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AN ARITHMETIC COMPACTIFICATION OF THE MODULI OF ABELIAN VARIETIES

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Dedicated to Professor Kunihiko Kodaira

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ABSTRACT. We compactify canonically the moduli scheme of abelian schemes over $\mathbf{Z}[\zeta_N, 1/N]$ by introducing the noncommutative level structures. Any degenerate abelian scheme on the boundary of the compactification is one of our models – projectively stable quasi-abelian schemes. A degenerate abelian scheme is asymptotically Kempf-stable if and only if it is a projectively stable quasi-abelian scheme.

0. INTRODUCTION.

This is a continuation of the previous report *Stability of degenerate abelian varieties* in the proceedings of Kinosaki symposium 1996.

The purpose of the present article is to report on a recent progress in the problem of arithmetic compactification of the moduli of abelian varieties.

We introduce the notion of projectively stable quasi-abelian schemes and prove Kempf-stability of their Hilbert points (Theorem 0.2). We also prove existence of the projective *reduced-fine-moduli scheme* of projectively stable quasi-abelian schemes (Theorem 0.4) over $\mathbf{Z}[\zeta_N, 1/N]$ where ζ_N is a primitive N-th root of unity. This is a natural geometric compactification of a moduli scheme of abelian varieties. See also [Alexeev96] for the principally polarised (torically) stable quasi-abelian varieties.

If we are given Faltings-Chai's degeneration data [FC90] there are two canonical choices of flat projective degenerating families of abelian varieties $(P, \mathcal{L})$ and $(Q, \mathcal{L})$ where $(Q, \mathcal{L})$ is the most naive choice and $(P, \mathcal{L})$ is the normalisation of $(Q, \mathcal{L})$ after

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some base change. The following is the stable reduction theorem of abelian varieties proved in [AN96].

Theorem 0.1. *Let R be a complete discrete valuation ring with the fraction field $k(\eta)$ and $(G_\eta, \mathcal{L}_\eta)$ be a polarised abelian variety over $k(\eta)$. Then after a suitable finite ramified cover* Spec $R' \to$ Spec R *it can be completed to a flat projective scheme $(P, \mathcal{L})$ (or $(Q, \mathcal{L})$) over R' with a relatively ample invertible sheaf $\mathcal{L}$ extending $\mathcal{L}_\eta$.*

We call the closed fibre $(P_0, \mathcal{L}_0)$ (resp. $(Q_0, \mathcal{L}_0)$) a torically stable quasi-abelian variety (abbr. SQAV) respectively a projectively stable quasi-abelian scheme (abbr. PSQAS). We note $(P_0, \mathcal{L}_0) \simeq (Q_0, \mathcal{L}_0)$ if the dimension is less than five. $(P_0, \mathcal{L}_0)$ is always reduced, while $(Q_0, \mathcal{L}_0)$ can be nonreduced if the dimension is greater than four. $(Q_0, \mathcal{L}_0)$ determines $(P_0, \mathcal{L}_0)$ uniquely but the converse is unknown.

By applying [Kempf78, Corollary 5.1] we will prove

Theorem 0.2. *Let $(Q_0, \mathcal{L}_0)$ be a projectively stable quasi-abelian scheme over an algebraically closed field k. Suppose the characteristic of k and $N := \deg \mathcal{L}_0/(g!)$ are coprime. If $\Gamma(Q, \mathcal{L}) \otimes k$ is very ample, then the n-th normalised Hilbert point of $(Q_0, \mathcal{L}_0)$ has a closed $SL(N, k)$-orbit,* [1] *and it is Mumford-semistable for any large n.*

It seems that we cannot expect any similar theorem for $(P_0, \mathcal{L}_0)$ except Mumford-semistability. The following is an analogue of [Gieseker82] and [Mumford77].

Theorem 0.3. *Let k be an algebraically closed field and K a finite abelian group of order N with $e_{\min}(K) \geq 3$* [2] *such that the characteristic of k and N are coprime. Suppose that a k-scheme (Z, L) is smoothable into an abelian variety (A, M) with $\ker \lambda(M) \simeq K \oplus K^\vee$ where $K^\vee := \mathrm{Hom}_{\mathbf{Z}}(K, \mathbf{G}_m)$. Then the following are equivalent.*

1. *(Z, L) is a projectively stable quasi-abelian scheme*
2. *$\mathrm{Aut}(Z, L)$ contains a subgroup of $SL_\pm(N, k)$ weight-one isomorphic to $G(K)$*[3]
3. *the n-th Hilbert point of (Z, L) is Kempf-stable for any large n*

where $G(K)$ is a central extension of $K \oplus K^\vee$ by the cyclic group μ_N of all the N-th roots of unity.

The group $G(K)$ is noncommutative of order N^3, which is a natural substitute for the classical (in general infinite) Heisenberg group of abelian varieties. The group algebra $k[K^\vee]$ of the dual group $K^\vee$ over k is an irreducible $G(K)$-module of weight one. The k-module $\Gamma(Q, \mathcal{L}) \otimes k$ is isomorphic to $k[K^\vee]$ as a $G(K)$-module. If $e_{\min}(K) \geq 3$, any PSQAS $(Q_0, \mathcal{L}_0)$ over k is embedded into the projective space $\mathbf{P}(k[K^\vee])$ (Theorem 5.3). Hence any PSQAS $(Q_0, \mathcal{L}_0)$ over k has a linearised action of $G(K)$. Thus

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[1] We call it Kempf-stable. See Section 7 for normalised Hilbert points.

[2] $e_{\min}(K) = e_1$ if $K \simeq \oplus_{i=1}^{g} \mathbf{Z}/e_i\mathbf{Z}$, $e_i | e_{i+1}$.

[3] $SL_\pm(N, k) = \{g \in GL(N, k); \det g = \pm 1\}$. See 6.3 for weight-one isomorphisms.

we are led to the notion of level $G(K)$-structures on PSQAS's generalising the classical notion of level structures on abelian varieties to formulate the moduli problem for PSQAS's.

Theorem 0.4. *Let K be a finite abelian group of order N with $e_{\min}(K) \geq 3$. The functor of g-dimensional projectively stable quasi-abelian schemes with level $G(K)$-structure is reductively-represented by a projective scheme $SQ_{g,K}$ over $\mathbf{Z}[\zeta_N, 1/N]$.*

We prove Theorem 0.4 with the help of Theorem 0.3 and Schur's lemma for irreducible $G(K)$-modules of weight one.

In Section 1 we discuss Hesse cubics as an example of projectively stable quasi-abelian schemes with rigid structures. We will show our main idea for the proof of Theorem 0.3 in this particular case in detail. In the rest of the article we discuss the general case without proofs.

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1. An example—Hesse cubics

1.1. Let us start with Hesse cubics to illustrate our theory. For simplicity we consider an algebraically closed field k of characteristic $\neq 3$ and ζ_3 a primitive cube root of 1. Let $K = K^\vee = \mathbf{Z}/3\mathbf{Z}$ and $H(K) := K \oplus K^\vee$ and Let $V[K]$ be the group algebra of $K^\vee$ over $\mathbf{Z}[\zeta_3, 1/3]$, that is, the algebra generated by $v(\beta)$ $(\beta \in K^\vee)$ subject to the group relation. Let $G(K)$ be a central extension of $H(K) := K \oplus K^\vee$ by $\mu_3 := \{c \in k; c^3 = 1\}$

$$1 \to \mu_3 \to G(K) \to H(K) \to 0$$

whose group law of $G(K)$ is defined by

$$(a, z, \alpha) \cdot (b, w, \beta) = (ab\beta(z), z + w, \alpha + \beta) \tag{1}$$

$$(a, b \in \mu_3, z, w \in K, \alpha, \beta \in K^\vee) \tag{2}$$

Then $G(K)$ operates upon $V(K)$ by

$$U(K)(a, z, \alpha)(v(\beta)) = a\beta(z)v(\beta + \alpha) \tag{3}$$

In other words, for $\beta = 0, 1, 2$

$$U(K)(1, 0, 1)(v(\beta)) = v(\beta + 1) = \sigma^*(v(\beta)) \tag{4}$$

$$U(K)(1, 1, 0)(v(\beta)) = \zeta^\beta v(\beta) \tag{5}$$

$$U(K)(a, 0, 0)(v(\beta)) = av(\beta) \tag{6}$$

Let $R := k[[s]]$, $I = sR$ the maximal ideal of R, $k(\eta) := k((s))$ the fraction field of R. Let $S := \text{Spec } R$, 0 the closed point of S, and η the generic point of S.

A Hesse cubic we consider first is a subscheme Z of $\mathbf{P}^1_R \times \mathbf{P}^2_R$ defined by

$$s^2(x_0^3 + x_1^3 + x_2^3) - 3x_0x_1x_2 = 0$$

where we have chosen s^2 in order to make theta series expression simpler.

The closed fibre Z_0 is a 3-gon, in other words, a union of 3 lines with 3 nodes, while the generic fibre Z_η is a smooth elliptic curve with a natural very ample sheaf $\mathcal{L}_\eta := O_{Z_\eta}(1)$.

The elliptic curve Z_η has 9 sections $= {}_3Z_\eta$ (all 3-torsion points)

$$e_0 : (x_0, x_1, x_2) = (0, 1, -1),\ e_1 := (-1, 0, 1),\ e_2 := (1, -1, 0)$$
$$e_3 := (0, 1, -\zeta),\ e_4 := (-\zeta, 0, 1),\ e_5 := (1, -\zeta, 0)$$
$$e_6 := (0, 1, -\zeta^2),\ e_7 := (-\zeta^2, 0, 1),\ e_8 := (1, -\zeta^2, 0)$$

Let $C_i : x_i = 0$ be a line and let $G := E \setminus C_1 \cup C_2$. Then G is a semi-abelian scheme with $G_\eta \simeq Z_\eta$ and $G_0 \simeq \mathbf{G}_m$ a split torus.

Let $K := \{e_0, e_3, e_6\} \simeq \mathbf{Z}/3\mathbf{Z}$ and $K^\vee := \{e_0, e_1, e_2\} \simeq \mathbf{Z}/3\mathbf{Z}$. Let $\lambda(\mathcal{L}_\eta) : G_\eta \to G_\eta^t \simeq G_\eta$ ($G_\eta^t := \text{Pic}^0(G_\eta)$ the dual abelian scheme of G_η) be a polarisation morphism. Since $\lambda(\mathcal{L}_\eta)$ is the multiplication by 3, we have

$$K(\mathcal{L}_\eta) := \ker \lambda(\mathcal{L}_\eta) = {}_3Z_\eta = K \oplus K^\vee$$

Let $G(\mathcal{L}_\eta)$ be a central extension

$$1 \to \mu_{3,\eta} \to G(\mathcal{L}_\eta) \to K(\mathcal{L}_\eta) \to 0$$

We call $G(\mathcal{L}_\eta)$ the finite Heisenberg group of $(Z_\eta, \mathcal{L}_\eta)$. We see $K(\mathcal{L}_\eta) \simeq K \oplus K^\vee$, $G(\mathcal{L}_\eta) \simeq G(K)$ as étale group schemes over $k(\eta)$ (hence essentially as discrete groups).

The R-free module $\Gamma(G, \mathcal{L})$ has a basis x_β ($\beta \in K^\vee = \{0, 1, 2\}$). There is a $G(K)$-isomorphism ϕ^* of $V(K) \otimes R$ and $\Gamma(G, \mathcal{L})$ defined by $\phi^*(v(\beta)) = x_\beta$ through the isomorphism $G(\mathcal{L}_\eta) \simeq G(K)$.

A remarkable fact is that $\Gamma(G_\eta, \mathcal{L}_\eta)$ is by Mumford an irreducible $G(\mathcal{L}_\eta)$-module, unique up to equivalence, such that the center $\mu_{3,\eta}$ acts as scalar multiplication. This implies by Schur's lemma that if we fix the matrix form of the action $U(K)$ of $G(\mathcal{L}_\eta)$, then the basis of $\Gamma(G_\eta, \mathcal{L}_\eta)$ is uniquely determined up to constant multiple, in other words, if we let y_β be another basis of $\Gamma(G_\eta, \mathcal{L}_\eta)$ with the same form of $U(K)$ as x_β, then $y_\beta = cx_\beta$ for some $c \in K$.

1.2. Now we extend the above action of $G(\mathcal{L}_\eta)$ to that over S. This is done in fact in an obvious manner in this case. However in order to suggest the construction in the general case we proceed as follows.

