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<th>Log Del Pezzo Surfaces of Rank One with Unique Singular Points</th>
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<td>Author(s)</td>
<td>Kojima, Hideo</td>
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<tr>
<td>Citation</td>
<td>代数幾何学シンポジウム記録 (1997), 1997: 96-110</td>
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<tr>
<td>Issue Date</td>
<td>1997</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/214672">http://hdl.handle.net/2433/214672</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
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<td>Textversion</td>
<td>publisher</td>
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0 Introduction

Let $k$ be an algebraically closed field of characteristic zero. Let $\overline{X}$ be a normal algebraic surface with a unique quotient singular point $P$. Let $f : X \rightarrow \overline{X}$ be a minimal resolution of $\overline{X}$ and let $D = \sum_{i=1}^{n} D_i$ be the reduced exceptional divisor with respect to $f$, where the $D_i$ are irreducible components. Then there exists uniquely an effective $\mathbb{Q}$-divisor $D^\# = \sum_{i=1}^{n} \alpha_i D_i$ such that $D^\# + K_X$ is numerically equivalent to $f^*(K_{\overline{X}})$. Since $P$ is a log terminal singular point, it follows that $0 \leq \alpha_i < 1$ for any $i$. Put $B_k (D) = D - D^\#$.

**Definition 0.1** The above pair $(X, D)$ is *almost minimal* if, for every irreducible curve $C$, either $(D^\# + K_X \cdot C) \geq 0$ or the intersection matrix of $C + D$ is not negative definite. The singular point $P$ is *almost minimal* in $\overline{X}$ if the pair $(X, D)$ is almost minimal.

Starting with arbitrary quotient singular points, we can construct almost minimal quotient singular points, though the singularities might be changed from the original ones.

In the present article, we assume that a pair $(X, D)$ is almost minimal and the logarithmic Kodaira dimension $\kappa(X - D) = -\infty$. Since $K_{\overline{X}}$ is then not numerically effective, there exists an extremal rational curve $\overline{\ell}$ on $\overline{X}$. Let

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*The author is partially supported by JSPS Research Fellowships for Young Scientists and Grant-in-Aid for Scientific Research, the Ministry of Education, Science and Culture.*
$\ell$ be the proper transform of $\bar{\ell}$ on $X$. By Miyanishi-Tsunoda [7, Lemma 2.7], one of the following two cases then takes place:

(A) The intersection matrix of $\ell + B_k(D)$ is negative semi-definite, but not negative definite. Furthermore, $(\bar{\ell}^2) = 0$.

(B) The Picard number $\rho(X)$ is equal to 1, and $-K_X$ is ample. Namely, $(X, D)$ is a logarithmic del Pezzo surface of rank one with contractible boundary (henceforce called a log del Pezzo surface of rank one, for short) (see [10] for the definition).

In the case (A), such pairs $(X, D)$ are completely classified easily (cf. [4, Theorem 1.1]). In the case (B), if $P$ is a rational double or triple singular point, then such pairs have been classified completely (see [9] and [11]). But for arbitrary quotient singularities, such pairs are not yet classified.

In the present article, we attempt to classify all the log del Pezzo surfaces of rank one with unique quotient singular points.

**Remark 0.2** By virtue of Keel-McKernan [3, Corollary 9.3], a log del Pezzo surface of rank one can have at most five singular points. On the other hand, there is no bound on the number of singularities in characteristic two.

**Terminology.** A $(-n)$-curve is a nonsingular rational curve with self intersection number $-n$. For the definition of rods, twigs and forks, we refer to [7]. A reduced effective divisor $D$ is called an NC (resp. SNC) divisor if $D$ has only normal (resp. simple normal) crossings. We employ the following notation:

$\rho(X)$: Picard number of $X$.

$F_n(n \geq 0)$: Hirzebruch surface of degree $n$.

$M_n(n \geq 0)$: the minimal section of $F_n$.

