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VIEW ON 4-DIMENSIONAL SMALL CONTRACTIONS AND FLIPS — LA TORRE PENDENTE —

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This note is a survey of the article [Kac3]. We work everything over $\mathbb{C}$.

The *Minimal Model Program* is a device for telling us a procedure how to approach a *minimal model* starting from a given algebraic variety, by a sequence of special birational transforms;

$$X \longrightarrow X_1 \dashrightarrow X_2 \cdots \longrightarrow X_n = X_{\min}$$

([Kaw1,2,3,6], [Kol], [Mo1,2], [Re0,1], [S1,2]). For surfaces, this mechanism has been classically known by the works of the Italian school (Enriques, Castelnuovo, et.al) in the 19 century, followed by the modernization of Zariski. In fact this case the process is quite simple, just to repeat contracting (-1)-*curves* to reach the minimal model $X_{\min}$ of X. In dimension 3 or more however it came a hard obstacle, the appearance of so-called *small contractions* or *flipping contractions*, meaning those birational contractions contracting cycles of codimensions at least 2 (Definition below). These are pretty complicated to handle with, and it turned out that the achievement of the whole program is concentrated on investigating this kind of birational morphisms (see [Re1], [Kaw2]). For a detailed explanation of the story see for instance [KaMaMa]. Here let us just give the precise definition of those small contractions:

Definition. (Flips)

Let $g\colon X \to Y$ be a proper birational morphism between normal algebraic varieties (or normal analytic spaces) of dimension n. Let $E := \operatorname{Exc} g$, $B := g(E)$.

$$\text{Assume} \quad \begin{cases} X \text{ has at worst } \textit{terminal singularities}([\text{Re1}]), \\ \dim E \leq n-2, \quad \text{and} \\ -K_X \text{ is } g\text{-ample} \quad (\textit{namely } (K_X \,.\, C) < 0 \text{ whenever } g(C) \text{ is a point.}) \end{cases}$$

Then g is called a *small contraction*, or a *flipping contraction*.

If there exists another proper birational morphism $X^+ \xrightarrow{g^+} Y$ (with the common target space Y), with $E^+ := \operatorname{Exc} g^+$, such that

$$\begin{cases} X^+ \text{ has at worst terminal singularities,} \\ \dim E^+ \leq n-2, \text{ and} \\ K_{X^+} \text{ is } g^+\text{-ample,} \end{cases}$$

then g^+ is called *the flip* of g.

By abuse of language sometimes the composite birational map (transform) $g^{+\,-1} \circ g : X \dashrightarrow X^+$ is also called a *flip*. —

The first concrete example of such a transform is given by P. Francia [Fra]. In general, to find such g^+ is a very hard question, and the Program, combined with *Kawamata–Shokurov contraction theorem*, says the existence problem of minimal models has been reduced, in an arbitrary dimension, to the following couple of statements called the *flip conjecture*:

Flip Conjecture.

(E) *(Existence) For a small contraction g the flip g^+ exists.*

(T) *(Termination) There is no infinite sequence of flips (starting from a projective X):*

$$X \dashrightarrow X^+ \dashrightarrow X^{++} \dashrightarrow \ldots.$$

This is still conjectural in $n \geq 4$, $n \geq 5$, respectively.

In $n = 3$, the statement (T) is first proved by Shokurov [S1], while (E) is also investigated by several people, especially Tsunoda, Shokurov, and Kawamata [Kaw3] proved this for the case of *semi-stable degenerations* of surfaces. By applying the criterion of [Kaw3], Mori [Mo2] then settled this for the general case in $n = 3$, and this way the existence of minimal models has been solved affirmatively in dimension 3.

In dimension 4 on the contrary, very little is known in this direction. Actually (T) is generalized by Kawamata–Matsuda–Matsuki ([KaMaMa] §5), while as for (E) nothing definite has been known since the characterization theorem of Kawamata [Kaw4] in 1989 for the *smooth* 4-fold case. To state his result let us make one convention;

Assumption. As long as the existence part (E) is concerned, the problem is local on Y, so we may assume that X is a sufficiently small analytic neighborhood of the compact, connected exceptional locus E.

Theorem 0.1 (Kawamata [Kaw4]).

Let $g\colon X \supset E \to Y \supset B$ be a small contraction as in the previous definition. Assume that X is a smooth 4-fold. Then

$$E \simeq \mathbb{P}^2, \quad \mathrm{Bs}\,|-K_X| = \emptyset, \quad N_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}.$$

Also the flip g^+ of g exists. —

Remark 0.2. The linear system $|-2K_Y|$ has a member with only rational singularities. —

To complete the Program however we ought to deal with the singular case, to be precise the case that X has terminal singularities. The aim of this talk is to give some generalization of Theorem 0.1 in this direction (see also [Kac2]). We should say although at this moment we are still in a primary stage, the first feature of this business has been mostly worked out by the main result today.

One of our main tool is the deformation theory for contractions, and the following conditions provide the test case on which this technique runs fairly well:

Assumption A.

(A-1) X has only isolated (terminal or rational) complete intersection singularities, and

(A-2) (Existence of "Good bi-elephants")

$|-2K_Y|$ contains a member which has a rational singularity.

Main Theorem ([Kac3]). *Let $X \supset E \xrightarrow{g} Y$ be a flipping contraction from a 4-fold X with Assumption A. Assume* $\operatorname{Sing} X \neq \emptyset$. *Then*

$$E \simeq \mathbb{P}^2, \quad \mathrm{Bs}\,|-K_X| = \emptyset, \quad N_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2).$$

Moreover it carries an inductive structure involving a chain of blow-ups (which we call 'La Torre Pendente', see 5.3), and in particular the flip g^+ exists.

Remark. Also the case that (A-2) fails we mostly worked out, and this is closely related to the classification of minimal resolutions of hyperplane-sections of $(1,-3)$-curves due to S. Katz–D. R. Morrison [KaMo] and Kawamata [Kaw7]. See §6 A.

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§1. $E \simeq \mathbb{P}^2$, $\mathrm{Bs}\,|-K_X| = \emptyset$.

In this section we collect several immediate consequences from a general theory.

Proposition 1.1. Under the Assumption (A-1),

(1) (Kawamata [Kaw4], Andreatta–Wiśniewski [AW])

$$\mathrm{Bs}\,|-K_X| = \emptyset,$$

(2)
$$E \simeq \mathbb{P}^2.$$

(2) follows from (1) plus the following generalization of Mori's dimension count of Hilbert schemes parametrizing rational curves on a variety:

Theorem 1.2 (J. Kollár [Ko3]).

Let X be an algebraic variety (or an analytic space) which has only complete intersection singularities and C a rational curve on X. Assume $C \not\subset \mathrm{Sing}\,X$. Then

$$\dim \mathrm{Hom}(\mathbb{P}^1, X)_{[\alpha]} \geq \dim X + (-K_X \,.\, C),$$

where $\alpha\colon \mathbb{P}^1 \longrightarrow C \subset X$ is the normalization.

The rest is proved exactly in the same manner as in Kawamata's proof [Kaw4].

