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DOES CHOW GROUP HAVE A SPACE STRUCTURE?

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ABSTRACT. Let S be a surface with $p_g(S) > 0$. In 1969, Mumford proved that its Chow group CH_*S is not "finite dimensional", so it is not representable by an algebraic variety. In this article, we give some evidence that Chow group might have some space structure, motivating us to try to define a new notion of "Spaces".

CONTENTS

1. A BRIEF HISTORY OF SPACE

1. Grothendieck defined the notion of schemes, which has been the standard notion of "spaces" for algebraic geometers for 30 years. The notion of schemes should be clear to the reader of this article.
2. M. Artin and Knutson [11] defined the notion of algebraic spaces. This notion is based on Yoneda's Lemma: Let $\mathcal{C}$ be a category, and consider the covariant functor from $\mathcal{C}$ to $\mathrm{Hom}(\mathcal{C}^{op}, \mathcal{S}ets)$ (the category of contravariant functors from $\mathcal{C}$ to $\mathcal{S}ets$) which sends X to $\mathrm{Hom}(?, X)$. Then Yoneda's Lemma says that this functor is fully faithful, namely, any category $\mathcal{C}$ can be embedded in the category $\mathrm{Hom}(\mathcal{C}^{op}, \mathcal{S}ets)$ as a subcategory. In particular, it implies that $\mathrm{Hom}(\mathcal{C}^{op}, \mathcal{S}ets)$ is the universal ambient category which contains $\mathcal{C}$ as a subcategory.

   So the idea of Artin and Knutson is to find "good objects" in this ambient category, which looks like spaces (for example, covered by schemes étale or flat locally) . In this way, they could

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represent Moishezon spaces which could not be represented by schemes.

3. Deligne and Mumford [3] proposed a different idea, namely the notion of algebraic stacks. They tried to solve the moduli problem (in particular, the moduli of stable curves, but their idea works for just any moduli) . Let M be the moduli of something (e.g., stalbe curves, stable vector bundles, etc.), then we expect that the set $\mathrm{Hom}(X, M)$ corresponds one-to-one to the family of that something, parametrized by X. They observed that the family of stable curves, for example, is not just a set, but could be understood better as a groupoid (a set with an extra structure, namely, some elements may have automorphisms). So they replaced the $\mathcal{S}ets$ in the definition of algebraic spaces with $\mathcal{G}roupoids$, to represent the moduli problem by space-like objects.
4. In 1969, Mumford found an evidence that Chow group cannot have any space structure like $1 \sim 3$ above in general (he called the phenomenon "infinite dimensionality" of the Chow group). For example, when S is a surface with $p_g(S) > 0$, then its Chow group does not have a space structure, as we will see later.

The goal of this article is to give an evidence that Chow groups still behave like spaces, and convince the reader that we need to extend the notion of spaces to understand the Chow groups.

2. Chow group

In this article, we work over the base field $\mathbb{C}$.

Definition 2.1. *Let X be an algebraic variety. Its* subvarieties *are closed integral subschemes of X (or the closures of scheme theoretic points of X with reduced scheme structures).*

Definition 2.2. Algebraic cycles *on X are formal finite linear combinations of subvarieties of X (with $\mathbb{Q}$-coefficients or $\mathbb{Z}$-coefficients), like $\alpha = \sum n_i[V_i]$. When all V_i's are d-dimensional, α is called d-cycle. The algebraic cycles on X form an additive group, and we denote it by $Z_*(X)$, or for d-cycles, $Z_d(X)$.*

One can parametrize algebraic cycles by algebraic varieties. For example, when V is a closed subscheme of $X \times T$ and assume that the morphism $V \to T$ is flat. Then for each $t \in T$, $V_t := V \cap (X \times \{t\}) \subset X$ determines an algebraic cycle (counting the multiplicities). Hilbert schemes are universal among such parametrizations. Chow varieties are also known to parametrize algebraic cycles.

