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Log Del Pezzo Surfaces of Rank One with Unique Singular Points

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0 Introduction

Let k be an algebraically closed field of characteristic zero. Let $\overline{X}$ be a normal algebraic surface with a unique quotient singular point P. Let $f : X \to \overline{X}$ be a minimal resolution of $\overline{X}$ and let $D = \sum_{i=1}^{n} D_i$ be the reduced exceptional divisor with respect to f, where the D_i are irreducible components. Then there exists uniquely an effective $\mathbf{Q}$-divisor $D^{\#} = \sum_{i=1}^{n} \alpha_i D_i$ such that $D^{\#} + K_X$ is numerically equivalent to $f^*(K_{\overline{X}})$. Since P is a log terminal singular point, it follows that $0 \leq \alpha_i < 1$ for any i. Put $\mathrm{Bk}\,(D) = D - D^{\#}$.

Definition 0.1 The above pair (X, D) is *almost minimal* if, for every irreducible curve C, either $(D^{\#} + K_X \cdot C) \geq 0$ or the intersection matrix of $C + D$ is not negative definite. The singular point P is *almost minimal* in $\overline{X}$ if the pair (X, D) is almost minimal.

Starting with arbitrary quotient singular points, we can construct almost minimal quotient singular points, though the singularities might be changed from the original ones.

In the present article, we assume that a pair (X, D) is almost minimal and the logarithmic Kodaira dimension $\overline{\kappa}(X - D) = -\infty$. Since $K_{\overline{X}}$ is then not numerically effective, there exists an extremal rational curve $\overline{\ell}$ on $\overline{X}$. Let

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ℓ be the proper transform of $\overline{\ell}$ on X. By Miyanishi-Tsunoda [7, Lemma 2.7], one of the following two cases then takes place:

(A) The intersection matrix of $\ell + \mathrm{Bk}\,(D)$ is negative semi-definite, but not negative definite. Furthermore, $(\overline{\ell}^2) = 0$.

(B) The Picard number $\rho(\overline{X})$ is equal to 1, and $-K_{\overline{X}}$ is ample. Namely, (X, D) is a logarithmic del Pezzo surface of rank one with contractible boundary (henceforce called a log del Pezzo surface of rank one, for short) (see [10] for the definition).

In the case (A), such pairs (X, D) are completely classified easily (cf. [4, Theorem 1.1]). In the case (B), if P is a rational double or triple singular point, then such pairs have been classified completely (see [9] and [11]). But for arbitrary quotient singularities, such pairs are not yet classified.

In the present article, we attempt to classify all the log del Pezzo surfaces of rank one with unique quotient singular points.

Remark 0.2 By virtue of Keel-McKernan [3, Corollary 9.3], a log del Pezzo surface of rank one can have at most five singular points. On the other hand, there is no bound on the number of singularities in characteristic two.

Terminology. A $(-n)$-curve is a nonsingular rational curve with self intersection number $-n$. For the definition of rods, twigs and forks, we refer to [7]. A reduced effective divisor D is called an NC (resp. SNC) divisor if D has only normal (resp. simple normal) crossings. We employ the following notation:

$\rho(X)$: Picard number of X.

$\mathbf{F}_n (n \geq 0)$: Hirzebruch surface of degree n.

$M_n (n \geq 0)$: the minimal section of $\mathbf{F}_n$.

$\#D$: the number of all irreducible components in Supp$\,(D)$.

1 Preliminary results

Let (X, D) be the minimal resolution of a log del Pezzo surface $\overline{X}$ of rank one with a unique singular point P. Then $\rho(X) = \#D + 1$. We have the following:

Lemma 1.1 (cf. [11, Lemma 1.1]) *Let (X, D) be as above. Then we have:*

(1) *$f^*(K_{\overline{X}}) \equiv D^\# + K_X$ and $-(D^\# + K_X)$ is nef and big. Moreover, for any irreducible curve F, $-(D^\# + K_X \cdot F) = 0$ if and only if F is a component of D.*

(2) *Any $(-n)$-curve with $n \geq 2$ is a component of D.*

(3) *X is a rational surface.*

Lemma 1.2 (cf. [10, Lemma 1.4]) *There is no (-1)-curve E such that, after contracting E and consecutively (smoothly) contractible curves in $E + D$, the divisor $E + D$ becomes an admissible rational rod or fork, where the adjective "admissible" means that each irreducible component of the image of $E + D$ has self intersection number ≤ -2.*

By Lemma 1.1(1), one can find an irreducible curve C such that $-(C \cdot D^\# + K_X)$ attains the smallest positive value.