§2. $\varepsilon_P(X \supset E)$, $N_{E/X}$, width g.

In this section we prepare three items measuring g, the numerical invariant $\varepsilon_P(X \supset E)$ (local), the normal bundle $N_{E/X}$ (global), and the width (local-global).

Definition 2.1. (Local Invariant $\varepsilon_P(X \supset E)$)

Let (X, P) be an isolated complete intersection singularity, and $E \subset X$ a smooth closed subspace. Define:

$$\varepsilon_P(X \supset E) := \dim_P \mathcal{E}xt^1_{\mathcal{O}_E}(\Omega^1_X \otimes \mathcal{O}_E, \mathcal{O}_E). \quad —$$

This relates Mori's invariant $i_P(1)$ which was used to solve Flip Conjecture in dimension 3. These in fact coincide when E is a curve. So ours provides a cohomological interpritation of $i_P(1)$. The advantage is that we can calculate this invariant explicitly out of the given set of defining equations. For simplicity we state it only for hypersurface singularities, and for the general case we refer the reader to [Kac3] §3.

Formula 2.2. *Let $(X, P) \supset E$ be as above. Assume that (X, P) is a hypersurface singularity. Write down the equation in $(\mathbb{C}^N_{(x,y)}, 0)$ as*

$$X = \{f(x, y) = 0\} \supset E = \{y_1 = \ldots = y_r = 0\},$$

$$f = \sum_{i=1}^{r} y_i \cdot g_i(x) + h(x, y) \qquad (h(x, y) \in (y)^2).$$

Then

$$\varepsilon_P(Z \supset E) = \mathrm{lgth}\, \mathbb{C}\{x\}/(g_1, \ldots, g_r).$$

Definition 2.3. (Normal Bundle)

Let I_E be the ideal of E in $\mathcal{O}_X$. Define:

$$N_{E/X} := \mathcal{H}om_{\mathcal{O}_E}(I_E/I_E^2, \mathcal{O}_E).$$

This is automatically a locally free $\mathcal{O}_E$-module of rank 2 so we call it the *Normal Bundle* of E in X.

We are able to determine $N_{E/X}$ uniquely as follows;

Theorem 2.4. *Under the assumptions (A-1), (A-2), assume furthermore* $\operatorname{Sing} X \neq \emptyset$. *Then*

$$N_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2).$$

To show the theorem first by Van de Ven's characterization of uniform verctor bundles [Va], it is enough to show

$$N_{E/X} \otimes \mathcal{O}_l \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \quad \text{(for all lines } l \subset E).$$

On the other hand thanks to the condition (A-2) we have

$$N_{E/X} \otimes \mathcal{O}_l \simeq \mathcal{O}_{\mathbb{P}^1}(a_l) \oplus \mathcal{O}_{\mathbb{P}^1}(b_l), \ \ |a_l - b_l| \leq 2.$$

So by taking a general $D \in |-K_X|$, $l := D \cap E$, it suffices to rule out the possibility $N_{l/D} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. This can be done by the following theorem;

Theorem 2.5 (Generalization of Yo. Namikawa's 'local moduli' [Nam3]).

Let

$$\begin{array}{ccccc} \mathcal{U} & \xrightarrow{\varphi} & \mathcal{V} & \longrightarrow & \Delta \\ \cup & & \cup & & \cup \\ U_t & \underset{\varphi_t}{\longrightarrow} & V_t & \longrightarrow & \{t\} \end{array}$$

be a 1-parameter family of birational contractions $\varphi_t \colon U_t \to V_t$ *between 3-folds;* $\dim U_t = \dim V_t = 3$, *over the disc* $\Delta = \{t \mid |t| < 1\}$. *Assume the following conditions:*

(1) $$C := \operatorname{Exc}\varphi \simeq \mathbb{P}^1 \times \Delta$$

such that the second projection $C \to \Delta$ *coincides with* $\varphi|_C$

$$\begin{array}{ccccc} \operatorname{Exc}\varphi & \simeq & \mathbb{P}^1 \times \Delta & \xrightarrow{\varphi|_{\operatorname{Exc}}} & \Delta \\ \cup & & \cup & & \cup \\ C_t := \operatorname{Exc}\varphi_t = & & \mathbb{P}^1 & \longrightarrow & \{t\} \end{array}$$

(2) $(K_{U_t} \cdot C_t) = 0$ *(for all* $t \in \Delta$*), and*

(3) $\mathcal{U}$ *has only isolated rational complete intersection singularities such that*

$$\emptyset \neq \operatorname{Sing}\mathcal{U} \subset C_0.$$

Then for $t \neq 0$,

$$N_{C_t/U_t} \not\simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}. \quad —$$

Remark. As argued above we are not allowed to put any condition on the singularity of the 3-fold U_0, but just on the singularity of the ambient 4-fold $\mathcal{U}$. Here recall an example of M. Reid [Re3]: there exists a 4-dimensional terminal isolated hypersurface singularity whose general hyperplane-section is a '*$K3$-singularity*', which is in fact an irrational singularity. This shows that in dimension 4, terminal singularities form a broad category, which makes 4-dimensional contractions more complicated to handle with. So now let us put extra condition on trial that U_0 has at worst terminal singularities (= cDV-singularities). Then the conclusion of the theorem is known as a special case of Yo. Namikawa's *local moduli* [Nam3], whose proof is mainly based on the structure of versal deformation spaces of Du Val singularities developed by Brieskorn [B] et.al. For the general case however this method is not applicable, and we run instead the *deformation theory* for contractions (see §3 for the precise formulations). The profit is that this methodology does not require any particular kind of assumptions on the given defining equations but enables us to discuss under enough generality. This thus brings us a hope to overcome complexity of 4-dimensional terminal singularities.

To define the third item *width*, we for a while turn back to 3-fold contractions, especially those which are called *flopping contractions* ([Re1], *cf.* [Ko2]).

2.6. Let $U \xrightarrow{\varphi} V$ be a birational contraction of a smooth 3-fold U with

$$C \simeq \operatorname{Exc}\varphi \simeq \mathbb{P}^1, \quad (K_U \,.\, C) = 0.$$

Such a contraction is called a *flopping contraction*. Assume moreover that

$$N_{C/U} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$$

($(0,-2)$-*curve*). Recall the fundamental theorem of M. Reid:

Theorem 2.7 (M. Reid [Re1]).

There exists an integer $m \geq 2$ *such that*

(a) The "pagoda" [Re1] *terminates after* m *successive blow-ups, or alternatively*

(b) $$(V, Q) \simeq \{x_1x_2 + x_3^2 + x_4^{2m} = 0\}. \quad —$$

Definition 2.8. (Reid [Re1], for dim 3)

Define the *width* of the contraction φ to be

$$\operatorname{width}\varphi := m. \quad —$$

Remark. Also for $(-1,-1)$-*curves*, i.e., those contractions $\varphi\colon U \supset C \simeq \mathbb{P}^1 \xrightarrow{\varphi} V \ni Q$ with $N_{C/U} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$, put width $\varphi := 1$. —

There is an interpretation of this invariant from another point of view:

Theorem 2.9 (H. Laufer, R. Friedman [Fri], H. Clemens, *cf.* Yo. Namikawa [Nam3]).

