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# Automorphisms of algebraic varieties.

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*Dedicated to* **MATSUMURA, Hideyuki**
a personal reflection.

In the period 1962 - 1995 the lives of MATSUMURA, Hideyuki, and myself intersected several times. This short note contains recollections, mathematical, but some also of a very personal nature. Maybe it is interesting for the reader to see the interaction of these two people, very different personalities and from very different cultural backgrounds, who are bound together by the love for mathematics and by mutual respect.

This is certainly not an "In Memoriam": to describe the person and the mathematics of Matsumura it would take much more space. But I thought that some of my personal recollections might disclose some aspects of his person and of his mathematics to you. In this way I hope to share with you some of the joy I experienced in the stimulating contact I had with this colleague and friend, whose loss came too early for all of us.

## 1 Stockholm, 1962.

Our first contact occurred in Stockholm, during the International Congress of Mathematicians. I had the opportunity to expose the results of my PhD-thesis in a short communication "Multiple algebraic curves" (a construction of the Picard scheme for algebraic curves, which need not be reduced and for algebraic schemes). After my talk a Japanese mathematician approached me with a question. The perfect way indeed this to find a person in such a mass meeting: just go to the talk, in this way you will discover whether you have found the right person.

This was my first contact with a Japanese mathematician (well, Professor Matsusaka was very kind to me when I was a young student, so perhaps I should say the second?). And it was the beginning of an intense and nice cooperation, with Matsumura, and later also with several other of my Japanese colleagues (all of the 10 different publications I did with 7 Japanese colleagues, were nice and exciting experiences for me).

Matsumura's question was about a construction he had and one detail of which he did not understand. He hoped I would come up with a suggestion.

## 2 Automorphisms give orbits.

Here is the elegant idea Matsumura had when he tried to construct the algebraic automorphism group of an algebraic variety.

**The problem:** Let $V$ be an algebraic variety (say over an algebraically closed field $k$, suppose for example $V$ is complete). Consider the group $\mathrm{Aut}(V)$ of all automorphisms of $V$. Try to construct an algebraic group $\mathbf{A}_V$ such that its geometric points correspond naturally with the group of automorphisms,

$$\mathbf{A}_V(k) \quad \cong \quad \mathrm{Aut}(V),$$

or, a formulation in more mathematical terms: define the functor $\mathcal{A}_V$, and show it is representable.

**Matsumura's idea:** Choose a finite set of points

$$P_1, \cdots, P_n \in V(k),$$

and construct the orbit map

$$f_n : \mathrm{Aut}(V) \to V^n \quad \text{by} \quad \varphi \mapsto (\varphi(P_1), \cdots, \varphi(P_n)) \in V^n.$$

Consider the subgroup $\mathrm{Aut}(V)^0$ of all automorphisms of $V$ algebraically equivalent to the identity, and restrict $f_n$ to this subgroup. If $K_n := \mathrm{Ker}(f_n \mid_{\mathrm{Aut}(V)^0}) \neq \{id\}$, choose a new point $P_{n+1}$ such that $\varphi \in K_n$, $\varphi(P_{n+1}) \neq P_{n+1}$ , and consider $f_{n+1}$. Clearly $K_{n+1} \subset K_n$, and $K_{n+1} \neq K_n$. Of course, you expect that after choosing enough points $\{P_i\}$ the map

$$f_n : \mathrm{Aut}(V)^0 \to V^n$$

is injective. Take the "image", that should produce the connected component $\mathbf{A}_V^0 \subset V^n$ containing the identity of the algebraic group we are looking for. A strikingly simple geometric construction, close to the nature of the problem, a prototype of a good idea!

**However,** as Matsumura pointed out to me in 1962 after describing this construction, it was not clear this would give the correct geometric approach. Let $\Theta_V$ be the tangential sheaf. A section $\sigma \in H^0(V, \Theta_V)$ is an infinitesimal automorphism of $V$. As a geometer you might expect that it can be "integrated" to a true automorphism, in other words is the natural map of the tangent space of $\mathbf{A}_V^0$ at the identity to $H^0(V, \Theta_V)$ an isomorphism ? Matsumura could show that

$$\dim \mathbf{A}_V^0 \quad \leq \quad \dim H^0(V, \Theta_V).$$

His question to me was whether I could see whether this could be proved to be an equality, or whether there is an explanation why this perhaps could not be an equality.