Lemma 1.3 *Suppose that $|C + D + K_X| \neq \emptyset$ and P is not a rational double point. Then D is an admissible rod, i.e., P is a cyclic quotient singular point.*

Proof. By [10, Lemma 2.1], there exists a unique decomposition of D as a sum of effective integral divisors $D = D' + D''$ such that:

(i) $(C \cdot D_i) = (D'' \cdot D_i) = (K_X \cdot D_i) = 0$ for any component D_i of D'.

(ii) $C + D'' + K_X \sim 0$.

If $D'' = 0$ then $D = D'$, where D' consists of (-2)-curves. This contradicts the hypothesis. Hence, $D'' \neq 0$ and $D = D''$ because Supp (D) is connected. By [11, Lemma 1.7(2)] D is an admissible rod. Q.E.D.

Suppose that $|C + D + K_X| \neq \emptyset$ and $D'' = 0$ in the proof of Lemma 1.3. Such a pair (X, D) is then completely determined. Namely, we have the following result.

Lemma 1.4 (cf. [9, Lemma 3]). *With the notation as above, suppose that* $|C + D + K_X| \neq \emptyset$ *and* $D'' = 0$. *Then* P *is a rational double point and the dual graph of* D *is one of the following Dynkin graphs:*

$$A_1,\ A_4,\ D_5,\ E_6,\ A_7,\ E_7,\ D_8,\ E_8,\ A_8.$$

Moreover, the above (X, D) *is obtained from the Hirzebruch surface* $\mathbf{F}_2$ *of degree* 2 *by a sequence of blowing-ups. The configuration or the weighted dual graph of* D *is given as in Appendices A, B and C.*

Lemma 1.5 (cf. [10, Lemma 2.2] and [2, Proposition 3.6]). *Suppose that* $|C + D + K_X| = \emptyset$. *Then* (X, D) *is* $(\mathbf{F}_n, M_n)$, *where* $n = -(D^2) \geq 2$, *or we may assume that* C *is a* (-1)*-curve.*

If $|C+D+K_X| = \emptyset$ then the divisor $C+D$ is an SNC-divisor, consisting of nonsingular rational curves and the dual graph of $C+D$ is a tree (cf. [6, Lemma 2.1.3]). Since Supp (D) is connected, C meets only one irreducible component of D transversally. More precisely, the following assertion holds.

Proposition 1.6 *Suppose that* $|C + D + K_X| = \emptyset$ *and* D *is an admissible rational fork. Let* D_0 *be the unique irreducible component of* D *such that* $(D_0 \cdot D - D_0) = 3$. *Then* C *does not meet* D_0.

Proof. Since D is a fork, C is a (-1)-curve by Lemma 1.5. Suppose that $(C \cdot D_0) = 1$. Then D_0 is a (-2)-curve by Lemma 1.2. Let $\mu : X \to \tilde{X}$ be the contraction of C and let $\tilde{D} = \mu(D)$. By an argument similar to [10, Lemma 6.4], we know that $(\tilde{X}, \tilde{D})$ is a log del Pezzo surface of rank one with non-contractible boundary (for the definition, see Miyanishi-Tsunoda [8]). By Lemma 2.6 and Theorems 4 and 6 in [8], $\tilde{D}$ consists of a fork and an admissible rational fork which are disjoint from each other. This is however a contradiction because Supp (D) (and hence Supp $(\tilde{D})$) is connected.

Q.E.D.

Lemma 1.7 *Let* $\Phi : X \to \mathbf{P}^1$ *be a* $\mathbf{P}^1$*-fibration on a nonsingular projective rational surface* X. *Suppose that there are two cross-sections* H_1 *and* H_2 *of* Φ *such that* $(H_i^2) \leq -2$ *for* $i = 1, 2$. *Let* $\mu : X \to \mathbf{F}_m$ *be the contraction of all* (-1)*-curves and consecutively contractible curves in singular fibers so that* $(\mu_*(H_1)^2) = -m = (H_1^2)$. *Then*

$$(\mu_*(H_2)^2) = m + 2(H_1 \cdot H_2).$$

Lemma 1.8 *Let $\mu : \mathbf{F}_n \to \mathbf{P}^1$ $(n \geq 0)$ be the ruling of the Hirzebruch surface of degree n and let M be an m-section $(m \geq 1)$ of μ different from M_n. Then $(M^2) \geq nm^2$.*

2 Results

Let $\overline{X}$ be a log del Pezzo surfaces of rank one with a unique singular point P and let (X, D) be the minimal resolution of $\overline{X}$. We have the following theorems:

Theorem 2.1 (The case of type E_n) *Suppose that P is a quotient singular point of type E_6, E_7 or E_8. Then the following assertions hold:*

(1) *If P is a rational double point, then there exists a (-1)-curve E such that the configuration of $E + D$ is one of* (2), (5) *and* (10) *in Appendix A.*