Let $U \supset C \simeq \mathbb{P}^1 \xrightarrow{\varphi} V \ni Q$ *be as in 2.6, then there exists a 1-parameter deformation* $\{U_t \xrightarrow{\varphi_t} V_t\}_{t\in\Delta}$ *of* φ *such that* φ_t *contracts* $m = \mathrm{width}\,\varphi$ *disjoint union of* $(-1,-1)$*-curves from a smooth* U_t $(t \neq 0)$. —

Now it is time to define the width also for our original 4-fold contraction.

Definition 2.10. (for dim 4)

Let $X \supset E \simeq \mathbb{P}^2 \xrightarrow{g} Y \ni Q$ be a flipping contraction of a 4-fold X satisfying the assumptions (A-1),(A-2). Take a general smooth member $D \in |-K_X|$, let $l := D \cap E$ (a line in E). By Theorem 2.4, $D \supset l \xrightarrow{g|_D} g(D) \ni Q$ gives a contraction of the $(0,-2)$-curve l, so define:

$$\mathrm{width}\ g := \mathrm{width}\ g|_D. \quad —$$

§3. Deformation Theory.

Let $X \supset E \simeq \mathbb{P}^2 \xrightarrow{g} Y \ni Q$ be a flipping contraction satisfying (A-1), (A-2), as usual. In this section we discuss deformations of g. There are two steps to describe the deformations, the first one is cohomological, the other is complex analytic.

3.0. (a) (Cohomological)

Recall the Grothendieck spectral sequence applied to the composite of two functors $\mathbb{R}g_*$ and $\mathbb{R}\mathcal{H}om(\ \cdot\ ,\mathcal{O}_X)$:

$$E_2^{pq} := R^p g_* \mathcal{E}xt^q_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X) \implies E^{p+q} := \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X)$$
$$\left(= \mathbb{H}^{p+q}(\mathbb{R}g_* \circ \mathbb{R}\mathcal{H}om)(\Omega^1_X,\mathcal{O}_X)\right).$$

Let us write down the edge sequence:

$$0 \longrightarrow R^1 g_* T_X \xrightarrow{*} \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X) \xrightarrow{\alpha} g_* \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X)$$
$$\longrightarrow R^2 g_* T_X \xrightarrow{**} \mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X).$$

(Needless to say, when X is smooth the arrows marked by $*$, $**$ are both isomorphisms, and $\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X,\mathcal{O}_X) = 0$.)

The homomorphism α describes the infinitesimal deformation of X in the first order. To see the actual holomorphic deformation of X we need the following complex analytic description;

(b) (Complex analytic) (Kuranishi space)

There is a natural holomorphic map

$$\mathrm{Def}\,X \xrightarrow{\gamma} \prod_{P_i \in \mathrm{Sing}\,X} \mathrm{Def}(X,P_i)$$

from the *global Kuranishi space* Def X to the product of the *local Kuranishi spaces* $\mathrm{Def}(X, P_i)$.

Fact. (1) $$(d\gamma)_0 = \alpha,$$

Zariski tangent space $\quad T_{\mathrm{Def}\,X,0} \simeq \mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X),$

$$T_{\mathrm{Def}(X,P_i),0} \simeq \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)_{P_i}.$$

(2) The 'Obstruction' lies in $\mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$. In particular, if $\mathrm{Ext}^2_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = 0$ then Def X is smooth (*unobstructedness*). —

Remark. (Kollár–Mori [KoMo], *cf.* Ran [Ra1]))

Given a deformation of X: $\mathcal{X} \to \Delta$, there exists an induced deformation $\mathcal{Y}$ of Y: $\mathcal{Y} \to \Delta$ and a morphism $\mathcal{G}\colon \mathcal{X} \to \mathcal{Y}$ (which is compatible with $\mathcal{X} \to \Delta$, $\mathcal{Y} \to \Delta$) such that $\mathcal{G}|_{X_0} = g$.

(This is essentially based on the vanishing $R^i g_* \mathcal{O}_X = 0$ $(i \geq 1)$.) —

In our specific case, the most satisfactory situation is achieved. The following theorem asserts that X has enough deformations;

Theorem 3.1. *Let g be satisfying (A-1) and (A-2). Then*

$$R^2 g_* T_X = \mathrm{Ext}^2(\Omega^1_X, \mathcal{O}_X) = 0. \qquad \square$$

(The proof requires the structure of $N_{E/X}$ (Theorem 2.4), where the assumption (A-2) is essentially used.)

Corollary 3.2. Def X *is smooth, and γ is surjective.* $\square$

Practically, this implies;

Corollary 3.3. *(Globalization of local deformations)*

Let $\{P_1, ..., P_m\} = \mathrm{Sing}\, X$. Assume that a local deformation $\mathcal{U}_i \to \Delta$ of a neighborhood $U_i \ni P_i$ in X for each i is given, then there exists a global deformation

$$\varphi\colon \mathcal{X} \to \Delta$$

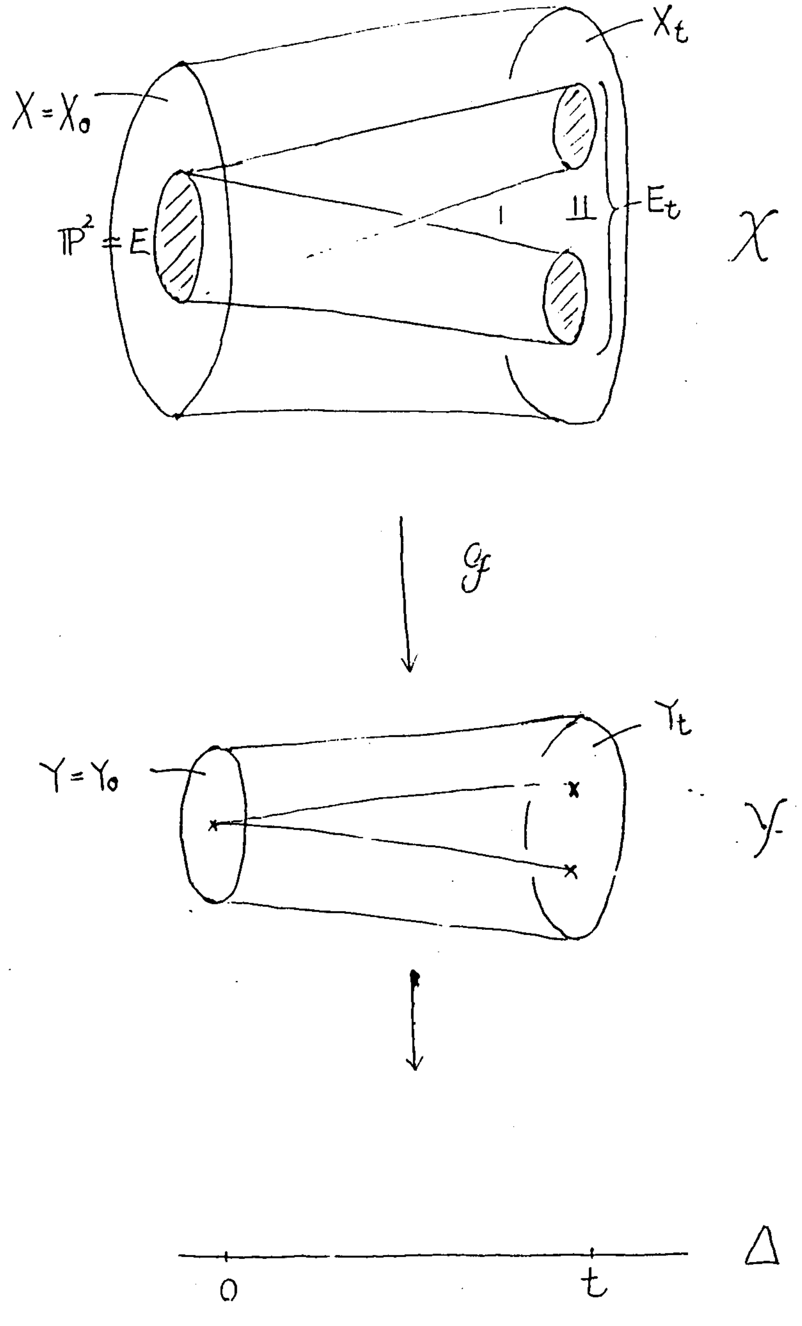
of X such that for a neighborhood $\mathcal{V}_i \ni P_i$ in $\mathcal{X}$,