$\#D$: the number of all irreducible components in $\text{Supp} (D)$.
1 Preliminary results

Let \((X, D)\) be the minimal resolution of a log del Pezzo surface \(\overline{X}\) of rank one with a unique singular point \(P\). Then \(\rho(X) = \#D + 1\). We have the following:

**Lemma 1.1** (cf. [11, Lemma 1.1]) Let \((X, D)\) be as above. Then we have:

1. \(f^*(K\overline{X}) \equiv D^# + K_X\) and \(-(D^# + K_X)\) is nef and big. Moreover, for any irreducible curve \(F\), \(- (D^# + K_X \cdot F) = 0\) if and only if \(F\) is a component of \(D\).

2. Any \((-n)\)-curve with \(n \geq 2\) is a component of \(D\).

3. \(X\) is a rational surface.

**Lemma 1.2** (cf. [10, Lemma 1.4]) There is no \((-1)\)-curve \(E\) such that, after contracting \(E\) and consecutively (smoothly) contractible curves in \(E + D\), the divisor \(E + D\) becomes an admissible rational rod or fork, where the adjective “admissible” means that each irreducible component of the image of \(E + D\) has self intersection number \(\leq -2\).

By Lemma 1.1(1), one can fix all irreducible curve \(C\) such that \(- (C \cdot D^# + K_X)\) attains the smallest positive value.

**Lemma 1.3** Suppose that \(|C + D + K_X| \neq \emptyset\) and \(P\) is not a rational double point. Then \(D\) is an admissible rod, i.e., \(P\) is a cyclic quotient singular point.

**Proof.** By [10, Lemma 2.1], there exists a unique decomposition of \(D\) as a sum of effective integral divisors \(D = D' + D''\) such that:

1. \((C \cdot D_i) = (D'' \cdot D_i) = (K_X \cdot D_i) = 0\) for any component \(D_i\) of \(D'\).
2. \(C + D'' + K_X \sim 0\).

If \(D'' = 0\) then \(D = D'\), where \(D'\) consists of \((-2)\)-curves. This contradicts the hypothesis. Hence, \(D'' \neq 0\) and \(D = D''\) because \text{Supp}(D) is connected. By [11, Lemma 1.7(2)] \(D\) is an admissible rod. Q.E.D.

Suppose that \(|C + D + K_X| \neq \emptyset\) and \(D'' = 0\) in the proof of Lemma 1.3. Such a pair \((X, D)\) is then completely determined. Namely, we have the following result.
Lemma 1.4 (cf. [9, Lemma 3]). With the notation as above, suppose that $|C + D + K_X| \neq \emptyset$ and $D'' = 0$. Then $P$ is a rational double point and the dual graph of $D$ is one of the following Dynkin graphs:

$$A_1, A_4, D_5, E_6, A_7, E_7, D_8, E_8, A_8.$$ 

Moreover, the above $(X, D)$ is obtained from the Hirzebruch surface $\mathbb{F}_2$ of degree 2 by a sequence of blowing-ups. The configuration or the weighted dual graph of $D$ is given as in Appendices A, B and C.

Lemma 1.5 (cf. [10, Lemma 2.2] and [2, Proposition 3.6]). Suppose that $|C + D + K_X| = \emptyset$. Then $(X, D)$ is $(F_n, M_n)$, where $n = -(D^2) \geq 2$, or we may assume that $C$ is a $(-1)$-curve.

If $|C + D + K_X| = \emptyset$ then the divisor $C + D$ is an SNC-divisor, consisting of nonsingular rational curves and the dual graph of $C + D$ is a tree (cf. [6, Lemma 2.1.3]). Since $\text{Supp}(D)$ is connected, $C$ meets only one irreducible component of $D$ transversally. More precisely, the following assertion holds.

Proposition 1.6 Suppose that $|C + D + K_X| = \emptyset$ and $D$ is an admissible rational fork. Let $D_0$ be the unique irreducible component of $D$ such that $(D_0 \cdot D - D_0) = 3$. Then $C$ does not meet $D_0$.