(2) *If P is not a rational double point, then the configuration of $C+D$ with the curve C as in the previous section is one of those except for* (2), (5) *and* (10) *in Appendix A.*

(3) *There exist a $\mathbf{P}^1$-fibration $\Phi : X \to \mathbf{P}^1$ and a component H of D such that H is a cross-section of Φ and the other components of $E + D$ (if P is a rational double point) or $C + D$ (if P is not a rational double point) are contained in a unique singular fiber of Φ.*

(4) *All the cases listed in Appendix A are realizable.*

Theorem 2.2 (The case of type D) *Suppose that P is a quotient singular point of type D and that the weighted dual graph is given in Figure* 1, *where $a_i \geq 2$ for $i = 0, 3, \ldots, r$.*

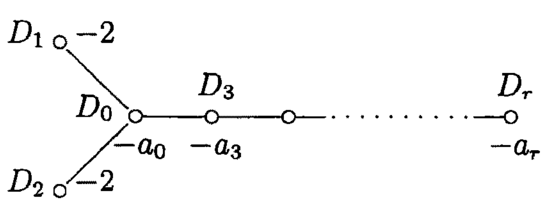


Figure 1

Then we have the following:

(1) *There exists a* (-1)*-curve* E *such that* $(E \cdot D) = (E \cdot D_1) = 1$. *The weighted dual graph of* $E + D$ *is given as in Appendix B.*

(2) *There exists a* $\mathbf{P}^1$*-fibration* $\Phi : X \to \mathbf{P}^1$ *such that* D_4 *is a cross-section of* Φ *and the other components of* D *are contained in at most two singular fibers of* Φ. *In particular, the* D_i $(0 \leq i \leq 3)$ *are* (-2)*-curves.*

(3) *All of the types of singularities listed in Appendix B are realizable.*

Theorem 2.3 *Suppose that* P *is a cyclic quotient singular point. Then we have the following:*

(1) *If* $(X, D) \neq (\mathbf{F}_n, M_n)$ *then the weighted dual graph of* D *is given by one of* (a), (b) *and* (c) *as in Appendix C.*

(2) *If the weighted dual graph of* D *is one of* (a) *and* (b) *in Appendix C, then there exist a* $\mathbf{P}^1$*-fibration* $\Phi : X \to \mathbf{P}^1$ *and an irreducible component* H *of* D *such that* H *is a cross-section of* Φ *and the other irreducible components of* D *are contained in singular fibers of* Φ. Φ *has at most two singular fibers.*

(3) *If the weighted dual graph of* D *is one of* (c) *in Appendix C, then there exists a* $\mathbf{P}^1$*-fibration* $\Psi : X \to \mathbf{P}^1$ *and two irreducible components* H_1 *and* H_2 *such that* H_1 *and* H_2 *are cross-sections of* D *and the other irreducible components of* D *are contained in singular fibers of* Ψ. Ψ *has exactly two singular fibers.*

(4) *All of the types of singularities listed in Appendix C are realizable.*

3 The proof of Theorem 2.2

In this section we give an outline of the proof of Theorem 2.2. For the proofs of Theorems 2.1 and 2.3, see [5].

The case where P is a rational double point follows from [9, Lemma 3]. Hence we treat the case where P is not a rational double point. Let C be an irreducible curve on X such that $-(C \cdot D^{\#} + K)$ attains the smallest positive value (cf. §1). By Lemmas 1.4 and 1.5, we may assume that C is a (-1)-curve and that $|C + D + K_X| = \emptyset$. Then C meets D only in one component, say D_k $(0 \le k \le r)$. By Proposition 1.6, $k \neq 0$, i.e., $k \geq 1$.

We consider the case $k = 1$ (similarly, $k = 2$). We then have the following:

Lemma 3.1 *Suppose that $k = 1$. Then $r \geq 4$, $a_0 = a_3 = 2$ and $S_0 := 2(C + D_0 + D_1) + D_2 + D_3$ defines a $\mathbf{P}^1$-fibration $\Phi : X \to \mathbf{P}^1$ such that Φ has at most two singular fibers. The weighted dual graph of $C + D$ is given as in Appendex B, where E is to be read as C.*

Proof. By Lemma 1.2, $a_0 = 2$. Similarly, $a_3 = 2$. Hence $S_0 := 2(C + D_0 + D_1) + D_2 + D_3$ gives rise to a $\mathbf{P}^1$-fibration $\Phi : X \to \mathbf{P}^1$. Since $\rho(X) = r + 2$ we know that $r \geq 4$ and D_4 is a cross-section of Φ.