$$\varphi|_{\mathcal{V}_i} = \text{Given } \mathcal{U}_i \to \Delta. \qquad \square$$

Remark. This fails if we drop (A-2). —

Fact. (Upper-semi-continuity)

For a general $t \in \Delta$, $X_t \xrightarrow{g_t} Y_t$ again gives a small contraction satisfying (A-1) and (A-2), contracting a certain number of disjoint $\mathbb{P}^2$'s. —



X_t
$X = X_0$
E_t
I
II
$\mathbb{P}^2 \cong E$
$\mathcal{X}$
$\mathcal{G}$
Y_t
$Y = Y_0$
$\mathcal{Y}$
Δ
0
t

Picture

Corollary 3.4 (the existence of a smoothing) (Relative 4-dimensional version of Yo. Namikawa [Nam1], *cf.* M. Gross [G1]).

There exists a defornation $\mathcal{X} \to \mathcal{Y} \to \Delta$ $(= (X_t \xrightarrow{g_t} Y_t)_{t\in\Delta})$ *such that* X_t *is smooth for* $t \neq 0$. □

Roughly speaking, in algebraic geometry deformation theory is divided into two categories; (i) Deformation of a variety X itself, this is parametrized by Kuranishi space Def X, and (ii) Deformation of an object attached with a *fixed* variety X (*e.g.* subschemes or coherent sheaves on X, morphisms $Z \to X$, etc.), these are parametrized by Hilbert schemes or those variants. In our context however we need a little more delicate treatment, that is, we ought to look at the behavior of E under deformations of X. To do this we formulate something to be called *deformations of pairs* (X, E), consisting of a variety X together with a subscheme E. This can be done in our specific situation by introducing 'extra-subscheme structure' on E affiliated with the given deformation of X, as follows;

3.5. (Subscheme structure E_0 for a deformation $\mathcal{X} \to \Delta$)

Let Exc $g_t =: E_t = \coprod_i (\mathbb{P}^2)_i$, and consider the structure morphism of the *relative Hilbert scheme*:

$$\mathrm{Hilb}_{\mathcal{X}/\mathcal{Y}/\Delta,[E_t]} \xrightarrow{\lambda} \Delta,$$

parametrizing deformations of E_t (*N.B.* not a connected component of E_t) inside the family $\mathcal{X} \to \mathcal{Y} \to \Delta$. λ is naturally an isomorphism, so define E_0 to be the closed subscheme associated to the point $\lambda^{-1}(0) \in \mathrm{Hilb}_{\mathcal{X}/\mathcal{Y}/\Delta,[E_t]}$. This is supported on $E = \mathrm{Exc}\, g$;

$$\mathrm{red}(E_0) = E \simeq \mathbb{P}^2.$$

So from now on let us distinguish E_0 from E. Also this subscheme E_0 depends on the given deformation $\rho\colon \Delta \to \mathrm{Def}\ X$, so sometimes write specifically $E_0 = E_0^\rho$ if any confusion is likely. (For instance in the case $\rho = 0$, *i.e.*, $\mathcal{X} = X \times \Delta$, then E_0^ρ is equal to the reduced E.)

Definition. (Universal subscheme E^*.)

Let $I_{E_0^\rho}$ be the defining ideal of E_0^ρ in $\mathcal{O}_X$ (or in $\mathcal{O}_{\mathcal{X}}$). Define

$$I_{E^*} = I_E^* := \bigcap_{\rho\colon \Delta \to \mathrm{Def}\, X} I_{E_0}^\rho. \quad —$$

Theorem 3.6 (Crucial observation).

(1) mult. $E_0^\rho =$ (*the number of connected components of* E_t) *for* $t \neq 0$,

(2) E_0^ρ *has no embedded primary components.* □

The assertion (1) says each branch of $\mathrm{Hilb}_{\mathcal{X}/\mathcal{Y}/\Delta,[E]} \longrightarrow \Delta$ (*N.B.* not $\mathrm{Hilb}_{\mathcal{X}/\mathcal{Y}/\Delta,[E_0]} \xrightarrow{\lambda} \Delta$ in turn) contributes the multiplicity of E_0 by 1. (It does not happen such as a degeneration of Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ onto a quadruple plane.)

3.7 (Mori's L-deformation versus our deformation).

In [Mo2] Mori gives a proof of the surjectivity of γ in dimension 3 by complex analytic methods, including an essential use of implicit function theorem (among some other more exquisit things). His mehod is to focus on a single irreducible component of $\operatorname{Exc} g_t$, and to disregard else. Ours on the other hand is more algebraic, what we do is to reduce the problem to the vanishing $R^2 g_* T_X = 0$ (which is no more automatic in dimension 4). We deal with all irreducible components simultaneously after a deformation by looking at the subscheme E_0^ρ, which inherits informations from all of those components. This reduces the task analyzing local classfication of singularities to an elementary commutative algebra (§4).

§4. Specify singularities $(X, P) \supset E$.

In this section we specify the possible singularities of X appearing on E. The following is the main result of this section:

Theorem 4.1. $\# \operatorname{Sing} X = 1$.
Let $\operatorname{Sing} X = \{P\}$, *then* $\varepsilon_P(X \supset E) = 1$.

There are three steps to show the theorem;

(a) Assume emb. codim.$(X, P) = 2$ to derive a contradiction,

(b) Assume (X, P) is a hypersurface singularity, to deduce $\varepsilon_P(X \supset E) = 1$, and

(c) $\# \operatorname{Sing} X = 1$.