## 3 Our first contact.

Our first meeting was a special experience for me, in a way completely different from previous ones. Although our languages, our cultures were so different, I had a sense of direct contact.

I could not have explained why; certainly the fact that Matsumura was such a gentle person had a lot to do with it. In his approach I felt very much at home, immediately. And of course, there was our shared background in mathematics. But yet: isn't it amazing: how can people, who were 5 minutes ago total strangers to each other, understand each other after such a short time, almost without words?

His basic idea was immediately clear to me. And his question was close to things I was trying to explore myself.

## 4 Group schemes.

Remember, this was 1962. The language of schemes was emerging. Aspects like this in algebraic geometry had been observed earlier, but it was only through the theory of Grothendieck that explanations and methods became available. I had been in the lucky situation of being guided by Andreotti and by Serre in this material, and I had become a little bit familiar with these ideas. My first mathematical reaction to this idea of Matsumura (beside admiration for his geometric insight, and for the fact that he managed to ask the right question) was to tell him about a quite similar situation.

**Igusa: "a fundamental inequality."** At that time I knew the results in the paper [3] by Igusa (just a note consisting of 4 pages !), and I more or less understood why such phenomena can occur. Igusa observed that the dimension of the Picard *variety* of a variety $V$ has dimension at most $\dim H^1(V, \mathcal{O}_V)$, and in that paper Igusa constructed an example where strict inequality occurs.

**Explanation via schemes.** Let me first indicate (in more modern terms) why in general you obtain "$\leq$": consider divisor classes on the scheme $V_\varepsilon := V \otimes k[\varepsilon]$, with $\varepsilon^2 = 0$ reducing to the trivial class on $V$ (this is canonically the tangent space to the Picard *functor*). This can be analyzed by using the exact sequence of sheaves of multiplicative groups on $V$ (here $\mathcal{O} = \mathcal{O}_V$):

$$1 \to (1 + \mathcal{O}{\cdot}\varepsilon)^\times \longrightarrow \mathcal{O}[\varepsilon]^* \longrightarrow \mathcal{O}^* \to 1$$

($\times$: denoting a multiplicative group, $*$: the multiplicative group of units in a sheaf of rings). Note that there is an isomorphism of sheaves of groups:

$$(1 + \mathcal{O}{\cdot}\varepsilon)^\times \quad \xrightarrow{\sim} \mathcal{O} \quad \text{by} \quad 1 + x{\cdot}\varepsilon \mapsto x$$

(this is the way we do differential calculus in an algebraic setting. I was quite startled to see this for the first time when I was young. It was Serre who showed it to me, saying "but people in algebraic number theory know this already for a long time!"). Hence we obtain (using the long exact sequence of cohomology):

$$H^1(V, (1 + \mathcal{O}{\cdot}\varepsilon)^\times) \cong H^1(V, \mathcal{O}) \quad \xrightarrow{\sim} \quad T_{Pic,0} = \mathrm{Ker}\left(H^1(V, \mathcal{O}[\varepsilon]^* \to H^1(V, \mathcal{O}^*)\right),$$

where "Pic" is the Picard *scheme*. Hence

$$T' := T_{Pic_{red},0} \subset T_{Pic,0} = H^1(V, \mathcal{O})$$

the tangent space to the Picard *variety* naturally is a subspace. In his paper Igusa shows that there exist cases where these are not equal! At that time (1955) it was kind of mysterious, why such phenomena could occur. But once schemes were invented and studied this became "obvious": sometimes naturally appearing schemes are not reduced (even in characteristic zero, think of some moduli schemes). Algebraic group schemes in characteristic zero are reduced, as Cartier observed (and the proof is easy, see [4], see [5], 11.4); hence in that case Picard variety = Picard scheme, and we get an equality. However in positive characteristic there are non-reduced group schemes (plenty of them! one can work a whole life time studying these), and Igusa's paper constructs an example of a variety (in fact a regular algebraic surface) having a non-reduced Picard scheme (it Picard variety has dimension 1, the tangent space to the Picard scheme has dimension 2), and hence an inequality

$$\dim Pic_{red} \quad < \quad \dim H^1(V, \mathcal{O}_V).$$

So, by analogy, I had the idea that indeed it was possible that there could exist a non-reduced automorphism group scheme, i.e. that

$$\dim \mathbf{A}_V^0 \quad \leq \quad \dim H^0(V, \Theta_V)$$

could be a true inequality for the automorphism variety of some $V$ in positive characteristic.