Definition 2.3. *When there is a family of algebraic cycles on X, parametrized by $\mathbb{P}^1$, then we identify any two algebraic cycles in the family. (This is an analogy of homotopy in algebraic topology, where we identify two objects parametrized by the closed interval $I = [0,1]$.) We generate equivalence relation by this indentification, and call it* rational equivalence. *The group of algebraic cycles on X modulo rational equivalence is called* Chow group, *and denoted by CH_*X. We denote the d-cycles part by CH_dX.*

Remark 2.4. Each subvariety of an algebraci variety is topological cycle, so algebraic cycles determine topological cycles. If two algebraic cycles are rationally equivalent, then they are parametrized by $\mathbb{P}^1$, so they are homotopic, hence homologically equivalent. Therefore, there is a natural map $CH_d(X) \to H_{2d}(X)$, which is called the cycle map. Cylce maps are not surjective nor injective in general. When X is connected, $H_0(X) = \mathbb{Z}$, and for $\alpha \in CH_0(X)$, $\deg(\alpha)$ is its image in $\mathbb{Z}$ by the cycle map.

Example 2.5. Let C be an algebraic curve. Then Theorem of Abel-Jacobi says that there is a morphism $S^nC \to J(C)$, where $S^nC = \overbrace{C \times C \times \cdots \times C}^{n-times}/\mathfrak{S}_n$ is the n-th symmetric product of C, and $J(C)$ the Jacobian variety of C. Moreover, when n is large enough, then S^nC is a projective bundle over $J(C)$.

Because there are no rational curves on Abelian varieties, two points on S^nC can be connected by a (chain of) rational curves if and only if they have the same image in $J(C)$. Using this fact, one can show that $CH_0(C) \simeq \mathbb{Z} \oplus J(C)$, and $J(C)$ is exactly the degree 0 part in $CH_0(C)$.

When X is an algebraic surface with $p_g(X) > 0$, in 1969 in [14], Mumford proved that $CH_0(X)$ is "infinite dimensional" in the following sense: Consider the set theoretic map $S^nX \to CH_0(X)$, and assume that $W \subset S^nX$ is mapped to one point in $CH_0(X)$, then Mumford proved that $\dim(W) \leq n$. If CH_0X is representable by an algebraic variety, then it implies that $\dim CH_0(X) \geq n$ (for all n!), hence the naming "infinite dimensional".

Remark 2.6. Conversely, when X is an algebraic surface with $p_g(X) = 0$, then Bloch conjectures that $CH_0(X) \simeq \mathbb{Z} \oplus Alb(X)$. In general, there is a canonical surjection $CH_0(X) \to \mathbb{Z} \oplus Alb(X)$. The kernel of this map is called the Albanese Kernel.

Side Story 2.7. *In some cases, Bloch's conjecture is verified. In particular, Minase-Mizukami [12] found an example of an algebraic surface X of general type with $p_g(X) = 0$ and $CH_0(X) = \mathbb{Z}$. Roughly, this* *means that any two points in the n-th symmetric product S^nX are connected by a chain of rational curves in the symmetric product. On the other hand, when X is a surface of general type, its symmetric products are also of general type (by Riemann-Roch), hence rational curves do not cover S^nX. So there is no rational curve through a generic point.*

What's going on?

The answer is this: Fix a point $P \in X$, which determines an imbedding $S^nX \to S^{n+1}X$ by "addition" of the point P. Iterating this imbedding, for any $n < N$, we have an imbedding $S^nX \to S^NX$. When N is large enough compared to n, then any two points in S^nX are connected by a chain of rational curves in S^NX. In other words, consider $S^\infty X$ as the inductive limit of the symmetric products, then $S^\infty X$ behaves like a unirational variety.

So Bloch's conjecture implies that the geometric genus p_gX of a surface X controlls the "Kodaira dimension" of $S^\infty X$.

Let us look closely the behavior of CH_0X when X is an algebraic surface with $p_g(X) > 0$. Let $X = C \times D$, where C and D are algebraic curves with $g(C) > 0$, $g(D) > 0$, hence $p_g(X) = g(C)g(D) > 0$, so by Mumford's result, $CH_0(X)$ is not representable by a finite dimensional variety. In this case, there is a canonical morphism $CH_0C \otimes_{\mathbb{Z}} CH_0D \to CH_0X$ which sends $[P] \otimes [Q] \mapsto [(P,Q)]$. It is surjective, because $CH_0(C \times D)$ is generated by $[(P,Q)]$.