Here (a) and (b) are essentially based on the same idea, and the proof is quite parallel. In fact in (a) we deal with a submodule $\mathcal{M}$ of $\mathbb{C}\{x_1, x_2\}^{\oplus 2}$, generated by 4 coefficient vectors, determined by 2 equations $\{f_1, f_2\}$, with 6 variables $(x_1, \cdots, x_6)$, while in (b) deal with an ideal $\mathcal{G}$ of $\mathbb{C}\{x_1, x_2\}$, generated by 3 coefficients, determined by a single equation f, with 5 variables $(x_1, \cdots, x_5)$. So in this section we only outline the proof of (b) (Proposition 4.2), and also (c) (Proposition 4.4).

Proposition 4.2.
Assume that (X, P) *is a hypersurface singularity. Then* $\varepsilon_P(X \supset E) = 1$.

First we may write down

$$\begin{aligned}(\mathbb{C}^5, 0) \supset (X, P) &= \{f(x_1, ..., x_5) = 0\} \\ \supset \quad E \quad &= \{x_3 = x_4 = x_5 = 0\},\end{aligned}$$

$$\begin{aligned}f(x) = g_3(x_1, x_2) \cdot x_3 &+ g_4(x_1, x_2) \cdot x_4 + g_5(x_1, x_2) \cdot x_5 \\ &+ h(x_1, ..., x_5), \qquad\qquad h \in (x_3, x_4, x_5)^2.\end{aligned}$$

Let $\mathcal{G} := (g_3, g_4, g_5) \subset \mathbb{C}\{x_1, x_2\}$. By abuse of notation, denote the pull backs of I_{E^*}, I_{E_0} by the surjection

$$\mathbb{C}\{x_1, \cdots, x_5\} \twoheadrightarrow \mathbb{C}\{x_1, \cdots, x_5\}/(f) \simeq \mathcal{O}_{X,P}$$

by the same symbol, so that

$$f \in I_{E^*} \subset I_{E_0} \subset \mathbb{C}\{x_1, \cdots, x_5\}.$$

Lemma. *(1)* $g_5 \in (g_3, g_4)$ *(after a suitable permutation of* $\{3,4,5\}$*).*

(2)
$$f(x) = \big(g_3(x_1,x_2) + h_3(x)\big) \cdot x_3 + \big(g_4(x_1,x_2) + h_4(x)\big) \cdot x_4 + x_5^k \qquad (k \geq 2)$$
$$(x_i \cdot h_i \in (x_3, x_4, x_5)^2).$$

(3)
$$I_{E^*} = (x_3, x_4, x_5^k).$$

Proof. (1) Recall that I_{E^*} has no embedded primes, and

$$\text{radical}(I_{E^*}) = (x_3, x_4, x_5).$$

If (1) is not true, then $x_3, x_4, x_5 \in I_{E^*}$, that is,

$$I_{E^*} = (x_3, x_4, x_5)$$

which is reduced, a contradiction.

(2) and (3) follow from (1). □

Proof of Proposition 4.2. Assume $\varepsilon_P(X \supset E) \geq 2$, to get a contradiction.

By Formula 2.2, $\varepsilon_P(X \supset E) = \text{lgth}\, \mathbb{C}\{x_1, x_2\}/\mathcal{G}$, so we may assume $x_2 \notin \mathcal{G}$, say. Consider a local deformation

$$\{f(x) + t \cdot x_2 = 0\}.$$

This can be globalized thanks to Corollary 3.3, let E_0 be the associated closed subscheme structure. By the previous Lemma,

$$I_{E_0} = (x_3, x_4, x_5^n) \qquad (n \geq 2),$$
$$I_{E_t} = (x_3 + t \cdot e_3, x_4 + t \cdot e_4, x_5^n + t \cdot e_5) \quad (\text{ for some } e_i(x,t)).$$

Write down the condition "$f + t \cdot x_2 \in I_{E_t}$";

(*)
$$f(x) + t \cdot x_2 = \xi_3 \cdot (x_3 + t \cdot e_3) + \xi_4 \cdot (x_4 + t \cdot e_4) + \xi_5 \cdot (x_5^n + t \cdot e_5)$$
$$(\text{for some } \xi_i(x,t)).$$

By comparing both-hand sides, it follows that

(4.2.1) $\xi_i \cdot e_i$ contains $c \cdot x_2$ as a monomial (c: unit), for some $i = 3, 4$ or 5.

Claim. In (4.2.1), $i = 5$.

Proof. : Assume $i = 4$, say, to get a contradiction. Rewrite (*);

$$f(x) + t \cdot x_2 = \xi_3 \cdot (x_3 + t \cdot e_3) + (c_1 \cdot x_2 + \xi_4')(x_4 + c_2 \cdot t) + \xi_5 \cdot (x_5^n + t \cdot e_5).$$

Put $t = 0$, then it is easily seen that $f(x)$ contains $c_1x_2x_4$ as a monomial (c_1: a unit), a contradiction to our assumption $x_2 \notin \mathcal{G}$. Hence the Claim.

Now we know that

(4.2.2) $\xi_5 \cdot e_5$ contains $c \cdot x_2$ as a monomial.

By Lemma, ξ_5 must be a unit. Thus I_{E_t} is expressed as;

$$I_{E_t} = (x_3 + t \cdot e_3, x_4 + t \cdot e_4, x_5^n + c' \cdot tx_2) \quad (c' : \text{ unit}).$$

This implies that E_t is irreducible, smooth for $t \neq 0$, and the multiplicity of E_0 is $n \geq 2$. These contradict Theorem 3.6. □

Corollary 4.3 ((A-1) + (A-2) $\Longrightarrow$ *cO.D.P*'s only).

$$\begin{aligned} (X, P) &\simeq \{x_1x_3 + x_2x_4 + x_5^m = 0\} \\ &\supset E = \{x_3 = x_4 = x_5 = 0\}. \qquad \square \end{aligned}$$

This suggests that possible singularities appearing on flipping contractions are rather limted.

The remaining thing to prove is the following;

Proposition 4.4. $\# \operatorname{Sing} X = 1. \quad m = \operatorname{width} g.$

Pf. Assume $P, P' \in \operatorname{Sing} X$ ($P \neq P'$) to get a contradiction. Consider the global deformation of X (Corollary 3.3) given locally by

$$\begin{cases} \{x_1x_3 + x_2x_4 + x_5^m + t = 0\} & (\text{near } P), \\ \{y_1y_3 + y_2y_4 + y_5^n + t^n = 0\} & (\text{near } P'). \end{cases}$$

Then

$$E_t \simeq \begin{cases} \{x_3 = x_4 = x_5^m + t = 0\} & (\text{near } P), \\ \{y_3 = y_4 = \prod_{i \in I} (y_5 + \zeta_n^i t) = 0\} & \\ \qquad (\# I = m \geq 2) & (\text{near } P'). \end{cases}$$

Let $\mathcal{E} := \bigcup_{t \in \Delta} E_t$, then

$$\operatorname{Sing} \mathcal{E} = \begin{cases} \emptyset & (\text{near } P), \\ E & (\text{near } P'), \end{cases}$$

which contradicts the nature that $\operatorname{Sing} \mathcal{E}$ has to be Zariski closed.