Let $\sigma : (x_0, x_1, x_2) \to (x_2, x_0, x_1)$ be a transformation of $\mathbf{P}^2$. We see $e_{3j+k} = \sigma^k(e_{3j})$ for $k = 0, 1, 2$. Let

$$G^\natural := \cup_{\beta \in K^\vee} \sigma^\beta(G)$$

and $K^\natural_S(\mathcal{L}_\eta)$ (resp. $K_S(\mathcal{L}_\eta)$) the closure (to be precise, the flat closure) of $K_S(\mathcal{L}_\eta)$ in $G^\natural$ (resp. G). Let $G^\natural_S(\mathcal{L}_\eta)$ (resp. $G_S(\mathcal{L}_\eta)$) be the central extension of $K^\natural_S(\mathcal{L}_\eta)$ (resp. $K_S(\mathcal{L}_\eta)$) by $\mu_{3,S} := \{a \in \mathbf{G}_{m,S}; a^3 = 1\}$

$$1 \to \mu_{3,S} \to G^\natural_S(\mathcal{L}_\eta) \to K^\natural_S(\mathcal{L}_\eta) \to 0$$
$$1 \to \mu_{3,S} \to G_S(\mathcal{L}_\eta) \to K_S(\mathcal{L}_\eta) \to 0$$

It is easy to see

$$\begin{aligned} G^\natural &= Z \setminus \mathrm{Sing}(Z_0) \\ \mathrm{Sing}(Z_0) &= (1,0,0) \cup (0,1,0) \cup (0,0,1) \\ K^\natural_S(\mathcal{L}_\eta) &= \{e_i; 0 \le i \le 8\} \simeq (\mathbf{Z}/3\mathbf{Z})^{\oplus 2}_S \\ K_S(\mathcal{L}_\eta) &= \{e_0, e_3, e_6\} \cup \{e_{i,\eta}; i \ne 0,3,6\} \simeq (\mathbf{Z}/3\mathbf{Z})_S \\ K^\natural_S(\mathcal{L}_\eta) \cap Z_0 &= \{e_i; 0 \le i \le 8\} \otimes (R/I) \simeq (\mathbf{Z}/3\mathbf{Z})^{\oplus 2} = 9 \text{ points} \\ K_S(\mathcal{L}_\eta) \cap Z_0 &= \{e_0, e_3, e_6\} \otimes (R/I) \simeq \mathbf{Z}/3\mathbf{Z} = 3 \text{ points} \end{aligned}$$

The theory of Mumford and Moret-Bailly says that

Lemma 1.3. *$\Gamma(G, \mathcal{L}) = \Gamma(Z, \mathcal{L}) = Rx_0 + Rx_1 + Rx_2$ is an irreducible $G^\natural_S(\mathcal{L}_\eta)$-module of weight one (= center acting as scalar multiplication) in the sense that any proper $G^\natural_S(\mathcal{L}_\eta)$-submodule $\Gamma(G, \mathcal{L})$ is of the form $J\Gamma(G, L)$ for an ideal J of R.*

The action of $G^\natural_S(\mathcal{L}_\eta)$ are the same as $U(K)$ of $G(\mathcal{L}_\eta)$.

1.4. By the theory in SGA III we see that the formal completion G_{for} is a formal split torus

$$G_{\mathrm{for}} \simeq \mathbf{G}_m \times \mathrm{Spf}\ R$$

over Spf R because G_0 is a split torus $\mathbf{G}_m$. We note then that $\mathcal{L}_{\mathrm{for}}$ is trivial on the split torus because any invertible sheaf on the split torus is trivial. Hence we see

$$x_\beta \in \Gamma(G, \mathcal{L}) \subset \Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}) = \prod_{x \in \mathbf{Z}} R w^x$$

where w is the coordinate such that $\mathbf{G}_m \simeq \mathrm{Spec}\, k[w^x; x \in \mathbf{Z}]$. By using the representation theory of $G_S(\mathcal{L}_\eta)$ (not of $G^\natural_S(\mathcal{L}_\eta)$) Faltings-Chai constructed a degeneration data (an algebraic analogue of coefficients of theta series). This means that (after some normalisations of various parameters) on the formal completion G_{for} the coordinate x_β can be expressed as theta series with a suitable parameter $t = s \cdot$(a unit in R)

$$x_\beta = \theta_\beta(t,w) = \sum_{m\in\mathbf{Z}} t^{(3m+\beta)^2} w^{3m+\beta} \quad (\beta = 0,1,2)$$

With the notation in §2 Faltings-Chai's degeneration data are given by

$$a(x) = t^{x^2}, \quad b(x,y) = t^{2xy} \tag{7}$$

In particular $a(x+y) = a(x)a(y)b(x,y) \in k(\eta)$. The parameters s and t are related by $s^2 = \lambda(t) = t^2 \cdot$(a unit in R) via the theta relation

$$\lambda(t)(\theta_0^3 + \theta_1^3 + \theta_2^3) - 3\theta_0\theta_1\theta_2 = 0$$

1.5. Starting from the degeneration data (7), we can construct two kinds of model families $(P,\mathcal{L})$ and $(Q,\mathcal{L})$. $(Q,\mathcal{L})$ is the algebraisation of the formal quotient

$$(\widetilde{Q}_{\text{for}}, O(1)_{\text{for}})/\{S_y; y \in 3\mathbf{Z}\}$$

where

$$\widetilde{Q} = \operatorname{Proj}\widetilde{R}, \quad \widetilde{R} := R[a(x)w^x\vartheta, x \in \mathbf{Z}]$$
$$S_y(a(x)w^x\vartheta) = a(x+y)w^{x+y}\vartheta \ (y \in 3\mathbf{Z})$$

Let $\widetilde{P}$ be the normalisation of $\widetilde{Q}$, $\widetilde{\mathcal{L}}_{\text{for}}$ the pull back of $O(1)$ to $\widetilde{P}$ and $(P,\mathcal{L})$ the algebraisation of the formal quotient of $(\widetilde{P}_{\text{for}}, \widetilde{\mathcal{L}}_{\text{for}})/\{S_y; y \in 3\mathbf{Z}\}$. Hence $(P,\mathcal{L})$ is the normalisation of $(Q,\mathcal{L})$. In this case $(P,\mathcal{L}) \simeq (Q,\mathcal{L})$. In dimension ≥ 5, P and Q can be different.

Definition 1.6. $(Q,\mathcal{L})$ (resp. the closed fibre $(Q_0,\mathcal{L}_0)$ of $(Q,\mathcal{L})$) is called a projectively stable quasi abelian scheme over S (resp. over k). For brevity we call each a PSQAS over S or k.

In order to explain our idea of the proof of Theorem 0.3 let us define;

Definition 1.7. Let $\bar{G}(K) := U(K)G(K)$ and $G(Z,\mathcal{L}) = G_S^\natural(\mathcal{L}_\eta)$. The sextuplet $(Z,\mathcal{L},G(Z,\mathcal{L}),V,\phi,\rho)$ is called a cubic curve over S with a rigid $\bar{G}(K)$-structure if

(i) $\phi : Z \to \mathbf{P}(V(K)\otimes R) \simeq \mathbf{P}_S^2$ is a closed S-immersion
(ii) $V = \Gamma(Z,\mathcal{L}) = \phi^*(V(K)\otimes R)$
(iii) ρ is an isomorphism of $G(K)_S$ onto $G(Z,\mathcal{L})$ given by

$$\rho(g) = G(\phi^*)U(K)(g) := (\phi^*)\cdot U(K)(g)\cdot(\phi^*)^{-1}$$

We denote the sextuplet $(Z,\mathcal{L},G(Z,\mathcal{L}),V,\phi,\rho)$ by $(Z,\phi,\rho)_{\text{RIG}}$. We call also the closed fibre $(Z_0,\phi_0,\phi_0)_{\text{RIG}}\otimes k$ a cubic curve over k with a rigid $\bar{G}(K)$-structure.

We note

Lemma 1.8. *A cubic curve* $(Z, \phi, \rho)_{\mathrm{RIG}}$ *(with a rigid* $\bar{G}(K)$*-structure) over* S *is* S*-isomorphic to a Hesse cubic*

$$\mu_0(x_0^3 + x_1^3 + x_2^3) - 3\mu_1 x_0 x_1 x_2 = 0$$

for some $[\mu_0, \mu_1] \in \mathbf{P}^1_S$.

Proof. The space $S^3(V(K)^\vee)$ of all cubic polynomials is 10 dimensional, which is decomposed into 8 one-dimensional $G(K)$-modules $V(\chi_k)$ $(k = 1, \cdots, 8)$ and 1 two-dimensional $G(K)$-module $V(\chi_0)$ where χ_k ranges over the set of all the 9 characters of $G(K)$. $V(\chi_0)$ is spanned by $x_0^3 + x_1^3 + x_2^3$ and $x_0x_1x_2$. Hence $(Z, \phi, \rho)_{\mathrm{RIG}}$ is isomorphic to either a Hesse cubic as above or a cubic curve C_k defined by $f_k \in V(\chi_k)$, where $f_1 = x_0^3 + \zeta_3 x_1^3 + \zeta_3^2 x_2^3$ for instance. It is easy to see that C_k is also isomorphic to a Hesse cubic by enlarging R if necessary. Let us forget these special cases because they have no nonconstant moduli. □

1.9. Now we will give a different proof of Lemma 1.8 by the argument applicable to the general case. Suppose we are given an arbitrary cubic curve $(Z, \mathcal{L})$ such that $(Z_\eta, \mathcal{L}_\eta)$ is a smooth cubic curve over S with $\ker \lambda(\mathcal{L}_\eta) \simeq H(K)_S$ where $K = \mathbf{Z}/3\mathbf{Z}$. Suppose that $(Z, \mathcal{L})$ has a rigid $\bar{G}(K)$-structure (ϕ, ρ_Z). In other words, we are given, first of all, the action ρ_Z of $G(Z, \mathcal{L}) \simeq G(K)_S$ extending that of $G(Z_\eta, \mathcal{L}_\eta) \simeq G(K)_\eta$ on $\Gamma(Z_\eta, \mathcal{L}_\eta) = \Gamma(Z, \mathcal{L}) \otimes k(\eta)$. $\Gamma(Z, \mathcal{L})$ is a unique irreducible $G(K)_S$-module of weight one by Lemma 1.3, there is a basis $x(\beta)$ $(\beta \in K^\vee)$ of $\Gamma(Z, \mathcal{L})$ such that

$$\rho_Z(a, z, \alpha)(x(\beta)) = a\beta(z)x(\beta + \alpha) \quad \forall (a, z, \alpha) \in G(K)_S \tag{8}$$

The embedding ϕ of $(Z, \mathcal{L})$ into $\mathbf{P}(V(K) \otimes R)$ is induced from the isomorphism $\phi^* : V(K) \otimes R \to \Gamma(Z, \mathcal{L})$. Since $V(K)$ is generated by $v(\beta)$ $(\beta \in K^\vee)$, $\phi^*(v(\beta))$ is a linear combination of $x(\beta)$. However the condition that (ϕ, ρ_Z) is a rigid $\bar{G}(K)$-structure means that ϕ^* is given by $x(\beta) = \phi^*(v(\beta))$.

Let us however suppose that we do not know the structure of $(Z_0, \mathcal{L}_0)$. What we are going to do is to show that $(Z_0, \mathcal{L}_0)$ is isomorphic to the closed fibre of the flat projective family $(Q, \mathcal{L})$ we can construct from the degeneration data of $(Z_\eta, \mathcal{L}_\eta)$.

Let $(G, \mathcal{L})$ be a semi-abelian scheme extending $(G_\eta, \mathcal{L}_\eta) := (Z_\eta, \mathcal{L}_\eta)$ over S, say a connected Néron model of G_η by taking a finite base change of S if necessary. By Faltings-Chai we have a degeneration data $a(x)$ and $b(x, y)$ so that we can construct as before two flat projective schemes $(P, \mathcal{L})$ and $(Q, \mathcal{L})$ such that

(i) $G \subset P$,
(ii) $(P_\eta, \mathcal{L}_\eta) \simeq (G_\eta, \mathcal{L}_\eta) \simeq (Q_\eta, \mathcal{L}_\eta)$
(iii) P is the normalisation of Q
(iv) $\Gamma(Q, \mathcal{L})$ is very ample

(v) $\Gamma(P,\mathcal{L}) \simeq \Gamma(G,\mathcal{L}) \simeq \Gamma(Q,\mathcal{L})$

(vi) the action of $G(Z_\eta,\mathcal{L}_\eta) \simeq G(K)_\eta$ on G_η extends to $(P,\mathcal{L})$ and $(Q,\mathcal{L})$ as the action of (an analogue of) $G_S^\sharp(\mathcal{L}_\eta) \simeq G(K)_S$ so that $\Gamma(Q,\mathcal{L})$ is an irreducible $G_S^\sharp(\mathcal{L}_\eta) \simeq G(K)_S$-module of weight one.

Hence again by Lemma 1.3 there is a basis $\theta(\beta)$ ($\beta \in K^\vee$), unique up to scalar multiple, which are transformed under $G(K)_S$ as in (3) and (8). In fact, we have from the construction of P and Q

$$\theta(\beta) = \sum_{\alpha\in 3\mathbf{Z}} a(\beta+\alpha)w^{\beta+\alpha} \tag{9}$$

The action of $G(Q,\mathcal{L})$ extending that of $G(Z_\eta,\mathcal{L}_\eta)$ is given by

$$\rho_Q(a,z,\alpha)(\theta(\beta)) = a\beta(z)\theta(\beta+\alpha) \tag{10}$$

We can show that a rigid $\bar{G}(K)$-structure (ϕ_η,ρ_η) on $(Q_\eta,\mathcal{L}_\eta)$ extends to a rigid $\bar{G}(K)$-structure (ψ,ρ_Q) on $(Q,\mathcal{L})$. This implies that ψ is the embedding of $(Q,\mathcal{L})$ into $\mathbf{P}^2_S$ given by $\theta(\beta) = \psi^*(v(\beta))$ where $v(\beta) \in V(K)$ as before.