Proof. Since $D$ is a fork, $C$ is a $(-1)$-curve by Lemma 1.5. Suppose that $(C \cdot D_0) = 1$. Then $D_0$ is a $(-2)$-curve by Lemma 1.2. Let $\mu : X \to \bar{X}$ be the contraction of $C$ and let $\bar{D} = \mu(D)$. By an argument similar to [10, Lemma 6.4], we know that $(\bar{X}, \bar{D})$ is a log del Pezzo surface of rank one with non-contractible boundary (for the definition, see Miyanishi-Tsunoda [8]). By Lemma 2.6 and Theorems 4 and 6 in [8], $\bar{D}$ consists of a fork and an admissible rational fork which are disjoint from each other. This is however a contradiction because $\text{Supp}(D)$ (and hence $\text{Supp}(\bar{D})$) is connected.

Q.E.D.

Lemma 1.7 Let $\Phi : X \to \mathbb{P}^1$ be a $\mathbb{P}^1$-fibration on a nonsingular projective rational surface $X$. Suppose that there are two cross-sections $H_1$ and $H_2$ of $\Phi$ such that $\langle H_i^2 \rangle \leq -2$ for $i = 1, 2$. Let $\mu : X \to F_m$ be the contraction of all $(-1)$-curves and consecutively contractible curves in singular fibers so that $(\mu_* (H_1)^2) = -m = (H_1^2)$. Then

$$(\mu_* (H_2)^2) = m + 2(H_1 \cdot H_2).$$
Lemma 1.8 Let $\mu : \mathbb{F}_n \to \mathbb{P}^1$ (n ≥ 0) be the ruling of the Hirzebruch surface of degree n and let $M$ be an $m$-section ($m \geq 1$) of $\mu$ different from $M_n$. Then $(M^2) \geq nm^2$.

2 Results

Let $\overline{X}$ be a log del Pezzo surfaces of rank one with a unique singular point $P$ and let $(X, D)$ be the minimal resolution of $\overline{X}$. We have the following theorems:

Theorem 2.1 (The case of type $E_i$) Suppose that $P$ is a quotient singular point of type $E_6$, $E_7$ or $E_8$. Then the following assertions hold:

1. If $P$ is a rational double point, then there exists a $(-1)$-curve $E$ such that the configuration of $E + D$ is one of (2), (5) and (10) in Appendix A.

2. If $P$ is not a rational double point, then the configuration of $C + D$ with the curve $C$ as in the previous section is one of those except for (2), (5) and (10) in Appendix A.

3. There exist a $\mathbb{P}^1$-fibration $\Phi : X \to \mathbb{P}^1$ and a component $H$ of $D$ such that $H$ is a cross-section of $\Phi$ and the other components of $E + D$ (if $P$ is a rational double point) or $C + D$ (if $P$ is not a rational double point) are contained in a unique singular fiber of $\Phi$.

4. All the cases listed in Appendix A are realizable.

Theorem 2.2 (The case of type $D$) Suppose that $P$ is a quotient singular point of type $D$ and that the weighted dual graph is given in Figure 1, where $a_i ≥ 2$ for $i = 0, 3, \ldots, r$.

![Figure 1](image-url)
Then we have the following:

(1) There exists a $(-1)$-curve $E$ such that $(E \cdot D) = (E \cdot D_1) = 1!$. The weighted dual graph of $E + D$ is given as in Appendix B.

(2) There exists a $\mathbf{P}^1$-fibration $\Phi : X \rightarrow \mathbf{P}^1$ such that $D_4$ is a cross-section of $\Phi$ and the other components of $D$ are contained in at most two singular fibers of $\Phi$. In particular, the $D_i (0 \leq i \leq 3)$ are $(-2)$-curves.

(3) All of the types of singularities listed in Appendix B are realizable.