To see the second assertion, consider the deformation

$$\{x_1x_3 + x_2x_4 + x_5^m + t = 0\},$$

then we will find $m = \operatorname{width} g$ by taking a general $D \in |-K_X|$, applying Theorem 2.9 (Laufer, Friedman, Clemens). □

§5. How to flip?

In this section we prove the existence of flip $g^+ : X^+ \to Y$ of $g : X \to Y$. Our strategy is to construct a sequence $X^{(m)} \to X^{(m-1)} \to \cdots \to X^{(1)} = \overline{X} \to X^{(0)} = X$ of blow-ups and to see that $X^{(m)}$ dominates the required flip, as is done for 3-fold flopping contractions by M. Reid [Re1]. So ours actually gives even a geometrically explicit way of constructing flips.

Proposition 5.1. *Let $X \supset E \simeq \mathbb{P}^2 \xrightarrow{g} Y \ni Q$ be as usual, satisfying (A-1), (A-2). Let*

$$\overline{X} \xrightarrow{f} X$$

be the blow-up of X with the center E. Let $F := \operatorname{Exc} f$. Then $-K_{\overline{X}}$ is $(g \circ f)$-ample, $\rho(\overline{X}/Y) = 2$, and the other extremal ray of $\overline{NE}(\overline{X}/Y)$ determines a flipping contraction

$$\overline{X} \supset \overline{E} \simeq \mathbb{P}^2 \xrightarrow{\overline{g}} \overline{Y} \ni \overline{Q}$$

satisfying the assumptions (A-1), (A-2), with

$$\operatorname{width} \overline{g} = \operatorname{width} g - 1.$$

Outline of Proof. A priori there are two possibilities. That is, besides the one in the conclusion of the theorem, there might be the case:

$(*)$ $-K_{\overline{X}}$ is $(g \circ f)$-nef but not $(g \circ f)$-ample, the other extremal ray determines a *flopping contraction which contracts rulings of a ruled surface isomorphic to* $\Sigma_1 := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$, sitting in F as a birational section of the fibration $F \xrightarrow{f|_F} E \simeq \mathbb{P}^2$ *(cf. [Ma]).*

This however is ruled out this way; consider the deformation $\mathcal{X}$ of X given locally by:

$$\{x_1 x_3 + x_2 x_4 + x_5^m + t^m = 0\}.$$

Then $\mathrm{Hilb}_{\mathcal{X}/\mathcal{Y}/\Delta,[E]}$ *consists of m irreducible components, all of which isomorphically* dominate Δ through:

$$\mathrm{Hilb}_{\mathcal{X}/\mathcal{Y}/\Delta,[E]} \longrightarrow \Delta.$$

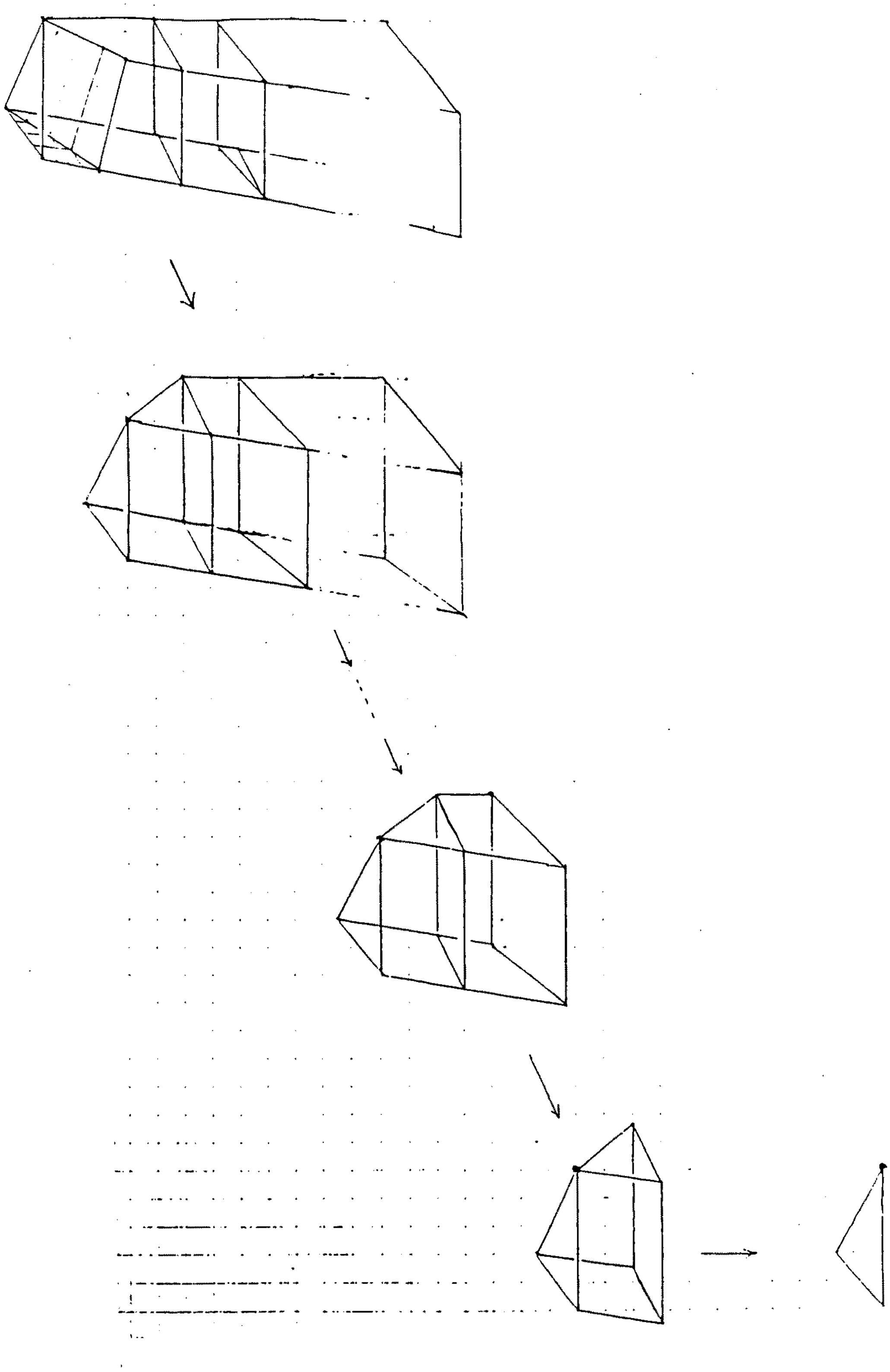
Take one irreducible component and consider the corresponding irreducible component $\mathcal{E}_1$ of $\mathcal{E} = \operatorname{Exc} \mathcal{G}$; $\mathcal{E}_1 \simeq \mathbb{P}^2 \times \Delta$. *Blow $\mathcal{X}$ up with the center $\mathcal{E}_1$.* Then we see that the rulings $(*)$ never go outside of $\overline{X}_0$, that is, they deform exactly with 1 dimensional parameters, while according to Theorem 1.2 (Mori, Kollár) they must deform with at least 2 dimensional parameters, a contradiction. □

Corollary 5.2 (Existence of the flip).