The embeddings ϕ and ψ induce natural morphisms $\mathrm{Hilb}(\phi)$ and $\mathrm{Hilb}(\psi)$ from $\mathrm{Spec}\,R$ into the Hilbert scheme $\mathrm{Hilb}^{P(n)}_{\mathbf{P}^2}$,

$$\mathrm{Hilb}(\phi) : \mathrm{Spec}\,R \to \mathrm{Hilb}^{P(n)}_{\mathbf{P}^2} \tag{11}$$

$$\mathrm{Hilb}(\psi) : \mathrm{Spec}\,R \to \mathrm{Hilb}^{P(n)}_{\mathbf{P}^2} \tag{12}$$

where $P(n) = 3n$. There exists an isomorphism $f : (Q_\eta,\mathcal{L}_\eta) \simeq (Z_\eta,\mathcal{L}_\eta)$ so that $f^* : \Gamma(Z_\eta,\mathcal{L}_\eta) \to \Gamma(Q_\eta,\mathcal{L}_\eta)$ is an isomorphism. By the construction of $(Q,\mathcal{L})$, f^* is $G(K)$-equivariant, that is, $f^*(\rho_z(g)(x)) = \rho_Q(g)f^*(x)$ ($\forall x \in \Gamma(Z_\eta,\mathcal{L}_\eta)$). Since both $\rho_Z(g)$ and $\rho_Q(g)$ have the same matrix expression, say, $B(g)$ with regards to the basis $x(\beta)$ and $\theta(\gamma)$, we see

$$FB(g) = B(g)F \quad (\forall g \in G(K)) \tag{13}$$

where F is the matrix expression of f^* with regards to $x(\beta)$ and $\theta(\gamma)$. Since $B(g) = U(K)(g)$ and $U(K)$ is an irreducible representation of $G(K)$, F is a scalar matrix by Schur's lemma. It follows that $f^*(x(\beta)) = c\theta(\beta)$ for some nonzero $c \in k(\eta)$.

Thus we have $\mathrm{Hilb}(\phi)_\eta = \mathrm{Hilb}(\psi)_\eta$, or equivalently $\phi(Z_\eta) = \psi(Q_\eta) \subset \mathbf{P}^2_\eta$. It follows from separatedness of $\mathrm{Hilb}^{P(n)}_{\mathbf{P}^2}$ that $\mathrm{Hilb}(\phi) = \mathrm{Hilb}(\psi)$.

This implies that $(Z,\mathcal{L}) \simeq (Q,\mathcal{L})$. In particular, $(Z_0,\mathcal{L}_0) \simeq (Q_0,\mathcal{L}_0)$.

This proves the following

Theorem 1.10. *Any cubic curve $(Z,\mathcal{L})$ over S with a rigid $\bar{G}(K)$-structure is isomorphic to a projectively stable quasi abelian scheme $(Q,\mathcal{L})$ which is constructed from the degeneration data of a smooth cubic curve $(Z_\eta,\mathcal{L}_\eta)$ over $k(\eta)$. In particular $(Z_0,\mathcal{L}_0) \simeq (Q_0,\mathcal{L}_0)$.*

The above argument works as well in arbitrary dimension and it is a key to the proof of Theorem 0.3 and Theorem 0.4.

1.11. The above argument also implies that the functor of cubic curves $(Z, \mathcal{L})$ with rigid $\bar{G}(K)$ structures or with an action of $G(K)$ on $\Gamma(Z, \mathcal{L})$ as above is represented by a projective scheme $SQ_{1,\mathbf{Z}/3\mathbf{Z}}$. In fact, the universal subscheme $(Z, \mathcal{L})$ is given by the Hesse cubic

$$\mu_0(x_0^3 + x_1^3 + x_2^3) - 3\mu_1 x_0 x_1 x_2 = 0$$

so that

$$SQ_{1,\mathbf{Z}/3\mathbf{Z}} \simeq \mathbf{P}^1_{\mathbf{Z}[\zeta_3, 1/3]} := \mathrm{Proj}(\mathbf{Z}[\zeta_3, 1/3][\mu_0, \mu_1])$$

TABLE 1. Stability of reduced cubic curves

curves (sing.)	stability	stab. gr.
smooth elliptic	properly stable	finite
3-gon	Kempf-stable/not properly stable	2-dim
irred. a node	semi-stable/not Kempf-stable	$\mathbf{Z}/2\mathbf{Z}$
a triple point	unstable	2-dim

1.12. Why does the above argument work ? Apart from the actual proofs, the real reason behind the argument is GIT-stability, we believe. The stability in [MFK,p.80] of a cubic curve is the stability of the third Hilbert point of it. By Table 1, a 3-gon is Kempf-stable, that is, by definition it has a closed $SL(3, k)$-orbit.

Lemma 1.13. *Let $(Z, \mathcal{L})$ be a cubic curve over k.*

(i) *If $\Gamma(Z, \mathcal{L})$ is an irreducible $G(K)$-module, then any n-th Hilbert point of $(Z, \mathcal{L})$ is Kempf-stable, that is, they have closed $SL(3, k)$-orbits*

(ii) *Any smooth cubic curve or 3-gon is Kempf-stable. It is therefore Mumford-semistable, but the 3-gon is not Mumford-stable.*

We will see by applying [Kempf78] that Lemma 1.13 follows from Lemma 1.3. As is concerned about construction of fine moduli schemes, we need no stability theorems like Lemma 1.13. However it is Theorem 0.3 that led us to the formulation of the functor $SQ_{g,K}$ in Sections 8-10.

2. Degeneration data

The purpose of this section is to sketch the description of degenerations of abelian varieties given by Faltings-Chai [FC90, II.4.1,5.1] and Moret-Bailly [MB85, IV-VI]. See also [AN96, Section 2]. We follows mainly the notation of [FC90].

2.1. Let R be a complete discrete valuation ring, $S = \text{Spec } R$. Let K be the fraction field of R, η the generic point of S and $k = R/I$ the residue field.

Suppose that we are given an abelian scheme G_η over K. Then by Grothendieck's Stable reduction theorem [SGA7] (See also [SGA7, Expóse I pp. 1-24]) G_η can be extended to a semiabelian scheme G over R as the connected Néron model of G_η by taking a finite extension of K if necessary. Moreover by taking a finite extension of K if necessary there exists an invertible sheaf $H \in \text{Pic}^0(G_\eta)$ such that $\mathcal{L}_\eta \otimes H$ is symmetric, namely $i^*(\mathcal{L}_\eta \otimes H) = \mathcal{L}_\eta \otimes H$ for the involution $i = [-1]_{G_\eta}$ of G_η.

Therefore we may assume from the start that $\mathcal{L}_\eta$ is symmetric, ample and rigidified along the unit section. The invertible sheaf $\mathcal{L}_\eta$ associates to some Cartier divisor, which extends uniquely to a smooth scheme G. Therefore $\mathcal{L}_\eta$ extends to G uniquely because G_0 is irreducible. See [MB85, II, 1.1]. On the other hand by [Raynaud70, p.158 XI, 1.2 and p.170 XI 1.13] $\mathcal{L}_\eta^{\otimes n}$ extends to G as an ample invertible sheaf for some $n > 0$ if $\mathcal{L}_\eta$ is symmetric and ample. Since $\mathcal{L}_\eta$ satisfies the condition in this case, the extension $\mathcal{L}$ is ample and symmetric.

Thus for a given an abelian scheme G_η over K we have a semiabelian S-scheme G of relative dimension g with generic fibre G_η (with a chosen unit section), $\mathcal{L}$ a rigidified relatively ample invertible sheaf on G. The special fibre G_0 is a semiabelian scheme over k, namely an extension of an abelian scheme A_0 of relative dimension g' by a torus T_0 of relative dimension g'', $g' + g'' = g$. We assume T_0 to be split by taking a finite base change of S, in other words, by taking a finite extension of K if necessary and the integral closure of R in it and by repeating the same construction as above.

2.2. Associated to G and $\mathcal{L}$ are the formal scheme $G_{\text{for}} = \lim G \otimes R/I^n$ and an invertible sheaf $\mathcal{L}_{\text{for}} = \lim \mathcal{L} \otimes R/I^n$. The scheme G_{for} fits into an exact sequence

$$1 \to T_{\text{for}} \to G_{\text{for}} \overset{\pi_{\text{for}}}{\to} A_{\text{for}} \to 0$$

where A_{for} is a formal abelian scheme. By the theory of cubical structures [Breen83] [MB85, p.40, Theorem 1.1 (ii)] there exists a unique cubical structure on $\mathcal{L}$ (viewed as a $\mathbf{G}_m$-torsor), which induces a cubical structure of the sheaf $\mathcal{L}_{\text{for}}$.

Then $\mathcal{L}_{\text{for}}$ is descended to a unique cubical ample invertible sheaf $\mathcal{M}_{\text{for}}$ on A_{for}, that is, $\mathcal{L}_{\rm for} = \pi^*_{\rm for}(\mathcal{M}_{\rm for})$ [4]. Since there exists an ample sheaf on $A_{\rm for}$, $A_{\rm for}$ is algebraisable. In other words by the algebraisation theorem of Grothendieck [EGA, III, 5.4.5] there exists an abelian S-scheme A with an ample invertible sheaf $\mathcal{M}$ such that the formal completion $(\hat{A}, \hat{\mathcal{M}})$ of $(A, \mathcal{M})$ is $(A_{\rm for}, \mathcal{M}_{\rm for})$.

By our assumption that T_0 is a k-split torus, $T_{\rm for}$ is a formal S-split torus by [SGA3, IX, Théorèm 3.6], [FC90, 2.2]. Let X be the character group of $T_{\rm for}$. Then by setting $T := \mathrm{Hom}_{\,\mathbf{Z}}(X, \mathbf{G}_m)$, T algebraises $T_{\rm for}$.

The sequence $1 \to T_{\rm for} \to G_{\rm for} \to A_{\rm for} \to 0$ is also algebraisable because the extension class of it is given by an element of $\mathrm{Ext}(A_{\rm for}, T_{\rm for}) \simeq \mathrm{Ext}(A, T)$ [FC90, p.34]. The dual abelian scheme G^t_η is also extended to a semiabelian S-scheme G^t by taking the connected Néron model [5] after taking a finite ramified cover of S if necessary. Then similarly we see that $G^t_{\rm for}$ is algebraisable. Namely there exists a semiabelian scheme $\tilde{G}^t$ such that $(\tilde{G}^t)_{\rm for} \simeq G_{\rm for}$. Thus we obtain the so called Raynaud extensions for $G_{\rm for}$ and $G^t_{\rm for}$

$$1 \to T \to \tilde{G} \xrightarrow{\pi} A \to 0$$

$$1 \to T^t \to \tilde{G}^t \xrightarrow{\pi^t} A^t \to 0$$

plus the homomorphisms $c : X \to A^t(R)$, $c^t : Y \to A(R)$ decoding them. In other words, $c \in \mathrm{Hom}(X, A^t(R)) \simeq \mathrm{Ext}(A, T)$ and $c^t \in \mathrm{Hom}(Y, A(R)) \simeq \mathrm{Ext}(A^t, T^t)$ describe the extension classes of semiabelian schemes $\tilde{G}$ and $\tilde{G}^t$ respectively.

2.3. Let $\lambda(\mathcal{L}_\eta) : G_\eta \to G^t_\eta$ be the polarisation morphism. Then by the universal property of the (connected) Néron model G^t of G^t_η we have an extension $\lambda : G \to G^t$ of $\lambda(\mathcal{L}_\eta)$. This gives rise to a formal morphism $\lambda_{\rm for} : G_{\rm for} \to G^t_{\rm for}$, which is algebraised into a morphism $\tilde{\lambda} : \tilde{G} \to \tilde{G}^t$ because $G_{\rm for}$ and $G^t_{\rm for}$ are quasi-projective [EGA, III, 5.4.1]. Since T is affine and A^t is projective, $\tilde{\lambda}(T)$ is the identity of $\tilde{A}^t$ so that we have a morphism $\lambda_T := \tilde{\lambda}_{|T} : T \to T^t$. Similarly we have a morphism $\lambda_A := \lambda(\mathcal{M}) : A \to A^t$ such that $\lambda_A \pi = (\pi^t)\tilde{\lambda}$.

Let $K(\mathcal{L}_\eta)$ be the kernel of $\lambda(\mathcal{L}_\eta)$. Let $\mathcal{G}(\mathcal{L}_\eta)$ (the Heisenberg group) be the central extension of $K(\mathcal{L}_\eta)$ by $\mathbf{G}_{m,K}$ with the commutator form equal to the Weil pairing $e^{\mathcal{L}_\eta}$. See [Mumford74]. There is an exact sequence

$$1 \to \mathbf{G}_{m,K} \to \mathcal{G}(\mathcal{L}_\eta) \to K(\mathcal{L}_\eta) \to 0$$

By [MB85, V, 2.5.5], we see that $\Gamma(G_\eta, \mathcal{L}_\eta)$ is an irreducible $\mathcal{G}(\mathcal{L}_\eta)$-module of weight one, unique up to isomorphism by taking a finite extension of K if necessary. The

[4]This is true because $T_{\rm for}$ is a split torus. Otherwise we need to take a symmetric invertible sheaf $\mathcal{L}_{\rm for} \otimes [-1]^* \mathcal{L}_{\rm for}$ for descent.

[5]We mean by the connected Néron model the identity connected component of the Néron model (with connected closed fibre).

$\mathcal{G}(\mathcal{L}_\eta)$-module structure of $\Gamma(G_\eta, \mathcal{L}_\eta)$ is known by [MB85, V, 3.4] and [Mumford74, §23] regardless of the characteristic of K and rank $K(\mathcal{L}_\eta)$.