**Theorem 2.3** Suppose that $P$ is a cyclic quotient singular point. Then we have the following:

(1) If $(X, D) \neq (F_n, M_n)$ then the weighted dual graph of $D$ is given by one of (a), (b) and (c) as in Appendix C.

(2) If the weighted dual graph of $D$ is one of (a) and (b) in Appendix C, then there exist a $\mathbf{P}^1$-fibration $\Phi : X \rightarrow \mathbf{P}^1$ and an irreducible component $H$ of $D$ such that $H$ is a cross-section of $\Phi$ and the other irreducible components of $D$ are contained in singular fibers of $\Phi$. $\Phi$ has at most two singular fibers.

(3) If the weighted dual graph of $D$ is one of (c) in Appendix C, then there exists a $\mathbf{P}^1$-fibration $\Psi : X \rightarrow \mathbf{P}^1$ and two irreducible components $H_1$ and $H_2$ such that $H_1$ and $H_2$ are cross-sections of $D$ and the other irreducible components of $D$ are contained in singular fibers of $\Psi$. $\Psi$ has exactly two singular fibers.

(4) All of the types of singularities listed in Appendix C are realizable.

**3 The proof of Theorem 2.2**

In this section we give an outline of the proof of Theorem 2.2. For the proofs of Theorems 2.1 and 2.3, see [5].

The case where $P$ is a rational double point follows from [9, Lemma 3]. Hence we treat the case where $P$ is not a rational double point. Let $C$ be an irreducible curve on $X$ such that $-(C \cdot D^# + K)$ attains the smallest positive value (cf. §1). By Lemmas 1.4 and 1.5, we may assume that $C$ is a $(-1)$-curve
and that \(|C + D + K_X| = \emptyset\). Then \(C\) meets \(D\) only in one component, say \(D_k\) \((0 \leq k \leq r)\). By Proposition 1.6, \(k \neq 0\), i.e., \(k \geq 1\).

We consider the case \(k = 1\) (similarly, \(k = 2\)). We then have the following:

**Lemma 3.1** Suppose that \(k = 1\). Then \(r \geq 4\), \(a_0 = a_3 = 2\) and \(S_0 := 2(C + D_0 + D_1) + D_2 + D_3\) defines a \(\mathbb{P}^1\)-fibration \(\Phi : X \to \mathbb{P}^1\) such that \(\Phi\) has at most two singular fibers. The weighted dual graph of \(C + D\) is given as in Appendix B, where \(E\) is to be read as \(C\).

**Proof.** By Lemma 1.2, \(a_0 = 2\). Similarly, \(a_3 = 2\). Hence \(S_0 := 2(C + D_0 + D_1) + D_2 + D_3\) gives rise to a \(\mathbb{P}^1\)-fibration \(\Phi : X \to \mathbb{P}^1\). Since \(\rho(X) = r + 2\) we know that \(r \geq 4\) and \(D_4\) is a cross-section of \(\Phi\).

Suppose that \(r \geq 5\). Let \(S_1\) be the singular fiber of \(\Phi\) containing \(D_\ell\) \((5 \leq \ell \leq r)\). Then,

\[
\rho(X) = r + 2 \geq 2 + (#S_0 - 1) + (#S_1 - 1) = #S_1 + 5.
\]

On the other hand, \(#S_1 \geq r - 3\) because \(S_1\) contains \(D_\ell\) \((5 \leq \ell \leq r)\) and some \((-1)\)-curves. Hence \(#S_1 = r - 3\). In particular, \(\text{Supp}(S_1)\) has a unique \((-1)\)-curve \(E_1\) and \(\Phi\) has no singular fibers other than \(S_0\) and \(S_1\). Put \(E := C\). The weighted dual graph of \(E + D\) is then given as in Appendix B. Q.E.D.