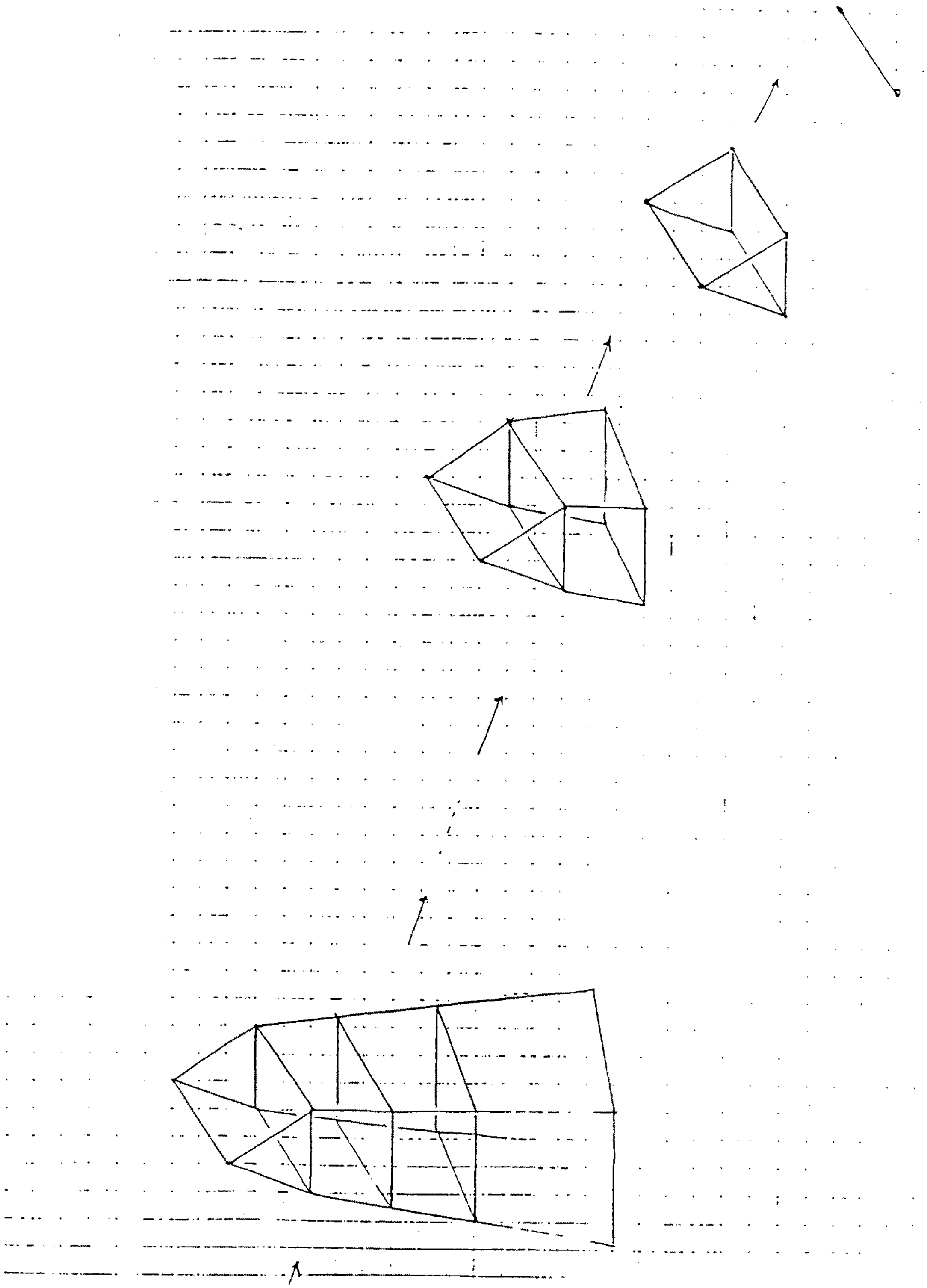
For g with the assuptions (A-1), (A-2), the flip g^+ exists.

Proof. Induction on $m = \operatorname{width} g$. The case $m = 1$ is nothing but Theorem 0.1 due to Kawamata [Kaw4]. □

To see a more concrete description, let us follow all the induction steps successively upstreams, then we eventually arrive $m = 1$, and get the following architecture:



La Torre Pendente (sinistra)



La Torre Pendente (destra)

5.3. This should be compared to M. Reid's *Pagoda* [Re1], not only because the pattern of the construction of the flip and the flop look similar, but this construction in fact contains Pagoda in the following way: In the above picture let us take a general smooth $D \in |-K_X|$ (so that $D \cap \operatorname{Sing} X = \emptyset$), and let

$$D = D^{(0)} \leftarrow D^{(1)} \leftarrow \cdots \leftarrow D^{(m)} \to \cdots \to D^{(1)+} \to D^{(0)+} = D^+$$

be the proper transforms of X in every stage. Then this indeed forms Pagoda [Re*], and in particular $D^+ \xrightarrow{g^+|_{D^+}} g^+(D^+)$ gives the *flop* of $D \xrightarrow{g|_D} g(D)$ (see also §2). As is easily seen Pagoda is symmetric with respect to the flop operation, while ours is no more symmetric with respect to the flip. So with a great esteem for Reid's humor of this lovely naming, by special grace we name this *La Torre Pendente*, meaning *The Leaning Tower* of Pisa, Italy.

§6. Concluding Remarks.

6.A. What if drop (A-2)?

Proposition 6.A.1. *Let $X \supset E \xrightarrow{g} Y \ni Q$ be a flipping contraction. Let us assume (A-1) but no (A-2). Then $E \simeq \mathbb{P}^2$ and*

$$N_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(-3). \qquad \square$$

Remark 6.A.2. Even the irreducibility of E in this case is not easy.

Question 6.A.3. Is Def X smooth, or equivalently, does the 'T^1-lifting' property (Ran [Ra2], Kawamata [Kaw5], Namikawa [Nam2], Gross [G2]) hold?

6.A.4 (Suggested to us by M. Gross).

Take a *hyperplane-section* $H := (s)_0$ for $(0 \neq) s \in H^0(I_E)$, so that $X \supset H \supset E$. M. Gross indicated that it might be helpful to understand X through the knowledge of H. Actually this makes things fairly controllable, and we are in progress based on this idea. Meanwhile, not even a single example of a contraction g as in Proposition 6.A.1 has yet been observed so far. In fact on this line we got a partial negative answer toward the existence of such flipping contraction (Corollary 6.A.10 below).