2.4. The space of theta functions $\Gamma(G_\eta, \mathcal{L}_\eta)$ on the generic fibre is embedded into $\Gamma(G_{\text{for}}, \mathcal{L}_{\text{for}}) \otimes_R K$. Since the latter has the torus action, every theta function $\theta \in \Gamma(G_\eta, \mathcal{L}_\eta)$ can be written as a Fourier series of eigenfunctions, and this series converges in the I-adic topology. The theorem of Faltings and Chai says that the coefficients of these Fourier series satisfy the same equations as in the classical complex analytic case.

2.5. First we consider the totally degenerate case, that is the case when A_0 (and hence A) is trivial. Then $G_{\text{for}} = T_{\text{for}}$ and $\tilde{G} = T$. The invertible sheaf $\mathcal{L}_{\text{for}}$ is trivial on T_{for}, and therefore

$$\Gamma(G_\eta, \mathcal{L}_\eta) = \Gamma(G, \mathcal{L}) \underset{R}{\otimes} k(\eta) \hookrightarrow \Gamma(G_{\text{for}}, \mathcal{L}_{\text{for}}) \underset{R}{\otimes} k(\eta) = \prod_{x \in X} k(\eta) \cdot w^x$$

Therefore, every theta function $\theta \in \Gamma(G_\eta, \mathcal{L}_\eta)$ can be written as a formal Fourier power series $\theta = \sum_{x \in X} \sigma_x(\theta) w^x$ with $\sigma_x(\theta) \in k(\eta)$.

Theorem 2.6. [Faltings–Chai90] *There exists a function $a : Y \to K^*$ and a bilinear function $b : Y \times X \to K^*$ with the following properties:*

(1) $b(y,z) = b(z,y) = a(y+z)a(y)^{-1}a(z)^{-1} \quad (\forall y, z \in Y)$
(2) $b(y,y) \in I$ $(\forall y \neq 0)$, *and for every* $n \geq 0$, $a(y) \in I^n$ *for almost all* $y \in Y$
(3) *The K-vector space $\Gamma(G_\eta, \mathcal{L}_\eta)$ is identified with the vector space of Fourier series θ that satisfy* $\sigma_{x+\phi(y)}(\theta) = a(y)b(y,x)\sigma_x(\theta)$.

Definition 2.7. The functions b and a can be extended respectively to $X \times X$ and X so that the previous relations between b and a are still true on $X \times X$. Then we define the functions $A : X \to \mathbf{Z}$, $B : X \times X \to \mathbf{Z}$ and $\bar{b}(y,x) \in R^*$, $\bar{a}(y) \in R^*$ by

$$B(y,x) = val_s(b(y,x)), \quad dA(\alpha)(x) = B(\alpha, x) + r(x)/2$$
$$A(x) = val_s(a(x)) = B(x,x)/2 + r(x)/2$$
$$b(y,x) = \bar{b}(y,x)s^{B(y,x)}, \quad a(x) = \bar{a}(x)s^{(B(x,x)+r(x))/2}$$

for some $r \in \mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{Z})$. We set $a_0 = \bar{a} \bmod I$ and $b_0 = \bar{b} \bmod I$. Therefore $a_0(x), b_0(x,y) \in k^*$ for any $x, y \in X$. B is positive definite by Theorem 2.6 (2).

3. Construction of $(P,\mathcal{L})$

3.1. We continue to consider the totally degenerate case. For simplicity we identify $\phi : Y \to X$ as the inclusion. We define

$$\begin{aligned}\widetilde{R} :&= R[a(x)w^x\vartheta; x \in X] \simeq R[\xi_x\vartheta; x \in X],\\ \xi_x :&= s^{B(x,x)/2+r(x)/2}w^x, \quad \xi_{x,c} := \xi_{x+c}/\xi_c\\ \zeta_{x,c} :&= s^{B(\alpha(\sigma),x)+r(x)/2}w^x \quad (x+c \in C(c,\sigma))\end{aligned}$$

where $\widetilde{R}$ is the graded algebra with $\deg(a(x)w^x\vartheta) = 1$ and $\deg a = 0$ for $a \in R$, while $\sigma \in \mathrm{Star}\,(c, \mathrm{Del}_B)$ is a maximal-dimensional Delaunay cell with $x + c \in C(c,\sigma)$.

Let $\widetilde{Q} := \mathrm{Proj}(\widetilde{R})$, and $\widetilde{P}$ the normalisation of $\widetilde{Q}$. We define an action S_y on $\widetilde{Q}$ by

$$S_y^*(a(x)w^x\vartheta) = a(x+y)w^{x+y}\vartheta \quad (y \in Y)$$

S_y induces a natural action of $\widetilde{P}$, which we denote by the same S_y. Let $\widetilde{\mathcal{L}}$ be $O_{\mathrm{Proj}}(1)$ on $\widetilde{Q}$ and its pull back to $\widetilde{P}$.

Theorem 3.2. *There exists a flat projective S-scheme $(P,\mathcal{L})$ such that the formal completion $(P_{\mathrm{for}},\mathcal{L}_{\mathrm{for}})$ of it along the closed fibre is isomorphic to $(\widetilde{P}_{\mathrm{for}},\widetilde{\mathcal{L}}_{\mathrm{for}})/Y$.*

Definition 3.3. If we take a suitable finite base change, we can assume P_0 to be reduced [AN96]. Then we call the closed fibre $(P_0,\mathcal{L}_0)$ of the flat projective family $(P,\mathcal{L})$ a polarised stable quasi-abelian variety over $k := R/I$.

Remark 3.4. The space $\Gamma(G_\eta,\mathcal{L}_\eta^n) = \Gamma(G,\mathcal{L}^n)\otimes K$ [6] is identified with the subspace of $\Gamma(G_{\mathrm{for}},\mathcal{L}_{\mathrm{for}}^n)\otimes K$ consisting of Fourier series $s = \sum_{y\in Y}\sigma_{\nu+ny}(s)w^{\nu+ny}$ such that

$$\sigma_{\nu+ny}(s) = a(y)^n b(y,x)\sigma_\nu(s), \sigma_\nu(s) \in K \quad (\forall \nu \in X, \forall y \in Y)$$

We see that $\Gamma(P_\eta,\mathcal{L}_\eta^n) = \Gamma(G_\eta,\mathcal{L}_\eta^n)$ so that $G_\eta \simeq P_\eta$.

We note that G is the semi-abelian scheme we started from, while P is the projective scheme we constructed with the degeneration data of G.

[6] G_0 is irreducible, so that the extension of $\mathcal{L}_\eta$ to G is unique. $\Gamma(G_\eta,\mathcal{L}_\eta^n) = \Gamma(G,\mathcal{L}^n)\otimes K$ follows from it.

3.5. Now we choose an embedding $G \subset P$. Let $G^\sharp := \cup_{x\in X/Y} S_x(G)$. Then $G^\sharp$ is a group scheme. Let $e(x) := S_x(e)$. Then $S_x = T_{e(x)}$ (translation of $G^\sharp$ by $e(x)$) on $G^\sharp$.

Let $K^\sharp_S(\mathcal{L}_\eta)$ be the flat closure of $K(\mathcal{L}_\eta)$ in $G^\sharp$. Then we see that $K^\sharp_S(\mathcal{L}_\eta)$ is finite.

Lemma 3.6. *We define a morphism* $\lambda(\mathcal{L}_0) : G^\sharp_0 \to \mathrm{Pic}^0(Q_0)$ *by*

$$\lambda(\mathcal{L}_0)(a) = T_a^*(\mathcal{L}_0) \otimes \mathcal{L}_0^{-1}$$

for any U*-valued point* a *of* $G^\sharp_0$*,* U *any* k*-scheme. Then* $K(Q_0, \mathcal{L}_0) = \ker \lambda(\mathcal{L}_0)$.

Definition 3.7. *The abelian Heisenberg group scheme* $K(P,\mathcal{L})$ is defined to be $H_S(\mathcal{L}_\eta)$. *The Heisenberg group scheme* $\mathcal{G}(P,\mathcal{L})$ is a central extension of $K(P,\mathcal{L})$ by $\mathbf{G}_{m,S}$, and the following is exact;

$$1 \to \mathbf{G}_{m,S} \to \mathcal{G}(P,\mathcal{L}) \to K(P,\mathcal{L}) \to 0.$$

We note $\mathcal{G}(P,\mathcal{L}) \otimes K = \mathcal{G}(G_\eta, \mathcal{L}_\eta) := \mathcal{G}(\mathcal{L}_\eta)$, $K(P,\mathcal{L}) \otimes K = K(G_\eta,\mathcal{L}_\eta) := K(\mathcal{L}_\eta)$.

We define $\mathcal{G}(P_0,\mathcal{L}_0) := \mathcal{G}(P,\mathcal{L}) \otimes k$ and $K(P_0,\mathcal{L}_0) := K(P,\mathcal{L}) \otimes k$.

4. The structure of $(Q, \mathcal{L})$

4.1. We consider the totally degenerate case. From Section 3 we recall

$$\widetilde{R} := R[a(x)w^x\vartheta; x \in X] \simeq R[\xi_x\vartheta; x \in X]$$
$$\xi_x := s^{B(x,x)/2+r(x)/2}w^x, \quad \xi_{x,c} := \xi_{x+c}/\xi_c$$
$$S_y^*(a(x)w^x\vartheta) = a(x+y)w^{x+y}\vartheta$$

Let $\widetilde{Q} := \mathrm{Proj}\ \widetilde{R}$ and $\widetilde{\mathcal{L}} := O_{\widetilde{Q}}(1)$.

The construction of the quotient $(Q,\mathcal{L}) := (\widetilde{Q},\widetilde{\mathcal{L}})/Y$ is quite similar to [Mumford72]. See also [AN96] and Theorem 3.2.

Theorem 4.2.

(1) *Let* $(\widetilde{Q}_0, \widetilde{\mathcal{L}}_0)$ *be the closed fibre of* $(\widetilde{Q}, \widetilde{\mathcal{L}})$*. Then* $\widetilde{Q}_0$ *is a scheme locally of finite type with infinitely many irreducible components. The restriction of* $\widetilde{\mathcal{L}}_0$ *to any irreducible component of* $\widetilde{Q}_0$ *is very ample.*
(2) $(\widetilde{Q}_0, \widetilde{\mathcal{L}}_0)/Y$ *is a projective scheme over* k.
(3) $(\widetilde{Q}_{\mathrm{for}}, \widetilde{\mathcal{L}}_{\mathrm{for}})/Y$ *is a flat projective formal* S*-scheme.*

(4) *There exists a flat projective S-scheme $(Q,\mathcal{L})$ such that the formal completion $(Q_{\mathrm{for}},\mathcal{L}_{\mathrm{for}})$ of it along the closed fibre is isomorphic to $(\tilde{Q}_{\mathrm{for}},\tilde{\mathcal{L}}_{\mathrm{for}})/Y$.*
(5) *$(P,\mathcal{L})$ is the normalisation of $(Q,\mathcal{L})$ (by a suitable base change).*

By Remark 3.4 and a similar consideration $(P_\eta,\mathcal{L}_\eta)\simeq(Q_\eta,\mathcal{L}_\eta)\simeq(G_\eta,\mathcal{L}_\eta)$.

Definition 4.3. We call the closed fibre $(Q_0,\mathcal{L}_0)$ of $(Q,\mathcal{L})$ *a projectively stable quasi-abelian scheme* over $k:=R/I$.

Definition 4.4. We define $K(Q,\mathcal{L}):=K(P,\mathcal{L})$ and $\mathcal{G}(Q,\mathcal{L}):=\mathcal{G}(P,\mathcal{L})$. Similarly we set $K(Q_0,\mathcal{L}_0):=K(P_0,\mathcal{L}_0)$ and $\mathcal{G}(Q_0,\mathcal{L}_0):=\mathcal{G}(P_0,\mathcal{L}_0)$.

The abelian Heisenberg group scheme $K(Q_0,\mathcal{L}_0)$ of $(Q_0,\mathcal{L}_0)$ operates upon Q_0 while the Heisenberg group scheme $\mathcal{G}(Q_0,\mathcal{L}_0)$ of $(Q_0,\mathcal{L}_0)$ operates upon $(Q_0,\mathcal{L}_0)$ so that upon $H^0(Q,\mathcal{L})\otimes k$, which is an irreducible $\mathcal{G}(Q_0,\mathcal{L}_0)$-module of weight one by Lemma 4.5. We also note

Lemma 4.5.
(1) $\Gamma(P,\mathcal{L})=\Gamma(Q,\mathcal{L})=\Gamma(G,\mathcal{L})$.
(2) *$\Gamma(Q,\mathcal{L})$ is an irreducible $\mathcal{G}(Q,\mathcal{L})$-module of weight one*
(3) *If k is algebraically closed, then $\Gamma(Q_0,\mathcal{L}_0)$ is an irreducible $\mathcal{G}(Q_0,\mathcal{L}_0)$-module of weight one.*

5. Projective embeddings

We consider only the case where L is a separable polarisation, that is, $d:=\deg L/(g!)$ is prime to the characteristic of $k:=R/I$. With the notation in Section 2 suppose that $d:=\deg L_\eta/(g!)$ is a separable polarisation of G_η. Then by the discussion in §2 $d_t:=\operatorname{rank} K(\mathcal{L}_\eta)^m$ and $d_a=\operatorname{rank} K(\mathcal{M})$ are prime to the characteristic of k. In particular, $\mathcal{M}$ is also a separable polarisation of the abelian scheme A, the abelian part of the Raynaud sequence of G.