From that time on we worked on the problem together. Once I went (back) to Pisa, where Matsumura was at that moment. We spent some happy days together. His family was with him at that time, and I enjoyed meeting them. I saw a sad example of how unsuspecting Hideyuki could be. Their personal belongings, shipped from Japan, arrived at Livorno harbor, and he went to pick them up. I was worried, and asked him not to leave the goods for a single moment out of sight once the goods were through the customs. He returned home saying that a kind man had offered to transport the trunk to Pisa for a small amount of money. What I had feared indeed happened: only a small quantity of the goods was still left in the trunk when it finally arrived in Pisa. The family was sad (from what was left I deducted that the rest must have been quite nice), but this gentle Hideyuki did not show much anger.

Some years later when he was working at Columbia University in New York, and I was visiting Harvard University our paths crossed again. Meeting between our families took place - happy memories! To our delight our joint work led to results, we were quite happy when we could finish our work:

## 5 Work in progress.

We tried to adapt Matsumura's original idea in such a way that a construction of the automorphism group scheme would result, with the property that indeed all infinitesimal automorphisms show up as tangent vectors to the automorphism group scheme. What had to be changed in the original approach? Clearly just geometric points $P_i \in V$ are not enough to test infinitesimal automorphisms. Hence we had the idea of using "infinitesimal points", better said the theory of higher jets, and we proved that the automorphism group functor $\mathcal{A}_V^0$ for a proper algebraic scheme $V$ is representable (i.e. the automorphism group scheme exists). We concluded by giving examples to show that indeed there are cases where this automorphism scheme is not reduced, i.e.

$$\dim \mathbf{A}^0_{V,red} \quad < \quad \dim H^0(V, \Theta_V), \quad \mathbf{A}_{V,red} \neq \mathbf{A}_V.$$

So we finally we had completely analyzed Matsumura's original question, we had found a satisfactory method to arrive at an answer in a satisfactory way:

**Theorem** (see [1], Th. (3.6/7)): *Let $k$ be a perfect field, and let $V$ be an algebraic $k$-scheme such that $\dim_k H^0(V, \Theta_V) < \infty$. Then the automorphism variety of $V$ exists. If $V$ is proper over $k$, the automorphism group scheme exists, and it is a group scheme locally of finite type over $k$.*

For a treatment using Hilbert schemes, see: [2], II, pp. 195-12/13, and IV, pp. 221-19/20.

In some fields joint work is quite usual, in other fields it may be rare. In mathematics we often see joint papers. We have the habit of even not remembering how much each author has contributed. In some of my joint publications I did most of the work, in other cases it was the other way around. Once I felt my contribution was getting smaller and smaller, and I finally proposed to have my name removed, the other author however insisted that my name would stay on. This is one of the nice aspects of our profession, the generosity in sharing ideas with other people.

## 6 Epilogue.

From then on we sometimes had a chance to meet and now and then we exchanged a letter or, later, an email.

On June 29, 1995 he wrote (careful as always) to me that he was "enjoying, to some extent..." the life he had at that moment, teaching elementary mathematics, and taking care of the library: he loved seeing and reading books.

Shortly after that I got the message that "...he fell off a mountain path in the "Northern Japan Alps" in trying to stop someone else who fell from above him." Recently, I saw the pictures which were in his camera on that fatal day August 7, 1995. I was moved to see the pictures he had taken that very morning. Some blurred, because he fell into the water, but on the whole, happy memories of mountain walks which he used to like so very much. So, just as a token of our mutual experiences, I wanted to tell something about the joint experiences of these two mathematicians. Even though I may not have quite succeeded in conveying what I felt about Matsumura, it may be clear that I feel that he gave me much more, than he may even have realized.

## References

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