Now we know that $CH_0(C) \simeq \mathbb{Z} \oplus J(C)$, we can decompose the tensor product $(\mathbb{Z} \oplus J(C)) \otimes_{\mathbb{Z}} (\mathbb{Z} \oplus J(D)) = \mathbb{Z} \oplus (J(C) \oplus J(D)) \oplus (J(C) \otimes_{\mathbb{Z}} J(D))$.

The first term $\mathbb{Z}$ maps isomorphically to the degree part $\mathbb{Z}$ in $CH_0(X)$, and the second term also maps isomorphically to $Alb(X)$. Because these two terms are finite dimensional, Mumford's result implies that the image of $J(C) \otimes_{\mathbb{Z}} J(D)$ is not 0. But one can observe that $J(C) \otimes_{\mathbb{Z}} J(D)$ is totally chaos. For example, pick a generic point $P \in J(C)$ and fix it, then the image of $J(D) \simeq \{P\} \otimes J(D)$ in $CH_0(X)$ is not 0. But when $Q \in J(D)$ is a torsion point, then $P \otimes Q = 0$ in $J(C) \otimes_{\mathbb{Z}} J(D)$. The induced topology on $J(C) \otimes_{\mathbb{Z}} J(D)$ is the trivial topology (the only open sets are ϕ and the whole set).

So we cannot expect any classical topology on CH_0X, even for generic point, hence it cannot be represented by schemes, algebraic spaces, nor algebraic stacks.

The rest of this article is devoted to show that there might be some space structure on $CH_0(X)$.

3. Correspondences

There are many homology theories for smooth projective algebraic varieties. For example, Chow groups, usual homology groups, K-theory, étale homology theory, etc. All these homology theories have common properties. For example, they are functorial; when $f : X \to Y$ is a morphism, then we have morphisms f^* and f_* for the homology theories. Also when $\alpha \in X$ is an algebraic cycle, then we have intersection product operation $\alpha \cdot ?$.

Let X and Y be varieties, and $f : X \to Y$ an morphism, $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ the projections, then for the homology theories above, we can recover f^* and f_* by $f^*? = \pi_{X*}([\Gamma_f] \cdot \pi_Y^* ?)$, and $f_* = \pi_{Y*}([\Gamma_f] \cdot \pi_X^* ?)$, where $[\Gamma_f]$ is the graph subvariety of f in $X \times Y$.

From the interpretation above, we realize that if we replace $[\Gamma_f]$ with any algebraic cycle $\alpha \in X \times Y$, we still have some morphism α_* and α^*. This leads to the following definition.

Definition 3.1. *Define* correspondence *from X to Y to be an algebraic cycle $\alpha \in CH_*(X \times Y)$. We have morphisms α_* and α^* for the homology theories, whenever they have push-forwards and pull-backs for smooth projective morphisms and intersection products with algebraic cycles. We denote it by $\alpha : X \vdash Y$.*

Remark 3.2. Samuel's theorem [17] says that rational equivalence is the finest equivalence relation in algebraic cycles, which admits pull-backs, push-forwards, and the intersection products. This justifies our choice that α is taken from the Chow group rather than the group of algebraic cycles.

Definition 3.3. *For correspondences $\alpha : X \vdash Y$ and $\beta : Y \vdash Z$, we define their composition $\beta \circ \alpha : X \vdash Z$ to be $\pi_{XZ*}(\pi_{XY}^* \alpha \cdot \pi_{YZ}^* \beta)$, where π_{XY}, π_{XZ} and π_{YZ} are the projections from $X \times Y \times Z$.*

Remark 3.4. It is easy to check that $(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$ and $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$.

Correspondences are generalizations of morphisms between smooth projective varieties, so we have a category whose objects are smooth projective varieties and whose morphisms are correspondences. In this category, the identity map is $[\Delta_X] \in CH_*(X \times X)$.

4. Motives

We need to introduce the notion of motives. In the rest of this paper, all the Chow group have rational coefficients.