Definition 5.1. Let K be a totally isotropic subgroup scheme of $K(Q_0,\mathcal{L}_0)$. Since rank K is prime to the characteristic to k, K is an étale group scheme so that $K(Q_0,\mathcal{L}_0)\simeq K\oplus K^\vee$, and $K=\oplus_{i=1}^g\mathbf{Z}/e_i\mathbf{Z}$ and $e_i|e_{i+1}$. The minimal (resp. maximal) elementary divisor $e_{\min}(K(Q_0,\mathcal{L}_0))$ (resp. $e_{\max}(K(Q_0,\mathcal{L}_0)))$ is defined by

$$e_{\min}(K)=e_{\min}(K(Q_0,\mathcal{L}_0))=e_1,\quad e_{\max}(K)=e_{\max}(K(Q_0,\mathcal{L}_0))=e_g.$$

Theorem 5.2. *Let A be an abelian variety over an algebraically closed field k, L an ample invertible sheaf on A with $\deg L/g! := (L^g)/g!$ prime to the characteristic of k and $K(L) := \ker \lambda(L)$. If $e_{\min}(K(L)) \geq 3$, L is very ample.*

Theorem 5.3. *Let $(Q_0, \mathcal{L}_0)$ be a polarised projectively stable quasi-abelian scheme over an algebraically closed field k, and $K(Q_0, \mathcal{L}_0)$ the abelian Heisenberg group. If $e_{\min}(K(Q_0, \mathcal{L}_0)) \geq 3$, then $\Gamma(Q, \mathcal{L}) \otimes k$ is very ample, a fortiori $\mathcal{L}_0$ is very ample.*

The proof of Theorem is basically the same as in dim ≤ 4. We omit the details. See [Nakamura97].

Definition 5.4. Let $k = R/I$, I maximal. Let $(Q_0, \mathcal{L}_0)$ be a projective stable quasi-abelian k-scheme. We call a k-submodule V of $\Gamma(Q_0, \mathcal{L}_0)$ *a Delaunay k-submodule* if $V = \Gamma(Q, \mathcal{L}) \otimes k$. We note that $\Gamma(Q, \mathcal{L}) \otimes k$ is generated by a sum of monomials ξ_a with $\Gamma(A_0, M_a)$ coefficients ($a \in \mathrm{Del}^{(0)}(Q_0, \mathcal{L}_0)$), and it is the unique irreducible $\mathcal{G}(Q_0, \mathcal{L}_0)$-submodule of $\Gamma(Q_0, \mathcal{L}_0)$ with the property. We note that it is also a very ample k-submodule of $\Gamma(Q_0, \mathcal{L}_0)$. By Theorem 4.5 we recall $\Gamma(Q, \mathcal{L}) \otimes k = \Gamma(P, \mathcal{L}) \otimes k$.

6. $G(K)$ AND $V(K)$

Let $\zeta := \zeta_N$ be a primitive N-th root of unity and $\mathcal{O}_N := \mathbf{Z}[\zeta, 1/N]$. [7]

Definition 6.1. Let K be a constant finite abelian group $\mathcal{O}_N$-scheme of rank N with $e_{\min}(K) \geq 3$. Let $K^\vee := \mathrm{Hom}_{\mathcal{O}_N}(K, \mathbf{G}_{m,\mathcal{O}_N})$ be the Cartier dual of K. We set $H := H(K) = K \oplus K^\vee$ and define $e_H : H \times H \to \mathbf{G}_{m,\mathcal{O}_N}$ by $e_H(z \oplus \alpha, w \oplus \beta) = \beta(z)\alpha(w)^{-1}$ where $z, w \in K$, $\alpha, \beta \in K^\vee$. We denote e_H by ℓ_K when it is necessary to emphasize dependence on K.

Let $\mu_N := \mathrm{Spec}\, \mathcal{O}_N[x]/(x^N - 1)$ be the group scheme of N-th roots of unity in $\mathcal{O}_N$. We define $G(K)$ by $G(K) := \{(a, z, \alpha); a \in \mu_N, z \in K, \alpha \in K^\vee\}$ endowed with a group law

$$(a, z, \alpha) \cdot (b, w, \beta) = (ab\beta(z), z + w, \alpha + \beta)$$

where $a, b \in \mu_N$, $z, w \in K$ and $\alpha, \beta \in K^\vee$. It is clear that $G(K)$ contains K as a level subgroup scheme, that is, the image of K in $H(K)$ is a maximally isotropic subgroup scheme with respect to e_H.

Let $V(K)$ be the group algebra $\mathcal{O}_N[K^\vee]$ of $K^\vee$ over $\mathcal{O}_N$, and an $\mathcal{O}_N$-basis $v(\chi)$ ($\chi \in K^\vee$) of $V(K)$. The group scheme $G(K)$ acts upon $V(K)$ by

$$U(K)(a, z, \alpha)(v(\chi)) = a\chi(z)v(\chi + \alpha).$$

[7] In fact, we can take $\zeta = \zeta_{e_{\max}}$ instead of ζ_N by a more careful argument.

where $a \in \mu_N$, $z \in K$ and $\alpha \in K^\vee$. We define a subgroup scheme $\bar{G}(K)$ of an algebraic group O_N-scheme $GL(V(K))$ by

$$\bar{G}(K) := \{U(K)(g); g \in G(K)\}$$

Lemma 6.2. *Let* Spec k *be a point of* Spec O_N. *Then* $V(K) \otimes k$ *is an irreducible* $G(K)$*-module of weight one, unique up to equivalence.*

Proof. We imitate the argument in [Mumford66]. Let V be a $G(K)$-k-module of weight one. Let $V(\chi)$ be the maximal k-submodule of V such that K operates on $V(\chi)$ by a character $\chi \in K^\vee$. There is a χ_0 such that $V(\chi_0) \neq 0$. Let $0 \neq v(\chi_0) \in V(\chi_0)$. Then we set $v(\chi) := (1, 0, \chi - \chi_0) \cdot v(\chi_0) \in V(\chi)$ and define V_0 to be the k-submodule of V spanned by $v(\chi)$ $(\forall \chi \in K)$. We see

$$\begin{aligned}(a, z, \alpha) \cdot v(\chi) &= (a, 0, \alpha) \cdot (1, z, 0) \cdot v(\chi)\\ &= a\chi(z)v(\chi + \alpha)\end{aligned}$$

This proves $V_0 \simeq V(K) \otimes k$. It follows that $V \simeq (V(K) \otimes k)^{\dim_k V / \dim_k V_0}$. □

Definition 6.3. Let R be a complete discrete valuation ring over O_N with $k = R/I$ and $S := \operatorname{Spec} R$. For a PSQAS $(Q, \mathcal{L})$ over S, we define $G(Q, \mathcal{L})$ to be the central extension of $K(Q, \mathcal{L})$ by μ_N with commutator form $e^\sharp_S$, hence the following is exact

$$1 \to \mu_{N,S} \to G(Q, \mathcal{L}) \to K(Q, \mathcal{L}) \to 0$$

If $(K(Q, \mathcal{L}), e^\sharp_S) \simeq (H(K)_S, \ell_{K,S})$ for a PSQAS $(Q, \mathcal{L})$ over S, then $G(K)_S$ is weight-one isomorphic to $G(Q, \mathcal{L})$, in other words (by definition), there is an isomorphism $\rho : G(K)_S \to G(Q, \mathcal{L})$ such that ρ is the identity on the centre $\mu_{N,S}$. Let $G(Q_\eta, \mathcal{L}_\eta) := G(Q, \mathcal{L}) \otimes k(\eta)$, $G(Q_0, \mathcal{L}_0) := G(Q, \mathcal{L}) \otimes k$.

We choose and fix an arbitrary weight-one isomorphism $\rho : G(K) \otimes k \simeq G(Q_0, \mathcal{L}_0)$. By Lemma 6.2 there is a k-isomorphism $\phi(\rho)^* : V(K) \otimes k \to V = \Gamma(Q, \mathcal{L}) \otimes k$ such that

$$U(\rho(g))(\phi(\rho)^*(w)) = \phi(\rho)^* U(K)(g)(w) \quad (\forall g \in G(K), \forall w \in V(K))$$

where U is the action of $G(Q_0, \mathcal{L}_0)$ on V. Let $\theta(\chi) := \phi(\rho)^*(v(\chi))$ $(\chi \in K^\vee)$. By Schur's lemma $\phi(\rho)^*$ is unique up to a scalar multiple so that there is a unique closed immersion $\phi(\rho)$ of $(Q_0, \mathcal{L}_0)$ into $\mathbf{P}(V(K) \otimes k)$ as above for a given ρ. The stabiliser group $\operatorname{Stab}(\phi(\rho)(Q_0))$ in $GL(V(K) \otimes k)$ contains $\bar{G}(K) \otimes k$.

Let $\phi : Q_0 \to \mathbf{P}(V(K) \otimes k)$ be any closed k-immersion. Then there is a unique $h \in GL(V(K) \otimes k)$ such that $\phi = h \cdot \phi(\rho)$. Then $\operatorname{Stab}(\phi(Q_0))$ contains $h\bar{G}(K)h^{-1}$.

Let $Z = \phi(Q_0)$ and $L := O_{\mathbf{P}(V(K) \otimes k)}(1)_{|Z}$. Then we can naturally identify $G(Z, L) = h\bar{G}(K)h^{-1}$.

7. Kempf-Stability

7.1. Let $(Q_0, \mathcal{L}_0)$ be a projectively stable quasi-abelian scheme over k. Suppose that $V^0 := \Gamma(Q, \mathcal{L}) \otimes k$ is very ample. Hence $(Q_0, \mathcal{L}_0)$ is a closed subscheme of the projective space $(\mathbf{P}, O_{\mathbf{P}}(1))$ where $\mathbf{P} = \mathbf{P}(V^0)$. Let I be the ideal of $O_{\mathbf{P}}$ defining $(Q_0, \mathcal{L}_0)$. Then by Serre vanishing theorem there exists a sufficiently large n_0 such that $H^1(\mathbf{P}, I \otimes O_{\mathbf{P}}(n)) = 0$ for $n \geq n_0$. Hence we have an epimorphism

$$\phi_n : S^n\Gamma(Q, \mathcal{L}) \otimes k \to \Gamma(Q_0, \mathcal{L}_0^n)$$

The epimorphism ϕ_n determines a point of the Grassmannian variety. Let $n(g) := n^g \deg(\mathcal{L}_0)$. By taking the Plücker coordinates we obtain a point $\overset{n(g)}{\wedge} \phi_n$ of the projective space $\mathbf{P}(\overset{n(g)}{\wedge} S^n\Gamma(Q, \mathcal{L}) \otimes k)$

$$\overset{n(g)}{\wedge} \phi_n : \overset{n(g)}{\wedge} S^n\Gamma(Q, \mathcal{L}) \otimes k \to \overset{n(g)}{\wedge} \Gamma(Q_0, \mathcal{L}_0^n) \simeq k.$$

We call $\overset{n(g)}{\wedge} \phi_n$ *the n-th normalised Hilbert point of* $(Q_0, \mathcal{L}_0)$, which we denote by $hilb_n(Q_0, \mathcal{L}_0)$. If $\Gamma(Q_0, \mathcal{L}_0) = \Gamma(Q, \mathcal{L}) \otimes k$, for instance if $Q_0 = P_0$, then $hilb_n(Q_0, \mathcal{L}_0)$ is just *the n-th Hilbert point of* $(Q_0, \mathcal{L}_0)$ in the usual sense.

We say that $hilb_n(Q_0, \mathcal{L}_0)$ is Kempf-stable if it has a closed $SL_{\pm}(V^0)$-orbit.

By Lemma 6.2 the following is a corollary to [Kempf78, Corollary 5.1].

Theorem 7.2. *Let $(Q_0, \mathcal{L}_0)$ be a polarised projectively stable quasi-abelian scheme over an algebraically closed field k. Suppose that the characteristic of k and* $\deg \mathcal{L}_0 := (\mathcal{L}_\eta^g)/g!$ *are coprime. If $\Gamma(Q, \mathcal{L}) \otimes k$ is very ample, then $hilb_n(Q_0, \mathcal{L}_0)$ is Kempf-stable for all large n.* [8] *In particular it is Mumford-semistable.*

Proof. Let $V^0 := \Gamma(Q, \mathcal{L}) \otimes k$. Let $SL_{\pm}(V^0)$ be a subgroup of $GL(V^0)$ consisting of elements with determinant ± 1. We note that closedness of the orbits for the actions of SL or $SL_{\pm}$ are equivalent to each other because $[SL_{\pm}(V^0); SL(V^0)]$ is finite. $G(Q_0, \mathcal{L}_0)$ operates on $\Gamma(Q_0, \mathcal{L}_0)$ keeping V^0 stable so that $hilb_n(Q_0, \mathcal{L}_0)$ is $G(Q_0, \mathcal{L}_0)$-invariant. Since V^0 is an irreducible $G(Q_0, \mathcal{L}_0)$ ($\simeq G(K) \otimes k$)-module by Lemma 6.2, $G(Q_0, \mathcal{L}_0)$ is contained in no parabolic subgroup of $SL_{\pm}(V^0)$. By applying [Kempf78, Corollary 5.1] to $SL_{\pm}(V^0)$, we see that $hilb_n(Q_0, \mathcal{L}_0)$ has a closed $SL_{\pm}(V^0)$ orbit. They are semistable in the sense of Mumford by [Seshadri77, p. 252, Proposition 6 (1)]. □

[8] Theorem 7.2 seems to be true without the assumption on $\deg \mathcal{L}_0$.