By Lemma 3.1, we may assume, in the subsequent arguments, that \(k \geq 3\). Since \(\rho(X) = r + 2\) and \((C \cdot D_\# + K_X) < 0\), the intersection matrix of \(C + D\) is neither negative definite nor negative semi-definite. Hence there exist an integer \(e > 0\) and an effective divisor \(\Delta_0\) such that \(\text{Supp}(\Delta_0) \subset \text{Supp}(D)\) and \(|eC + \Delta_0|\) defines a \(\mathbb{P}^1\)-fibration \(\Phi : X \to \mathbb{P}^1\). Put \(S_0 = eC + \Delta_0\) and put \(\Delta = (\Delta_0)_{\text{red}}\).

We consider the case where \(\text{Supp}(\Delta)\) is not a linear chain. Then we have the following:

**Lemma 3.2** Suppose that \(\text{Supp}(\Delta)\) is not a linear chain, i.e., \(\Delta = D_0 + D_1 + D_2 + D_3 + \cdots + D_i\) for some \(3 \leq i \leq r\). The following assertions hold:

1. \(i = r - 1\) and \(D_r\) is a 2-section of \(\Phi\). Furthermore, \(\Phi\) has no singular fibers other than \(S_0\).

2. There exists a \((-1)\)-curve \(E\) such that \((E \cdot D) = (E \cdot D_1) = 1\) or \((E \cdot D) = (E \cdot D_2) = 1\). Hence the weighted dual graph of \(E + D\) is given as in Appendix B.
Proof. (1) Since \( \rho(X) = r + 2 \), it follows that \( i < r \). Since \( C + \Delta \) can be contracted to a nonsingular rational curve with self-intersection number zero, the multiplicity of \( D_i \) in \( S_0 \) is equal to 2. Hence \( D_{i+1} \) is a 2-section of \( \Phi \). Suppose that \( i \leq r - 2 \). Let \( S_1 \) be a singular fiber of \( \Phi \) containing \( D_{i+2}, \ldots, D_r \). By an argument similar to the proof of Lemma 3.1, we know that \( S_1 \) contains a unique \((1)\)-curve \( E_1 \) and \( \Phi \) has no singular fibers other than \( S_0 \) and \( S_1 \). The weighted dual graph of \( \text{Supp}(S_1) \) is then the one in Figure 2 in Appendix B. Since \( D_{i+2} \) and \( D_r \) are terminal components of \( \text{Supp}(S_1) \), the multiplicity of \( D_{i+2} \) in \( S_1 \) is equal to one. Then \( (E_1 \cdot D_{i+1}) = 1 \). Since the multiplicity of \( E_1 \) in \( S_1 \) is then equal to one, there exists another \((1)\)-curve in \( \text{Supp}(S_1) \). This is a contradiction.

(2) Let \( f : X \to \mathbb{F}_m \) be a sequence of contractions of \((1)\)-curves in a unique singular fiber \( S_0 \) of \( \Phi \). Then \( m = 0 \) or \( 1 \). Indeed, if \( m \geq 2 \) then the proper transform \( f'(M_m) \) of \( M_m \) on \( X \) is a nonsingular rational curve with \( (f'(M_m))^2 \leq -2 \) and \( f'(M_m) \) is not contained in \( \text{Supp}(D) \). This contradicts Lemma 1.1(2).

In order to prove the assertion, we consider two cases \( m = 1 \) and \( m = 0 \) separately.

Case: \( m = 1 \). Put \( \ell = f_*(S_0) \). Then there exists a unique fundamental point \( Q \) of \( f \) on \( \ell \). Let \( E' \) be the proper transform of \( M_1 \) on \( X \). Since \( f_*(D_r) \sim 2M_1 + \alpha \ell \) with \( \alpha \geq 2 \) we have \( (f_*(D_r))^2 = -4 + 4\alpha \geq 4 \). Furthermore, by the above contractions, \( f_*(D_r) \) remains as a nonsingular rational curve. Since

\[
-2 = (f_*(D_r))^2 + (f_*(D_r) \cdot K_{\mathbb{F}_m}) = 2(\alpha - 3),
\]

we have \( \alpha = 2 \) and \( (f_*(D_r) \cdot M_1) = 0 \). Since \( E' \) is a \((1)\)-curve by Lemma 1.1, it follows that \( (E' \cdot D) = (E' \cdot D_k) = 1 \), where \( k = 1 \) or \( 2 \).