Conjecture 4.1. *(Grothendieck-Murre) Let X be an n-dimensional projective complex manifold. Consider the cycle map*

$$CH_n(X \times X) \to H_{2n}(X \times X, \mathbb{Q}) = \bigoplus H_i(X, \mathbb{Q}) \otimes H_{2n-i}(X, \mathbb{Q})$$

and look at the image of the diagonal subvariety $[\Delta_X]$, and see how it decomposes. In $H_{2n}(X \times X, \mathbb{Q})$, $[\Delta_X] = a_o + a_q + \cdots + a_{2n}$ where $a_i \in H_i(X, \mathbb{Q}) \otimes H_{2n-i}(X, \mathbb{Q})$. Grothendieck [5] conjectured that all these a_i's are algebraic cycles, namely there are algebraic cycles $\tilde{a}_i \in CH_n(X \times X)$ such that they are mapped to a_i by the cycle map.

Murre [15] conjectured that these preimages $\tilde{a}_i$ can be taken so that

$$\tilde{a}_i \circ \tilde{a}_j = \begin{cases} 0 & (i \neq j), \\ \tilde{a}_i & (i = j). \end{cases}$$

Murre's conjecture implies that $id = [\Delta_X]_* = (a_0)_* + (a_1)_* + \cdots + (a_{2n})_*$, so it gives a "canonical" decomposition for any homology theory, corresponding to the dimension decomposition of the topological homology group.

Example 4.2. When $\dim X = 1$, Murre's conjecture holds: $[\Delta_X] = [P \times X] + ([\Delta_X] - [P \times X] - [X \times P]) + [X \times P]$, where P is any closed point on X. This decomposition gives the decomposition of the Chow group into

$$CH_*(X) = \mathbb{Z} \oplus J(X) \oplus CH_1(X)$$

where the first two terms are decomposition of $CH_0(X)$.

The correspondences $\tilde{a}_i$ gives "a direct summand" of the variety in terms of "universal" homology theory.

Definition 4.3. *Define a motive to be a pair (X, α) where X is a smooth projective variety, and $\alpha \in CH_*(X \times X)$ such that $\alpha \circ \alpha = \alpha$ as correspondences.*

In this definition, the condition $\alpha \circ \alpha = \alpha$ roughly means that α is a projector to a "direct summand" of X. Hence the motive (X, α) is a "slice" of X.

Definition 4.4. *For motives (X, α) and (Y, β), we define their direct sum and tensor product by*

$$(X, \alpha) \oplus (Y, \beta) := (X \amalg Y, \alpha \amalg \beta)$$

and

$$(X,\alpha)\otimes(Y,\beta) := (X\times Y, \alpha\times\beta).$$

Definition 4.5. *Morphism between motives $f : (X,\alpha) \to (Y,\beta)$ is a correspondence $f : X \vdash Y$ which satisfies $f = f\circ\alpha = \beta\circ f$. When $M = (X,\alpha)$ is a motive, then its identity morphism is $\alpha : X \vdash X$.*

Definition 4.6. *When X is a smooth projective variety, then the* motive of X *is defined to be $M_X := (X,[\Delta_X])$ (the "whole slice"). When $f : X \to Y$ is a morphism of smooth projective varieties, then it induces $[f] : M_X \to M_Y$.*

Definition 4.7. *When X is a curve, the motives $(X,[X\times P])$, $(X,[\Delta_X]-[P\times X]-[X\times P])$ and $(X,[P\times X])$ are denoted as $h^0(X)$, $h^1(X)$ and $h^2(X)$ respectively.*

Definition 4.8. *Let $M = (X,\alpha)$ be a motive, and H be a homology theory, then we define the homology of M by $H(M) := \alpha_*(H(X))$.*

Example 4.9. Let C be a curve, and $h^i(C)$ be as in Definition 4.6. Then $H_*(h^i(C)) = H_i(X)$ for the topological homology. Also

$$CH_*(h^i(X)) = \begin{cases} \mathbb{Z}\subset CH_1(C) & (i=0)\\ J(C)\subset CH_1(C) & (i=1)\\ \mathbb{Z}\subset CH_0(C) & (i=2)\end{cases}$$

5. Bivariant Space

Some motives behave like the motives of some varieties. We define such motives as "bivariant spaces", defined as follows.

Definition 5.1. Bivariant Space *is a motive M with the morphisms*
i) the diagonal map $\Delta_M : M \to M\otimes M$
ii) the structure map $\pi_M : M \to M_{pt}$
which makes the following diagrams commute.