The (normalised) Hilbert points of $(Q_0, \mathcal{L}_0)$ are not necessarily properly stable, for instance a 3-gon of rational curves.

8. Rigid $\bar{G}(K)$-structures

8.1. In what follows we consider only separable polarisations $\mathcal{L}_\eta$, that is, $d := \deg \mathcal{L}_\eta/(g!)$ is prime to the characteristic of $k := R/I$. Then $d_t :=$ rank $K(\mathcal{L}_\eta)^m$ and $d_a =$ rank $K(\mathcal{M})$ are prime to the characteristic of k. In particular, $\mathcal{M}$ is also a separable polarisation of the abelian scheme A, the abelian part of the Raynaud sequence of G.

Definition 8.2. Let $\mathcal{O}_N = \mathbf{Z}[\zeta, 1/N]$ and K a constant finite abelian group $\mathcal{O}_N$-scheme of rank N. Let Spec k be a (not necessarily closed) point of Spec $\mathcal{O}_N$. Suppose $e_{\min}(K) \geq 3$. A triple $(Q_0, \mathcal{L}_0, V_0)$ is called a g-dimensional K-symplectic projectively stable quasi-abelian scheme over k or a K-symplectic PSQAS over k if

(1) $(Q_0, \mathcal{L}_0)$ is a g-dimensional projectively stable quasi-abelian scheme over k, a closed fibre of some $(Q, \mathcal{L})$ in Theorem 4.2,
(2) $G(Q_0, \mathcal{L}_0) \otimes \bar{k}$ is weight-one isomorphic to $G(K) \otimes \bar{k}$.
(3) $V_0 \otimes \bar{k}$ is the theta $\bar{k}$-module of $(Q_0, \mathcal{L}_0) \otimes \bar{k}$ [9]

where $\bar{k}$ is the algebraic closure of k. See Definition ?? resp. Definition 5.4 for $G(Q_0, \mathcal{L}_0)$ resp. theta modules.

Lemma 8.3. *Let T be an irreducible $\mathcal{O}_N$-scheme, $\pi : (A, L) \to T$ a polarised abelian T-scheme, $A^t := \mathrm{Pic}^0(A/T)$ and $\lambda(L) : A \to A^t$ the polarisation morphism. Assume that L_s is a separable polarisation for any geometric point $s \in T$. Then there exists a finite étale covering $f : T^* \to T$ and a constant finite abelian subgroup T^*-scheme K_{T^*} of $\ker \lambda(L_{T^*})$ such that $(\ker \lambda(L_{T^*}), (e^L)_{T^*}) \simeq (K \oplus K^\vee, \ell_K)_{T^*}$ where e^L is the Weil pairing on $\ker \lambda(L)$.*

Proof. We have an exact sequence of group schemes

$$0 \to \ker \lambda(L) \to A \xrightarrow{\lambda(L)} A^t \to 0.$$

By the assumption $\lambda(L)_s = \lambda(L_s)$ is étale for any geometric point s of T so that $\lambda(L)$ is étale. Therefore $\ker \lambda(L)$ is étale and finite over T. Let T' be one of the

[9]If the following conjecture for $N = 1$ is affirmatively solved, this datum is removed
Conjecture: $H^q(Q_0, \mathcal{L}_0^N) = 0$ for $q, N > 0$.

irreducible components of $\ker\lambda(L)$ with $T' \not\simeq T$. Then $\ker\lambda(L)_{T'}$ has a new section over T'. By repeating the same argument we see that there exists an irreducible O_N-scheme T^* étale and finite over T such that $\ker\lambda(L_{T^*}) = \ker\lambda(L)_{T^*}$ is a constant finite group scheme. The Weil pairing $(e^L)_{T^*}$ is a symplectic bilinear form on $\ker\lambda(L_{T^*})$ with values in μ_N, which is therefore constant on T^*. Hence there exists a totally isotropic constant subgroup T^*-scheme K_{T^*} of $\ker\lambda(L)_{T^*}$ such that $(\ker\lambda(L_{T^*}), e^L_{T^*}) \simeq (K\oplus K^\vee, \ell_K)_{T^*}$. This proves Lemma. □

Lemma 8.4. *Let T be an irreducible O_N-scheme, $\pi : (A, L) \to T$ a polarised abelian T-scheme. Suppose that there exists a constant finite abelian subgroup T-scheme K_T of $\ker\lambda(L)$ such that $(\ker\lambda(L), e^L) \simeq (K\oplus K^\vee, \ell_K)_T$. Then there exists an invertible O_T-module M with trivial $G(K)_T$-action such that $\pi_*(L) \simeq M\otimes_{O_N} V(K)$.*

Proof. Let $G(A, L)$ be the central extension of $\ker\lambda(L)$ with commutator form e^L. By the assumption there exists a weight one isomorphism $\rho : G(K)_T \to G(A, L)$ of group T-schemes.

Let s be a closed point of T, U an affine open subset of T with $s \in U$. It follows that through ρ $\pi_*(L)$ is a $G(K)_T$-module of weight one with the centre of $G(K)_T$ acting upon $\pi_*(L)$ by scalar multiplication. By Lemma 6.2 $\pi_*(L)\otimes k(s) \simeq V(K)\otimes_{O_N} k(s)$ as $G(K)_T\otimes k(s)$-modules for any closed point $s\in T$. Therefore the action of an abelian group $K_T(T)$ on $\pi_*(L)$ is diagonalised locally because any eigenvalue of the action of $K_T(T)$ belongs to O_T. This implies that there exists an open affine covering $\{U_i\}$ of T such that $\pi_*(L)\otimes_{O_T} O_{U_i} \simeq O_{U_i}\otimes_{O_N} V(K)$ as $G(K)_{U_i}$-modules. Let $U(K)_i = 1_{O_{U_i}}\otimes U(K)$ be the action of $G(K)_{U_i}$ on $O_{U_i}\otimes_{O_N} V(K)$. It follows that there is a one-cocycle $\ell_{jk}\in H^1(\{U_{jk}\}, GL(O_T\otimes_{O_N} V(K)))$ such that

$$U(K)_j(g)\ell_{jk} = \ell_{jk}U(K)_k(g) \quad (\forall g \in G(K)_T(T))$$

on $U_{jk} := U_j \cap U_k$. Hence $U(K)(g)\ell_{jk} = \ell_{jk}U(K)(g)$. Therefore ℓ_{jk} is a scalar matrix by Schur's lemma, so that $\ell_{jk}\in H^1(\{U_{jk}\}, O_T^*)$, which defines an invertible O_T-module M. This proves Lemma. □

Corollary 8.5. *Let anything be as in Lemma 8.4. Then there exist a closed T-immersion $\phi : A \to \mathbf{P}(V(K)\otimes_{O_N} O_T)$, a weight-one isomorphism $\rho : G(K)_T \to G(A, L)$ and an invertible O_T-module M such that*

(1) $L = \phi^*(O_{\mathbf{P}(V(K))}(1)\otimes_{O_N} O_T)$,
(2) $\pi_*(L) = \phi^*(M\otimes_{O_N} V(K))$,
(3) $\rho\otimes k = G(\phi^*)\cdot(U(K)\otimes k)$ *for any geometric point* Spec k *of* T

where $G(\phi^*)(g) := \phi^* g(\phi^*)^{-1}$ *for* $g \in \bar{G}(K) := U(K)G(K)$ *and* $\Gamma(O_{\mathbf{P}(V(K))}(1))$ *is identified with* $V(K)$.

Definition 8.6. Let Spec k be a point of Spec O_N, (Z, L, V) a K-symplectic PSQAS over k. A rigid $\bar{G}(K)$-structure (ϕ, ρ) on (Z, L, V) is a pair of a closed k-immersion $\phi : Z \to \mathbf{P}(V(K) \otimes k)$ and a weight-one isomorphism $\rho : G(K) \otimes_{O_N} k \to G(Z, L)$ such that

(1) $L = \phi^*(O_{\mathbf{P}(V(K)\otimes k)}(1))$,
(2) $\phi^* : V(K) \otimes k \simeq V$ is a k-linear isomorphism,
(3) $\rho = G(\phi^*) \cdot (U(K) \otimes_{O_N} k)$

where $G(\phi^*)(g) = (\phi^*)g(\phi^*)^{-1}$ for any $g \in G(K)$.

If a K-symplectic PSQAS (Z, L, V) has a rigid $\bar{G}(K)$-structure (ϕ, ρ), then $\phi = \phi(\rho)$ by the remark in 6.3. Evidently L and V are uniquely determined by ϕ. We denote $(Z, L, G(Z, L), V, \phi, \rho)$ by $(Z, \phi, \rho)_{\mathrm{RIG}}$.

Lemma 8.7. *Let* Spec k *be a point of* Spec O_N. *Any* K*-symplectic PAQAS* (Z, L, V) *over* k *has a unique rigid* $\bar{G}(K)$*-structure* ϕ.

Proof. By definition we are given an isomorphism $\rho : G(K) \simeq G(Z, L)$. It suffices to choose a closed k-immersion $\phi(\rho) : Z \to \mathbf{P}(V(K) \otimes_{O_N} k)$ by $\phi(\rho)^*(v(\chi)) = \theta(\chi)$ $(\chi \in K^\vee)$ with the notation in Definition 6.3. Uniqueness follows from Lemma 6.2. □

Definition 8.8. Let $(Z_i, L_i, G(Z_i, L_i), V_i, \phi_i, \rho_i)$ be k-PSQAS's with rigid $\bar{G}(K)$-structures $(i = 1, 2)$. $(Z_i, L_i, G(Z_i, L_i), V_i, \phi_i, \rho_i)$ are isomorphic as k-PSQAS's with rigid $\bar{G}(K)$-structures if there is a k-isomorphism $f : Z_1 \simeq Z_2$ such that

(1) $L_1 = f^*L_2$, $V_1 = f^*V_2$,
(2) $G(\phi_1^*) = G(f^*) \cdot G(\phi_2^*)$

where $G(f^*)(g) = f^*g(f^*)^{-1}$ for any $g \in G(Z_2, L_2)$.

In this case we write $(Z_1, \phi_1, \rho_1)_{\mathrm{RIG}} \simeq (Z_2, \phi_2, \rho_2)_{\mathrm{RIG}}$. By (2) we have $G(Z_1, L_1) = G(f^*)G(Z_2, L_2)$.

Lemma 8.9. *Let* $(Z_i, \phi_i, \rho_i)_{\mathrm{RIG}}$ *be* k*-PSQAS's* $(i = 1, 2)$. *Then* $(Z_1, \phi_1, \rho_1)_{\mathrm{RIG}} \simeq (Z_2, \phi_2, \rho_2)_{\mathrm{RIG}}$ *iff there is a* k*-isomorphism* $f : Z_1 \simeq Z_2$ *with* $\phi_1 = \phi_2 \cdot f$.

Proof. First we prove if part. Definition 8.8 (1) is clear from the uniqueness of theta modules. (2) is clear.

Next we prove only if part. By Definition 8.8 (1) and by the very-ampleness of V_i there is an $h \in GL(V(K) \otimes k)$ such that $\phi_1^* \cdot h = f^* \cdot \phi_2^* \in \mathrm{Hom}(V(K) \otimes k, V_1)$. Hence $G(\phi_1^*)G(h) = G(f^*) \cdot G(\phi_2^*)$. It follows from Definition 8.8 (2) that $G(h)$ is the identity, *i.e.*, $h \cdot g = g \cdot h$ for any $g \in \bar{G}(K)$. By Schur's lemma, h is a scalar multiple of the identity so that $\phi_1 = \phi_2 \cdot f$. □

Corollary 8.10. *If* $e_{\min}(K) \geq 3$, *then* $\mathrm{Aut}\,((Z, \phi, \rho)_{\mathrm{RIG}})$ *is trivial.*

9. The scheme $SQ_{g,K}$

9.1. Let K be a constant finite abelian group $\mathcal{O}_N$-scheme with $e_{\min}(K) \geq 3$ and $N := \mathrm{rank}\,_{\mathcal{O}_N}K$. Let $P(n) := n^g N$, $S_N := \mathrm{Spec}\,\mathcal{O}_N$ and let $H_{g,K} := \mathrm{Hilb}_{\mathbf{P}}^{P(n)}$ be the Hilbert scheme parametrising all projective subschemes of $\mathbf{P} := \mathbf{P}(V(K))$ with their Hilbert polynomial $P(n)$, $(Z_{g,K}, L_{g,K})$ the universal subscheme over $H_{g,K}$ and $\pi : Z_{g,K} \to H_{g,K}$ the natural morphism. Let $i : (Z_{g,K}, L_{g,K}) \to \mathbf{P}(V(K)) \times_{S_N} H_{g,K}$ be the natural (given) closed immersion of the universal subscheme $Z_{g,K}$ over $H_{g,K}$. We remark that $H_{g,K}$ and $Z_{g,K}$ are $\mathcal{O}_N$-schemes by [FGA, 221, Théorème 3.1]. In this section we will define an $\mathcal{O}_N$-subscheme $SQ_{g,K}$ of $H_{g,K}$ which ought to be the moduli scheme. See [MFK94, Proposition 7.3, pp.132-134].