Proposition 6.A.5. *$S_H := \operatorname{Sing} H$ is purely 1-dimensional, and $S_H \in |\mathcal{O}_{\mathbb{P}^2}(3)|$. (Here a general S_H maynot necessarily be reduced or irreducible.)*

Question 6.A.6. Let $\Lambda := \{S_H\}_H \subset |\mathcal{O}_{\mathbb{P}^2}(3)|$ be the sublinear system consisting of all such S_H's. How does Λ look like? How much is $\dim \Lambda$?

6.A.7. Let $D \in |-K_X|$ be a general member, then $D \supset l := D \cap E \xrightarrow{g|_D} g(D) \ni Q$ gives a contraction of the $(1,-3)$-*curve* l (see also [L]). Now because of the surjectivity of the homomorphism $H^0(\mathcal{O}_X) \longrightarrow H^0(\mathcal{O}_D)$, an information of hyperplane-sections of $D \supset l$ inherits that of $X \supset E$. So it is natural to ask the same question as Question 6.A.6 for $(1,-3)$-curves first.

The following gives an example in dimension 3 that the linear system in question does not move at all:

Observation 6.A.8. *Let $U \supset C \simeq \mathbb{P}^1 \longrightarrow V \ni Q$ be a contraction of $(1,-3)$-curve C; $N_{C/U} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$. According to* [KaMo] *(cf.* [Kaw7]*), for a general $s_{\mathrm{gen}} \in H^0(I_C)$, the minimal resolution graph* [Re1] *corresponding to $(s_{\mathrm{gen}})_0$ is of type either D_4, E_6, E_7 or E_8. Let us assume that it is E_8, then again by* [KaMo] *either $\#\,\mathrm{Sing}(s_{\mathrm{gen}})_0 = 2$ or 3 set theoretically. Let us assume it is 3. Then there exists a set of points $\{x, y, z\} \subset C$ such that $\mathrm{Sing}(s)_0 = \{x, y, z\}$, independent of the choice of any $s \neq 0$.* □

I quite recently learned from H. Takagi the following, which is the firsthand result on the plot of 6.A.4;

Proposition 6.A.9 (H. Takagi).

Let $X \supset E \simeq \mathbb{P}^2 \xrightarrow{g} Y \ni Q$ be as in Proposition 6.A.1, and H as in 6.A.4. Then

(1) A general H has at most canonical singularities.

(2) Λ has a fixed component.

This, combined with the purity of S_H (Proposition 6.A.5), and an argument of [Kaw7], proves:

Corollary 6.A.10. *Let $X \supset E \simeq \mathbb{P}^2 \xrightarrow{g} Y \ni Q$ be as in Proposition 6.A.1. We say that g is of type D_4, E_6, E_7, E_8 if the graph of the $(1,-3)$-curve $D \supset l := D \cap E$ for a general $D \in |-K_X|$ is of the corresponding type.*

Then g cannot be of type D_4. □

6.B. What if drop (A-1)?

Question 6.B.1. Generalize our framework to the case X is a '*LCIQ*' (=locally complete intersection quotient) 4-fold (in the sense of J. Kollár [Ko4]), that is, X has only isolated singularities and for each $P \in \mathrm{Sing}\, X$ there exists a local finite cover

$$(X, P) \longleftarrow (\widetilde{X}, \widetilde{P})$$

which is étale in codimension 1 such that $(\widetilde{X}, \widetilde{P})$ is a complete intersection singularity.

This is motivated by the following example;

Example 6.B.2 (Mukai, Reid [Re2], see [Kac3] §8).

There exists a flipping contraction $X \supset E \xrightarrow{g} Y \ni Q$ from a 4-fold X with only an isolated singularity, $\mathrm{Sing}\, X = \{P\}$, and $(X, P) \simeq \frac{1}{2}(1,1,1,1)$ (that is, the quotient singularity of $(\mathbb{C}^4, 0)$ divided by the involution $z \mapsto -z$). The exceptional locus E is isomorphic to a singular quadric cone, so this time Theorem 1.2 (Kollár) no longer holds. This contraction is constructed as one which factors a certain extremal contraction from a 4-fold to a 3-fold admitting a 2-dimensional fiber. See [Kac3] §8 for a detailed description (*cf.* [Kac1]).

Here is a relevant question;

Question 6.B.3. *Let $X \supset C$ be an analytic space, containing a (complete) rational curve C, with $C \not\subset \operatorname{Sing} X$. Assume that for any $P \in \operatorname{Sing} X \cap C$, the analytic germ (X, P) does not admit a cover $(X, P) \longleftarrow (\widetilde{X}, \widetilde{P})$ which is étale in codimension 1 and is of degree > 1. Then is the same formula as in Theorem 1.2 hold?*

6.B.4. In the general case, the *Bug-eyed cover* (Kollár [Ko4]) or the associated *algebraic stack* (Artin, Deligne–Mumford); $X^b \longrightarrow X$, might perhaps say something. (See [KeM^c], [BF], [Vi].) Especially, is it possible to construct something like an '*equivariant deformation*' of f^b?

6.C (Gross' covering trick).

6.C.1 (M. Gross).

M. Gross observed an alternative way of producing our series of contractions. Start from the Kawamata contraction $X \supset E \simeq \mathbb{P}^2 \xrightarrow{g} Y \ni Q$ as in Theorem 0.1; X is a smooth 4-fold, and $N_{E/X} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$. Take a hyperplane-section H as in 6.A.4. If H is chosen general enough, then H has only one singular point which is O.D.P. Regard X as the total space of the deformation of H:

$$\begin{array}{ccc} X & \longrightarrow & \Delta \\ \cup & & \cup \\ H & \longrightarrow & 0 \end{array}$$

Take the base-change by $\Delta \to \Delta$, $t \mapsto t^m$, to get a flipping contraction g satisfying (A-1) plus (A-2), with width $g = m$. Also if we take H which has a singularity along a line on E, then we get an example of a flipping contraction $\widetilde{X} \supset E' \simeq \mathbb{P}^2 \to \widetilde{Y}$ where X has 1-dimensional singular locus, a line of E'.

In general, the condition (A-2) is preserved through this operation, so a contraction g as in 6.A.1, if any, is considered to be sitting on an entirely different lines.

6.D (Ando's description of g, (Y, Q)).

Ando gave a description of a flipping contraction g with (A-1), (A-2), as well as the singularity (Y, Q) and the flip g^+ which we classified, by means of an explicit coordinate expression, after H. Laufer [L] in dimension 3.

Theorem 6.D.1 (T. Ando).

Let $X \supset E \simeq \mathbb{P}^2 \xrightarrow{g} Y \ni Q$ be satisfying (A-1), (A-2), with width $g = m$. *Then*

$$(Y, Q) \simeq \{x_1x_6 = x_3x_5,\ x_4x_5 = (x_3 - x_2^m)x_6,\ x_1x_4 = (x_3 - x_2^m)x_3\} \subset (\mathbb{C}^6, 0).$$

Blow up of Y with the ideal (x_3, x_4, x_6) (resp. (x_1, x_3)) recovers g (resp. g^+). □

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