Case: \( m = 0 \). Let \( \ell \) and \( Q \) be the same as in the case \( m = 1 \). Put \( \overline{D_r} := f_*(D_r) \). Since \( \overline{D_r} \sim 2M_0 + \alpha \ell \) with \( \alpha > 0 \), we have \( (\overline{D_r} \cdot M_0) = \alpha > 0 \). Since \( \overline{D_r} \) is a nonsingular rational curve and since

\[
-2 = (\overline{D_r} \cdot \overline{D_r} + K_{\mathbb{F}_0}) = 2(\alpha - 2),
\]

we have \( \alpha = 1 \). Let \( \ell' \) be a fiber of the second \( \mathbb{P}^1 \)-fibration on \( \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \) through \( Q \). The proper transform \( f'(\ell') \) of \( \ell' \) is then a \((1)\)-curve satisfying

\[
(f'(\ell') \cdot D) = (f'(\ell') \cdot D_k) = 1
\]
Finally, we consider the case where $\text{Supp}(\Delta)$ is a linear chain. Analyzing singular fibers of $\mathbf{P}^1$-fibrations and using Lemmas 1.7 and 1.8, we have the following.

**Lemma 3.3** With the notation and assumptions as above, let $D_i$ and $D_{i_0}$ be the terminal components of the linear chain $\Delta$, where $i_0 < i$. Then the following assertions hold:

1. $i_0 = 5$ and $i = r$.
2. There exists a $(-1)$-curve $E$ ($\neq C$) such that $(E \cdot D) = (E \cdot D_k) = 1$ where $k = 1$ or 2. The weighted dual graph of $E + D$ is given as in Appendix B.

For the proof of Lemma 3.3, see [5, §3].

**Appendix**

**A The case of type $E_n$**

In the following list of configurations, a solid line stands for a component of $\text{Supp}(D)$; the self-intersection number $-2$ of a $(-2)$-curve of $D$ is omitted; a line with $*$ on it is not contained in any fiber of the vertical $\mathbf{P}^1$-fibration $\Phi : X \to \mathbf{P}^1$.

*Type $E_6$*

![Diagram of Type E6 configurations](image)

(1) $D_0, D_1, \ldots, D_6$  
(2) $D_0, D_1, D_2, D_3$
(Type $E_7$)

(3)

(4)

(5)

(6)

(7)

(8)

(9)

(10)

(Type $E_8$)
B The case of type $D$

In the above list of the weighted dual graphs, $m \geq 2$ and the subgraph denoted by the encircled $A_a$ ($a \geq 1$) is given as in the following Figure 2, where $m_1 \geq 2$ and $m_i \geq 1$ for $2 \leq i \leq a$. 
$A_1$:

$A_a (a \geq 2)$:

Figure 2
C Cyclic quotient singularities

We omit the case where \((X, D) = (F, M)\). In the above list of the weighted dual graphs, the subgraph denoted by the encircled \(A_\alpha (\alpha \geq 1)\) is given as in Figure 2, where \(E_1\) is to be read as \(E\), and the subgraphs denoted by the encircled \(T_\beta (\beta \geq 0)\) and \(\tilde{T}_\gamma (\gamma \geq 0)\) are given as in the following Figure 3, where \(m_1, \tilde{m}_1 \geq 3\) and \(m_i, \tilde{m}_{i'} \geq 1\) for \(2 \leq i \leq \beta, 2 \leq i' \leq \gamma\).
Figure 3
References