(1)
$$\begin{array}{ccc} M & \xrightarrow{\Delta_M} & M\otimes M \\ {\scriptstyle id_M}\downarrow & & \downarrow{\scriptstyle id_M\otimes\pi_M} \\ M & = & M \end{array}$$

(2)
$$\begin{array}{ccc} M & \xrightarrow{\Delta_M} & M\otimes M \\ {\scriptstyle id_M}\downarrow & & \downarrow{\scriptstyle \pi_M\otimes id_M} \\ M & = & M \end{array}$$

$$
\begin{array}{ccc}
M & \xrightarrow{\Delta_M} & M \otimes M \\
{\scriptstyle \Delta_M}\downarrow & & \downarrow{\scriptstyle id_M \otimes \Delta_M} \\
M \otimes M & \xrightarrow[\Delta_M \otimes id_M]{} & M \otimes M \otimes M
\end{array} \tag{3}
$$

Example 5.2. When X is a smooth projective variety, then $(X, [\Delta_X])$ has a canonical bivariant space structure.

Definition 5.3. *Let $f : M \to N$ be a morphism of motives. When M and N are bivariant spaces, then f is a morphism of bivariant spaces when the following diagrams commute.*

$$
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
{\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_N} \\
M_{pt} & = & M_{pt}
\end{array} \tag{4}
$$

$$
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
{\scriptstyle \Delta_M}\downarrow & & \downarrow{\scriptstyle \Delta_N} \\
M \otimes M & \xrightarrow[f \otimes f]{} & N \otimes N
\end{array} \tag{5}
$$

Example 5.4. When $f : X \to Y$ is a morphism of smooth projective varieties, then $[f] : M_X \to M_Y$ is a morphism of bivariant spaces. In general, morphisms of bivariant spaces between motives of smooth varieties do not always come from morphisms of schemes [8], [9].

Theorem 5.5. *Let X be a smooth projective variety, and A an Abelian variety. Then we have*

$$
\mathrm{Hom}_{\ BivariantSpaces}(M_X, M_A) \simeq \mathrm{Hom}_{\ Varieties}(X, A)/Torsion
$$

where torsion is the group of the constant morphisms to the torsion points.

The proof is in [8]. Rough idea goes like this: Let $\alpha : M_X \to M_A$ be a morphism of bivariant spaces. Consider $\alpha \in CH_*(X \times A)$ as an element of the Chow group, and consider $X \times A$ to be a relative Abelian scheme over X. Then we can define the Pontrjagin products in $CH_*(X \times A)$ by the group scheme structure, and can define

$$
\log(\alpha) := (\alpha - 1) - \frac{(\alpha-1)^2}{2} + \frac{(\alpha-1)^3}{3} - \frac{(\alpha-1)^4}{4} + \cdots
$$

where the power is defined by the Pontrjagin products. Then the diagram (4) implies that all but finite terms vanish (hence this infinite sum makes sense), and the diagram (5) implies that $2_{X\times A*}\log(\alpha) = 2\alpha$, where $2_{X\times A} = id_X \times 2_A : (X \times A) \to (X \times A)$ the multiplication by 2. Then Mukai-Beauville's Fourier transform [13] and [1] sends $\log(\alpha)$ to $\mathcal{F}(\log(\alpha))$, which is a class of topologically trivial line bundle on $X \times \hat{A}$, which determines a morphism from X to A.

We have to divide it by the torsion, because we tensor the Chow group with $\mathbb{Q}$.

Theorem 5.6. *(Shermenev [18]) Let C be a curve, and take the motive $h^1(C)$ as in Definition 4.7. Then the motive of the Jacobian variety is isomorphic to the symmetric algebra of $h^1(C)$, namely we have*

$$M_{J(C)} \simeq Sym^* h^1(C).$$

Recall that $CH_*(h^1(C)) = J(C)$ by Example 4.9. We can recover the space structure of its Chow group (as an algebraic space, extending the category of smooth projecitve varieties) by Theorems 5.5 and 5.6.