Since $e_{\min}(K) \geq 3$, any K-symplectic PSQAS $(Q_0, \mathcal{L}_0, V_0)$ over k is a closed point of $H_{g,K}$ by choosing any isomorphism $V_0 \simeq V(K) \otimes_{\mathcal{O}_N} k$ in view of Theorem 5.3.

Let U be the open maximal subscheme of $H_{g,K}$ such that π is smooth, which is a $\mathcal{O}_N$-subscheme of $H_{g,K}$ by [EGA, IV, Corollaire 6.8.7]. Suppose that a fibre of π over a geometric point s of U is an abelian variety with $\ker \lambda(L_{g,K,s}) \simeq H(K) \otimes_{\mathcal{O}_N} k(s)$. Let U_1 be a connected component of U containing s, and $Z_1 := Z_{g,K} \times_{H_{g,K}} U_1$. By the base change U_2 of U_1 we may assume $Z_2 := Z_1 \times_{U_1} U_2$ has a section e over U_2. For instance choose $U_2 = Z_1$.

By [MFK94, Theorem 6.14] Z_2 is an abelian scheme over U_2 with e unit section. It follows from Lemma 8.3 that any geometric fibre of Z_2, a fortiori, of Z_1 is an abelian variety with $\ker \lambda(L_{g,K,s}) \simeq H(K) \otimes_{\mathcal{O}_N} k(s)$. Since U_1 is an $\mathcal{O}_N$-scheme, there exists an open $\mathcal{O}_N$-subscheme $U_{g,K}$ of U such that any abelian variety fibre of $H_{g,K}$ with $\ker \lambda(L_{g,K,s}) \simeq H(K) \otimes_{\mathcal{O}_N} k(s)$ is isomorphic to a geometric fiber of π over $U_{g,K}$. Let $W_{g,K}$ be the closure of $(U_{g,K})_{\mathrm{red}}$ in $H_{g,K}$ with reduced structure.

Remark 9.2. Let $\pi : (A, L) := (Z_{g,K}, L_{g,K}) \times_{H_{g,K}} U_3 \to U_3$ be a polarised abelian scheme over an irreducible component U_3 of $U_{g,K}$. By Lemmas 8.3, 8.4 and Corollary 8.5, there exist an $\mathcal{O}_N$-scheme T, a closed T-immersion $\phi : A_T \to \mathbf{P}(V(K))_T$ and a weight-one isomorphism $\rho : G(K)_T \to G(A, L)$ such that $\rho = G(\phi^*)U(K)_T$. Hence there is an $\mathcal{O}_N$-morphism $\mathrm{Hilb}(\phi) : T \to U_{g,K}$ such that

$$(\phi(A), O_{\mathbf{P}}(1)_{|\phi(A)}) \otimes k(s) = \mathrm{Hilb}(\phi)(s) \in U_{g,K}(k(s))$$
$$\rho \otimes k(s) = G((\phi \otimes k(s))^*)(U(K)_T \otimes k(s))$$

for any geometric point s of T.

Definition 9.3. By Remark 9.2 (plus some argument) there exists an $\mathcal{O}_N$-subscheme $A_{g,K}$ of $U_{g,K}$ such that

(1) $G((Z_{g,K}, L_{g,K}) \times_{H_{g,K}} A_{g,K}) \otimes k(s) = \bar{G}(K) \otimes_{\mathcal{O}_N} k(s)$ for any geometric point s of $A_{g,K}$ and

(2) any geometric fiber of π over $U_{g,K}$ is isomorphic to a geometric fiber of π over $A_{g,K}$.

The natural representation $U(K) : G(K) \to \bar{G}(K)$ induces a weight-one isomorphism $\rho_{g,K} : G(K)_{A_{g,K}} \simeq G((Z_{g,K}, L_{g,K}) \times_{H_{g,K}} A_{g,K})$. We also note by Remark 9.2 that for any abelian scheme $(A, L) \in U_{g,K}(k)$ over a closed field k, there is a closed immersion $\phi : A \to \mathbf{P}(V(K) \otimes k)$ such that $(\phi(A), O_{\mathbf{P}}(1)_{|\phi(A)}) \in A_{g,K}(k)$.

We define $SQ_{g,K}$ to be the closure of $A_{g,K}$ in $H_{g,K}$, *i.e.*, the minimal (reduced) closed $\mathcal{O}_N$-subscheme of $H_{g,K}$ containing $(A_{g,K})_{\mathrm{red}}$. Let $Z^{SQ}_{g,K} := Z_{g,K} \times_{H_{g,K}} SQ_{g,K}$, and let $\pi_{g,K} : Z^{SQ}_{g,K} \to SQ_{g,K}$ be the natural projection.

Theorem 9.4. *$SQ_{g,K}$ is a projective $\mathcal{O}_N$-subscheme of $W_{g,K}$ pointwise fixed by $\bar{G}(K)$ such that for any geometric point* Spec k *of* Spec $\mathcal{O}_N$,

(1) $A_{g,K}(k) = \left\{ (Z, L) \in W_{g,K}(k);\ \begin{array}{l} (Z, i, U(K)) \textit{ is an abelian variety} \\ \textit{with a rigid } \bar{G}(K)\textit{-structure} \end{array} \right\} / k\textit{-isom.}$

(2) $SQ_{g,K}(k) = \left\{ (Z, L) \in W_{g,K}(k);\ \begin{array}{l} (Z, i, U(K)) \textit{ is a PSQAS over } k \\ \textit{with a rigid } \bar{G}(K)\textit{-structure} \end{array} \right\} / k\textit{-isom.}$

where i is the natural inclusion of Z into $\mathbf{P}(V(K) \otimes k)$.

Proof. By Definition 9.3 $A_{g,K}$ is an $\mathcal{O}_N$-subscheme of $W_{g,K}$ satisfying the condition that $(Z,L) \in A_{g,K}(k)$ is an abelian variety with $G(Z,L) = \bar{G}(K)$, hence $(Z,i,U(K))$ is an abelian variety with a rigid $\bar{G}(K)$-structure. Thus (1) is true. It follows that (Z,L) is fixed by the action of $\bar{G}(K)$ so that $A_{g,K}(k)$ is pointwise fixed by the induced action of $\bar{G}(K)$ upon $H_{g,K}$. It follows from it that $SQ_{g,K}$ is an $\mathcal{O}_N$-subscheme of $W_{g,K}$ pointwise fixed by $\bar{G}(K)$.

Let R be a complete discrete valuation ring with fraction field $k(\eta)$, η the generic point of $S := \operatorname{Spec} R$. Suppose that we are given a flat R-subscheme (Z,L) of $(Z_{g,K} \times_{H_{g,K}} SQ_{g,K}) \otimes_{\mathcal{O}_N} R$ such that (Z_η, L_η) is a polarised abelian scheme (possibly with no unit section). Let i be the inclusion immersion of (Z,L) into $\mathbf{P}(V(K) \otimes R)$. The subgroup scheme $\bar{G}(K)_S$ of $SL_{\pm}(V(K) \otimes R)$ stabilises (Z,L). By choosing a suitable ramified cover of S if necessary we may assume by Theorem 0.1 and Definition ?? that we have a projective flat family (W,M) with a rigid $\bar{G}(K)$-structure (ϕ,ρ) over S such that $(W_\eta, M_\eta) \simeq (Z_\eta, L_\eta)$ and the closed fibre (W_0, M_0) is a PSQAS. Since we start from the given (Z,L) in order to construct (W,M), we may assume that $(Z_\eta, i_\eta, U(K)_\eta)_{\mathrm{RIG}} \simeq (W_\eta, \phi_\eta, \rho_\eta)_{\mathrm{RIG}}$. In fact, the rigid structure $(W,\phi,\rho)_{\mathrm{RIG}}$ was constructed by extending $(W_\eta, \phi_\eta, \rho_\eta)_{\mathrm{RIG}}$. This part is clear from Sections 2-4, Paragraph 3.5 and Definition ??. Hence there is by Lemma 8.9 a $k(\eta)$-isomorphism $f : Z_\eta \to W_\eta$ such that $i_\eta = \phi_\eta \cdot f$, that is, $i_\eta(Z_\eta) = \phi_\eta(W_\eta)$. Since i and ϕ are closed S-immersions respectively, they induce natural morphisms $\mathrm{Hilb}(i)$ and $\mathrm{Hilb}(\phi)$ from $\operatorname{Spec} R$ into $SQ_{g,K}$ by the universal property of $H_{g,K}$. It follows from $i_\eta = \phi_\eta \cdot f$ that we have $\mathrm{Hilb}(i_\eta) = \mathrm{Hilb}(\phi_\eta) : \operatorname{Spec} k(\eta) \to SQ_{g,K}$. Since $SQ_{g,K}$ is projective (separated), we have $\mathrm{Hilb}(i) = \mathrm{Hilb}(\phi)$ [EGA, II, 7.2.3]. This implies that $i(Z) = \phi(W)$, *a fortiori*, $i_0(Z_0) = \phi_0(W_0)$. Hence $Z_0 \simeq W_0$, $(Z_0, i_0, U(K))$ is a k-PSQAS with a rigid $\bar{G}(K)$-structure by the uniqueness of $G(Z_0, L_0)$, which follows from Lemma 3.6. This proves (2). □

Definition 9.5. Now we define the relative n-th Hilbert point $hilb_n$. Let $\pi : Z_{g,K} \to H_{g,K}$ be the natural morphism, and let

$$\mathcal{V} := O_{H_{g,K}} \otimes_{\mathcal{O}_N} V(K), \quad \mathcal{V}_n := \pi_*(O_{Z_{g,K}}(n)).$$

We note that $\mathcal{V}_n$ is locally free of rank $n(g) := n^g N$ for sufficiently large $n \geq n_0$. Let $\phi_n : S^n\mathcal{V} \to \mathcal{V}_n$ be the natural epimorphism for $n \geq n_0$. Thus we have a morphism $hilb_n := \overset{n(g)}{\wedge} \phi_n$ of $H_{g,K}$ into the projective space $\mathbf{P}_{\mathrm{large}} := \mathbf{P}(\overset{n(g)}{\wedge} S^n V(K))$. For any large $n \geq n_0$ $hilb_n$ is a closed $\mathcal{O}_N$-immersion of $H_{g,K}$. For a geometric fibre (Z,L) of π we call $hilb_n(Z,L)$ *the n-th Hilbert point of* (Z,L).

Theorem 9.6. *Let* $\operatorname{Spec} k$ *be a geometric point of* $\operatorname{Spec} \mathcal{O}_N$ *and* $(Z, L) \in H_{g,K}(k)$. *Suppose* $e_{\min}(K) \geq 3$ *and that* (Z, L) *is smoothable into an abelian variety* (A, M) *with* $\ker \lambda(M) \simeq H(K) \otimes k$. *Then the following are equivalent.*

(1) (Z, L) *is a* K*-symplectic PSQAS over* k,
(2) $\operatorname{Aut}(Z, L)$ *contains a subgroup of* $SL_{\pm}(V(K) \otimes k)$ *weight-one isomorphic to* $G(K) \otimes k$,
(3) $hilb_n(Z, L)$ [10] *is Kempf-stable for any large* $n \geq n_0$.

Proof. (2) follows from (1) by Definition 6.3. (3) follows from (2) by [Kempf78, Corollary 5.1]. It remains to prove that (3) implies (1). We choose and fix a large n. Suppose (3) and that (Z, L) is smoothable into an abelian variety with $\ker \lambda \simeq K \oplus K^{\vee}$. It follows $(Z, L) \in W_{g,K}(k)$. By (3) $hilb_n(Z, L)$ is Mumford-semistable. By [Seshadri77, p. 269, Remark 8] there is a categorical quotient

$$W_{g,K} \cap hilb_n^{-1}(\mathbf{P}_{\text{large}}(\text{SemiStable}))/SL(V(K))$$

which is a projective $\mathcal{O}_N$-scheme. By [Seshadri77, p. 269, Theorem 4] and the $(Q, \mathcal{L})$-version of Stable reduction theorem (Theorem 0.1) the closure of the $SL(V(K) \otimes k)$-orbit of (Z, L) intersects the $SL(V(K) \otimes k)$-orbit of a PSQAS (W, M). By the assumption (3) the orbit of (Z, L) is closed. Therefore the orbit of (Z, L) is that of (W, M), hence (Z, L) is a PSQAS. □

10. Reduced-fine-moduli

Definition 10.1. Let $\mathcal{O}_N := \mathbf{Z}[\zeta, 1/N]$. For a contravariant functor F over $\mathcal{O}_N$ a reduced $\mathcal{O}_N$-scheme M is said to be a reduced-fine-moduli scheme over $\mathcal{O}_N$ of F or we say that F is reductively-represented over $\mathcal{O}_N$ by M if the following conditions are satisfied;

(a) $f_M(T) : F(T) \to \operatorname{Hom}_S(T, M)$ is a bijection for a reduced $\mathcal{O}_N$-scheme T
(b) $f_M(T) \cdot F(h) = \operatorname{Hom}(h, M) \cdot f_M(U)$ for an $\mathcal{O}_N$-morphism $h : T \to U$ of reduced $\mathcal{O}_N$-schemes T, U
(c) if there is another reduced $\mathcal{O}_N$-scheme N satisfying (a) and (b), then there exists a unique $\mathcal{O}_N$-morphism $\psi : M \to N$ such that $f_N = \operatorname{Hom}(\psi) \cdot f_M$.

