathfrak{E}_p) = -(5/64)p(p-1)(p^2-6p+1+4a_p^2),$$

where an odd integer $a_p$ is determined(uniquely upto sign) by the equation:$p = a_p^2 + b_p^2$ with an even integer $b_p$. This implies at once the following characterization of the prime number $p = 5$ among all primes $p \equiv 1(\mathrm{mod}.4)$

**Corollary.** *The following four facts are equivalent*

$p = 5$, $\mathbb{F}_p$ *does not contain four elements* $i, j, k, l$ *whose graph is isomorphic to square,*
$c_4(\mathfrak{E}_p) = 0$, $\quad$ *cod*$(Sing(\mathfrak{E}_p), \mathbb{P}) \geq 5$.

It is very difficult to find rank two bundle on $\mathbb{P}_4$. The following may be a common idea of many geometers.

There are no rank two indecomposable bundle on $\mathbb{P}_4$
which are not a satellite bundle of the HM-bundle

In view of [D],[DS] "satellite" may be undersood to be a twist or the pullback by means of a finite map:$\mathbb{P}_4 \to \mathbb{P}_4$.

**Case of rank three.** This is constructed as follows(cf.[Sa 2]). Let $x = (x_0, x_1, x_2, x_3, x_4)$ be the homogeneous coordinates of $\mathbb{P} = \mathbb{P}_4(\mathbb{C})$. We set $L_i = \mathcal{Z}(x_i)$ We work with the four hyperplanes: $L^1 = \cup_{i\in\Delta} L_i$ with $\Delta = \{1,2,3,4\}$.Also , to each $i \in \Delta$, we attach a vector in $\Gamma(\mathfrak{O}_{\mathbb{P}}^{\oplus 3}(4))$

$$\mathfrak{f}_1 = {}^t[\psi_1,\ x_3x_2,\ x_3x_4],\ \mathfrak{f}_2 ={}^t[\psi_2, x_4x_1, x_4x_3]$$
$$\mathfrak{f}_3 = {}^t[\psi_3,\ x_1x_4,\ x_1x_2]\ ,\mathfrak{f}_4 =\ t[\psi_4,\ x_2x_3,\ x_2x_1],$$

where

$$\psi_1 = x_0^2 + x_2x_4,\ \psi_2 = x_0^2 + x_1x_3,\ \psi_3 = x_0^2 - x_2x_4 \text{ and } \psi_4 = x_0^2 - x_1x_3.$$

Now form an $\mathfrak{O}_{\mathbb{P}}$-submodule $\mathfrak{E}$ of $\mathfrak{O}_{\mathbb{P}}^{\oplus 3}(4)$ as follows:first we set

$$\mathfrak{E}_{|\mathbb{P}-L^1} = \mathfrak{O}_{\mathbb{P}-L^1}^{\oplus 3}(4).$$

Next let $x$ be an arbitrary point of $L^1$. Then the $\mathfrak{O}_{\mathbb{P},x}$-module $\mathfrak{E}_x$ is defined as follows. An element $frakf \in \mathfrak{O}_{\mathbb{P}}^{\oplus 3}(4)_x$ is in $\mathfrak{E}_x$ if the following holds for any $i \in \Delta$ satisfying $x \in L_i$.

$$\text{There is an element } c_i \in \mathfrak{O}_i(2)_x \text{ so that } \ \zeta_{|i} = c_i\mathfrak{f}_{i|i}. \tag{2}$$

*Remark.* The pair $(L_i, \mathfrak{f}_i)$ defines an elementary transformation in the sense of Mayuyama-Sumihiro. There is no doubt that it is possible to take this theory as a starting point for the construction of $\mathfrak{E}$

**Multiplier.** Take distinct elements $i, j \in \Delta$.Then (2) ensurres the relation

$$c_i\mathfrak{f}_i = c_j frakf_j \text{ on } \ L_{ij} = L_i \cap L_j.$$

Analyzing this we can write

$$c_i = \varphi_{i,ij}(c_{ij}),\ \ c_j = \varphi_{j,ij}(c_{ij}) \text{ on } L_{ij}$$

where, writing $\varphi_{i,j}$ and $\varphi_{j,i}$ for $\varphi_{i,ij}$ and $\varphi_{j,ij}$, these are given by the following matrix.

$$\begin{pmatrix} 0 & 1 & \psi_3 & 1 \\ 1 & 0 & 1 & \psi_4 \\ \psi_1 & 1 & 0 & 1 \\ 1 & \psi_2 & 1 & 0 \end{pmatrix}$$

where the $(i,j)$ component $= \varphi_{i,j}(i \neq j)$. This is the *muktiplier* of $/frakE$. In constructing sheaves by using a hyperplane configuration, such a matrix is defined. This is the most basic notion in our construction, which enables us a through use of combinatorial arguments. In the caase of the HM-bundle the multiplier is very clear, and, as was pointed out by Hulek, it is an example of Moore matrix of the HM-bundle. Now we see that

$$(L_{ij} - L^3) \cap Z(\varphi_{i,j}) \cap Z(\varphi_{j,i}) \subset Sing(\mathfrak{E}),$$

where $L^d = \cup_{I \subset \Delta \|I = d} L_I$ with $L_I = \cap_{i \in I} L_i$. This statement is valid in a full generality. Moreover, if the rank of the sheaf $\mathfrak{F}$ in question equals 2, then we have:

$$L_{ijk} \cap (Z(\varphi_{i,j}) \cap Z(\varphi_{j,i})) \subset Sing(\mathfrak{F})$$

On the otherhand, in the case of $\mathrm{rk}(\mathfrak{F}) = 3$, this condition can be cancelled. Our naive idea is to use this as best as possible. In doing this our suspection is that

Four point graph, which is isomorphic to square

must be advatageous in the case of rank three in contrast to the case of rank two.

*Remark.* The rank three bundle $E$ is globally generated and

$$c(\mathfrak{E}, t) = 1 + 4t + 8t^2 + 8t^3$$

Thus the second Chern class is represented a surface of degree 8. We(with Abo and Decker), more precisely finally Decker, proved that it is a (non geometrical) elliptic conic bundle. In view of the situation for the irregular surfaces in $BbbP_4$, this is an encouraging fact. Also I hope that this is an evindence for our impression stated at the beginning of this article.

*Remark.* We begin this article eith the example of the null correlation bundle defined on the projective spaces of odd dimension. In connection with this I may ask

If there is an indecomposabel bundle of rank $(2n-1)$ on $\mathbb{P}_{2n}$

which is a generalization of the rank three bundle $\mathfrak{E}$

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