6. Representing the space

Now, let us come back to the case $X = C \times D$ where C and D are smooth projective curves of positive genera, and consider the motive $M = h^1(C) \otimes h^1(D)$. We assume that there are no non-trivial morphisms between the Jacobians $J(C)$ and $J(D)$, so that $CH_*(M)$ is exactly the Albanese Kernel in $CH_0(X)$. This part is non-zero and "infinite dimensional" by Mumford's theorem.

Proposition 6.1. *Let us consider the wedge product of the motive M. Then we have* $\bigwedge^k M \simeq \begin{cases} M_{pt} & (k = 4g(C)g(D)) \\ 0 & (k > 4g(C)g(D)) \end{cases}$

Proof. Shermenev's result implies that

$$Sym^i h^1(C) = \begin{cases} M_{pt} & (i = 2g(C)) \\ 0 & (i > 2g(C)) \end{cases}$$

and similarly for D. From this, one can prove the proposition mimicking the proof that the tensor product of two finite dimensional vector spaces is again finite dimensional. For details, see [7] □

On the other hand, $Sym^k M \neq 0$ for any large k, hence in order to mimic the case of curves, it is natural to use the exterior algebra rather than the symmetric algebra.

Proposition 6.2. *The exterior algebra $\bigwedge^* M$ has a structure of bivariant space.*

Proof. Let $\check{M}$ be the dual motive of M (which turns out to be isomorphic to M). Then we have the canonical morphism $\bigwedge^i \check{M} \otimes \bigwedge^j \check{M} \to \bigwedge^{i+j} \check{M}$, whose dual gives $\bigwedge^k M \to \bigoplus_{i+j=k} \bigwedge^i M \otimes \bigwedge^j M$, which determines the diagonal map $\bigwedge^* M \to \bigwedge^* M \otimes \bigwedge^* M$. Also the isomorphism $\bigwedge^{4g(C)g(D)} M \simeq M_{pt}$ determines π_M. One can easily check the axioms. □

We hope that $CH_*(M)$ is a space, and the bivariant space $\bigwedge^* M$ at least approximates the space structure. We have some evidence:

Theorem 6.3. 1. *We have a canonical isomorphism*

$$\mathrm{Hom}_{\;BivariantSpace}(M_{pt}, \bigwedge^{*} M) \simeq CH_*(M)$$

Hence, when Y is a smooth projective variety, for a point $y \in Y$ and a morphism of bivariant spaces $f \in \mathrm{Hom}_{BivariantSpace}(M_Y, \bigwedge^* M)$, $f(y)$ is defined as the element of $CH_*(M)$ which corresponds to the composition $M_{\{y\}} \to M_Y \to \bigwedge^* M$.

2. *For $f \in \mathrm{Hom}_{BivariantSpace}(M_Y, \bigwedge^* M)$, $f = 0$ if and only if $f(y) = 0$ for all the closed points $y \in Y$.*
3. *When $f \in \mathrm{Hom}_{BivariantSpace}(M_Y, \bigwedge^* M)$, there exist two morphisms $F, G : \tilde{Y} \to S^n(X)$, and a birational map $\tilde{Y} \to Y$ such that for any $y \in Y$, take any preimage $\tilde{y} \in \tilde{Y}$, then $F(\tilde{y}) - G(\tilde{y}) = f(y) \in CH_0(M) \subset CH_0(X)$.*

   Conversely, when $F, G : \tilde{Y} \to S^n(X)$ are morphisms, with $\tilde{Y} \to Y$ birational, and if $F(\tilde{y}) - G(\tilde{y})$ as above is always contained in $CH_*(M)$, then there exists unique $f : Y \to bigwedge^* M$ as bivariant spaces such that $F(\tilde{y}) - G(\tilde{y}) = f(y)$.

Proof. For the most part, we can simply mimic the proof of Theorem 5.5. In this case, we do not have to divide by the torsion, because Roitman's theorem [16] implies that there are no torsion in $CH_*(M)$. For $\alpha \in \mathrm{Hom}_{BivariantSpace}(M_Y, \bigwedge^* M)$, the Fourier transform of $\log(\alpha)$ determines an element $\beta \in CH^2(Y \times C \times D)$, such that $\beta = \sum n_i[V_i]$ with each V_i generically flat over Y. This information determines two rational maps (gathering the positive coefficient part and the negative coefficient part) $F, G : Y \to S^n X$. For details, see [10] □

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