It is clear that a reduced-fine-moduli scheme is unique if there exists.

[10] This is understood as a normalised Hilbert point.

Definition 10.2. Let $\operatorname{Spec} k$ be a point of $\operatorname{Spec} O_N$. A level $G(K)$-structure (ϕ, ρ) on a PSQAS (Z, L, V) over k is a pair of a closed k-immersion $\phi : Z \to \mathbf{P}(V(K) \otimes_{O_N} k)$ and a weight-one isomorphism $\rho : G(K) \otimes_{O_N} k \to G(Z, L)$ such that

(1) $L = \phi^*(O_{\mathbf{P}(V(K)\otimes_{O_N} k)}(1))$,

(2) $\phi^* : V(K) \otimes_{O_N} k \simeq V$ is a k-linear isomorphism,

(3) ρ is equivalent to $G(\phi^*)U(K)$ in $\operatorname{Hom}_{k\text{-gr.sch.}}(G(K) \otimes_{O_N} k, GL(V))$.

We denote a PSQAS (Z, L, V) with a level $G(K)$ structure (ϕ, ρ) by $(Z, \phi, \rho)_{\mathrm{LEV}}$.

Two k-PSQAS's $(Z_i, L_i, G(Z_i, L_i), V_i, \phi_i, \rho_i)$ $(i = 1, 2)$ with level $G(K)$-structures are isomorphic if there is a k-isomorphism $f : Z_1 \simeq Z_2$ such that

(1) $L_1 = f^*L_2$, $V_1 = f^*V_2$,

(2) $\rho_1 = G(f^*) \cdot \rho_2$

where $G(f^*)(g) = f^*g(f^*)^{-1}$ for any $g \in G(Z_2, L_2)$.

If we are given $(Z, \phi, \rho)_{\mathrm{LEV}}$, by Lemma 6.2, there is a unique k-closed immersion $\phi(\rho) : Z \to \mathbf{P}(V(K) \otimes_{O_N} k)$ such that $G(\phi(\rho)^*)U(K) = \rho$, *i.e.*,

$$\rho(g)(\phi(\rho)^*(w)) = \phi(\rho)^*U(K)(g)(w) \quad (\forall g \in G(K), \forall w \in V(K))$$

We note that $(Z, \phi(\rho), \rho)_{\mathrm{RIG}}$ is a unique PSQAS with a rigid $\bar{G}(K)$-structure such that $(Z, \phi(\rho), \rho)_{\mathrm{LEV}} \simeq (Z, \phi, \rho)_{\mathrm{LEV}}$.

Definition 10.3. Given a noetherian O_N-scheme T, $(Q, \mathcal{L}, G(Q, \mathcal{L}), \mathcal{V}, \phi, \rho)$ is called a projectively stable quasi-abelian T-scheme of relative dimension g with a level $G(K)$-structure if

(i) Q is a flat proper T-scheme with a relatively ample invertible sheaf $\mathcal{L}$,

(ii) ϕ is a closed T-immersion of Q into $\mathbf{P}(V(K) \otimes_{O_N} O_T)$,

(iii) $G(Q, \mathcal{L})$ is a finite flat group T-scheme operating upon $(Q, \mathcal{L})$

(iv) $\rho : G(K)_T \to G(Q, \mathcal{L})$ is a weight-one isomorphism of group T-schemes

(v) for any geometric point s of T, (Q_s, ϕ_s, ρ_s) is a projectively stable quasi-abelian scheme of dimension g over $k(s)$ with a level $G(K)$-structure, [11]

(vi) $\mathcal{L} = \phi^*(O_{\mathbf{P}(V(K))}(1) \otimes_{O_N} O_T)$ and $\mathcal{V} = \phi^*(V(K) \otimes_{O_N} O_T)$.

We denote $(Q, \mathcal{L}, G(Q, \mathcal{L}), \mathcal{V}, \phi, \rho)$ by $(Q, \phi, \rho)_{\mathrm{LEV}}$ for brevity. If further any fibre (Q_s, ϕ_s, ρ_s) in (iv) is a PSQAS with a rigid $\bar{G}(K)$-structure, then we call the sextuplet $(Q, \mathcal{L}, G(Q, \mathcal{L}), \mathcal{V}, \phi, \rho)$ a projectively stable quasi-abelian T-scheme with a rigid $\bar{G}(K)$-structure and we denote it by $(Q, \phi, \rho)_{\mathrm{RIG}}$.

Definition 10.4. For T-PSQAS's $(Q_i, \phi_i, \rho_i)_{\mathrm{LEV}}$ $(i = 1, 2)$ with level $G(K)$-structures, we define $(Q_1, \phi_1, \rho_1)_{\mathrm{LEV}} \simeq (Q_2, \phi_2, \rho_2)_{\mathrm{LEV}}$ if there exist a T-isomorphism $f : Q_1 \to Q_2$ and an O_T-invertible sheaf M such that

[11]This implies $G(Q_s, \mathcal{L}_s) = G(Q, \mathcal{L}) \otimes k(s)$ for any point $s \in T$.

(1) $\phi_1^*(V(K)\otimes_{O_N} O_T) = M\otimes_{O_T} f^*\phi_2^*(V(K)\otimes_{O_N} O_T)$,
(2) $\rho_1 = G(f^*)\cdot\rho_2$

For T-PSQAS's $(Q_i,\phi_i,\rho_i)_{\mathrm{RIG}}$ $(i=1,2)$ with rigid $\bar{G}(K)$-structures, we define $(Q_1,\phi_1,\rho_1)_{\mathrm{RIG}} \simeq (Q_2,\phi_2,\rho_2)_{\mathrm{RIG}}$ by one of the following equivalent conditions; [12]

(1) $(Q_1,\phi_1,\rho_1)_{\mathrm{LEV}} \simeq (Q_2,\phi_2,\rho_2)_{\mathrm{LEV}}$
(2) there is a T-isomorphism $f: Q_1\to Q_2$ such that $\phi_1=\phi_2\cdot f$,
(3) there is a T-isomorphism $f: Q_1\to Q_2$ such that
$\phi_1^*(V(K)\otimes_{O_N} O_T) = f^*\phi_2^*(V(K)\otimes_{O_N} O_T)$ and $G(\phi_1^*) = G(f^*)\cdot G(\phi_2^*)$

We define the functors $\mathcal{SQ}_{g,K}$ and $\mathcal{SQ}_{g,K}^{\mathrm{RIG}}$ as follows. For any noetherian S-scheme T, we set

$$\begin{aligned}\mathcal{SQ}_{g,K}(T) = {} & \text{the set of projectively stable quasi-abelian}\\ & T\text{-schemes } (Q,\phi,\rho)_{\mathrm{LEV}} \text{ of relative dimension } g\\ & \text{with level } G(K)\text{-structures modulo } T\text{-isom}\end{aligned}$$

$$\begin{aligned}\mathcal{SQ}_{g,K}^{\mathrm{RIG}}(T) = {} & \text{the set of projectively stable quasi-abelian}\\ & T\text{-schemes } (Q,\phi,\rho)_{\mathrm{RIG}} \text{ of relative dimension } g\\ & \text{with rigid } \bar{G}(K)\text{-structures modulo } T\text{-isom}.\end{aligned}$$

Theorem 10.5. *Let $N=\operatorname{rank} K$. If $e_{\min}(K)\geq 3$, then the functor $\mathcal{SQ}_{g,K}$ is reductively represented by a projective scheme $SQ_{g,K}$ over $\mathbf{Z}[\zeta_N, 1/N]$.*

Proof. First we prove $\mathcal{SQ}_{g,K}\simeq\mathcal{SQ}_{g,K}^{\mathrm{RIG}}$. Suppose we are given a T-PSQAS $(Q,\phi,\rho)_{\mathrm{LEV}}$ with a level $G(K)$-structure. Let $\{U_i = \operatorname{Spec} R_i\}$ be an open affine covering and $\mathcal{V}_{U_i} = V_i\otimes_{R_i} O_{U_i}$ for some R_i-free module V_i. We have a collection of weight-one isomorphisms $\rho_i : G(K)_{U_i}\to G(Q_{U_i}, L_{U_i})$. By the condition (iv) and by Lemma 6.2, by choosing a finer open covering of $\{U_i\}$ if necessary, there is a collection of $A_i\in GL(V_i)$ such that $\rho_i(g) = A_iG(\phi^*)U(K)(g)A_i^{-1}$ $(\forall g\in G(K)_{U_i}(U_i))$. Let $\phi(\rho)_i^* := A_i\phi^*$. Then $\phi(\rho)_i^*$ induces a closed U_i-immersion of Q_{U_i} into $\mathbf{P}(V(K)\otimes_{O_N} O_{U_i})$ such that $\rho_i(g) = G(\phi(\rho)_i^*)U(K)(g)$. On $U_i\cap U_j$, we have $\rho_i(g)=\rho_j(g)$ so that $G(\phi(\rho)_i^*) = G(\phi(\rho)_j^*)$. Hence by Schur's lemma there is a scalar $\ell_{ij}\in GL(V(K)\otimes_S O_{U_{ij}})$ such that $\phi(\rho)_i^* = \ell_{ij}\phi(\rho)_j^*$.

It follows that there is a closed T-immersion $\phi(\rho)$ of Q into $\mathbf{P}(V(K))$ and an invertible O_T-module $M := \{\ell_{ij}\}\in H^1(O_T^*)$ such that $\phi(\rho)^*(V(K)\otimes_{O_N} O_T) = \mathcal{V}\otimes M$ and $\rho = G(\phi(\rho)^*)U(K)$. Then $(Q,\phi(\rho),\rho)_{\mathrm{RIG}}$ is a unique T-PSQAS with a rigid $\bar{G}(K)$-structure such that $(Q,\phi(\rho),\rho)_{\mathrm{LEV}}\simeq(Q,\phi,\rho)_{\mathrm{LEV}}$. It follows $\mathcal{SQ}_{g,K}\simeq\mathcal{SQ}_{g,K}^{\mathrm{RIG}}$.

[12] Equivalence is proved in the same manner as in Lemma 8.9.

It remains to prove that $\mathcal{SQ}^{\mathrm{RIG}}_{g,K}$ is *reductively* represented by $SQ_{g,K}$. Let T be a reduced $\mathcal{O}_N$-scheme. Suppose we are given a projectively stable quasi-abelian T-scheme $(Z, \phi, \rho)_{\mathrm{RIG}}$ with a rigid $\bar{G}(K)$-structure. Then $\phi : Z \to \mathbf{P}(V(K) \otimes_{\mathcal{O}_N} O_T)$ is a closed immersion of Z so that we have a natural T-morphism $\mathrm{Hilb}(\phi) : T \to H_{g,K}$ which factors through $SQ_{g,K}$ by Theorem 9.4 (2). Moreover $Z = \mathrm{Hilb}(\phi)^*(Z^{SQ}_{g,K}) = Z^{SQ}_{g,K} \times_{SQ_{g,K}} T$ by the universal property of $Z_{g,K}$. Hence the map $(Z, \phi, \rho)_{\mathrm{RIG}} \mapsto \mathrm{Hilb}(\phi)$ is bijective. This shows $\mathcal{SQ}^{\mathrm{RIG}}_{g,K}(T) = \mathrm{Hom}_S(T, SQ_{g,K})$. The second condition (b) in Definition 10.1 is clear. Suppose that another M satisfies the conditions (a) and (b) in Definition 10.1. Since there is a flat projective scheme $Z^{SQ}_{g,K}$ over $SQ_{g,K}$, we have by the condition (a) for M a unique $\mathcal{O}_N$-morphism $\psi : SQ_{g,K} \to M$ such that $\mathrm{Hom}_{\mathcal{O}_N}(T, M) = \mathrm{Hom}(\psi) \cdot \mathrm{Hom}_{\mathcal{O}_N}(T, SQ_{g,K})$ for any reduced T. This proves that $\mathcal{SQ}^{\mathrm{RIG}}_{g,K}$ is reductively represented by $SQ_{g,K}$. □

10.6. Let $V = \mathbf{Z}x_0 + \mathbf{Z}x_1 + \mathbf{Z}x_2$. Let $\mathbf{P} := \mathbf{P}(S^3(V^\vee))$ be the projective space of ternary cubic forms on V. Let U be an open subscheme of $\mathbf{P}$ consisting of curves with at worst a unique nodal singularity. The categorical quotient of U by $SL(3)$ is $\mathbf{P}^1_{\mathbf{Z}}$, which is a coarse moduli scheme of the functor of smooth elliptic curves and a rational curve with a node. However as we saw above, we have a different kind of compactification $SQ_{1,(\mathbf{Z}/3\mathbf{Z})}$, a reduced-fine-moduli scheme over $\mathbf{Z}[\zeta_3, 1/3]$ of one-dimensional PSQAS's with level $G(\mathbf{Z}/3\mathbf{Z})$-structures. The universal subscheme $Z^{\mathrm{SQ}}_{1,(\mathbf{Z}/3\mathbf{Z})}$ is given by the Hesse cubic

$$\mu_0(x_0^3 + x_1^3 + x_2^3) - 3\mu_1 x_0 x_1 x_2 = 0$$

which is known as Shioda's elliptic modular surface of level three. We note that $SQ_{1,(\mathbf{Z}/3\mathbf{Z})} \simeq \mathbf{P}(\mathbf{Z}[\zeta_3, 1/3][\mu_0, \mu_1])$.

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