Suppose that $r \geq 5$. Let S_1 be the singular fiber of Φ containing D_ℓ $(5 \le \ell \le r)$. Then,

$$\rho(X) = r + 2 \geq 2 + (\#S_0 - 1) + (\#S_1 - 1) = \#S_1 + 5.$$

On the other hand, $\#S_1 \geq r - 3$ because S_1 contains D_ℓ $(5 \le \ell \le r)$ and some (-1)-curves. Hence $\#S_1 = r - 3$. In particular, $\mathrm{Supp}(S_1)$ has a unique (-1)-curve E_1 and Φ has no singular fibers other than S_0 and S_1. Put $E := C$. The weighted dual graph of $E + D$ is then given as in Appendix B. Q.E.D.

By Lemma 3.1, we may assume, in the subsequent arguments, that $k \geq 3$. Since $\rho(X) = r + 2$ and $(C \cdot D^{\#} + K_X) < 0$, the intersection matrix of $C + D$ is neither negative definite nor negative semi-definite. Hence there exist an integer $e > 0$ and an effective devisor Δ_0 such that $\mathrm{Supp}\,(\Delta_0) \subset \mathrm{Supp}\,(D)$ and $|eC + \Delta_0|$ defines a $\mathbf{P}^1$-fibration $\Phi : X \to \mathbf{P}^1$. Put $S_0 = eC + \Delta_0$ and put $\Delta = (\Delta_0)_{\mathrm{red}}$.

We consider the case where $\mathrm{Supp}\,(\Delta)$ is not a linear chain. Then we have the following:

Lemma 3.2 *Suppose that* $\mathrm{Supp}\,(\Delta)$ *is not a linear chain, i.e., $\Delta = D_0 + D_1 + D_2 + D_3 + \cdots + D_i$ for some $3 \le i \le r$. The following assertions hold:*

(1) *$i = r - 1$ and D_r is a 2-section of Φ. Furthermore, Φ has no singular fibers other than S_0.*

(2) *There exists a (-1)-curve E such that $(E \cdot D) = (E \cdot D_1) = 1$ or $(E \cdot D) = (E \cdot D_2) = 1$. Hence the weighted dual graph of $E + D$ is given as in Appendix B.*

Proof. (1) Since $\rho(X) = r + 2$, it follows that $i < r$. Since $C + \Delta$ can be contracted to a nonsingular rational curve with self-intersection number zero, the multiplicity of D_i in S_0 is equal to 2. Hence D_{i+1} is a 2-section of Φ. Suppose that $i \leq r - 2$. Let S_1 be a singular fiber of Φ containing $D_{i+2}, \ldots, D_r$. By an argument similar to the proof of Lemma 3.1, we know that S_1 contains a unique (-1)-curve E_1 and Φ has no singular fibers other than S_0 and S_1. The weighted dual graph of Supp (S_1) is then the one in Figure 2 in Appendix B. Since D_{i+2} and D_r are terminal components of $\mathrm{Supp}(S_1)$, the multiplicity of D_{i+2} in S_1 is equal to one. Then $(E_1 \cdot D_{i+1}) = 1$. Since the multiplicity of E_1 in S_1 is then equal to one, there exists another (-1)-curve in Supp (S_1). This is a contradiction.
(2) Let $f : X \to \mathbf{F}_m$ be a sequence of contractions of (-1)-curves in a unique singular fiber S_0 of Φ. Then $m = 0$ or 1. Indeed, if $m \geq 2$ then the proper transform $f'(M_m)$ of M_m on X is a nonsingular rational curve with $(f'(M_m)^2) \leq -2$ and $f'(M_m)$ is not contained in Supp (D). This contradicts Lemma 1.1(2).

In order to prove the assertion, we consider two cases $m = 1$ and $m = 0$ separately.

Case: $m = 1$. Put $\ell = f_*(S_0)$. Then there exists a unique fundamental point Q of f on ℓ. Let E' be the proper transform of M_1 on X. Since $f_*(D_r) \sim 2M_1 + \alpha\ell$ with $\alpha \geq 2$ we have $(f_*(D_r)^2) = -4 + 4\alpha \geq 4$. Furthermore, by the above contractions, $f_*(D_r)$ remains as a nonsingular rational curve. Since

$$-2 = (f_*(D_r)^2) + (f_*(D_r) \cdot K_{\mathbf{F}_m}) = 2(\alpha - 3),$$

we have $\alpha = 2$ and $(f_*(D_r) \cdot M_1) = 0$. Since E' is a (-1)-curve by Lemma 1.1, it follows that $(E' \cdot D) = (E' \cdot D_k) = 1$, where $k = 1$ or 2.

Case: $m = 0$. Let ℓ and Q be the same as in the case $m = 1$. Put $\overline{D_r} := f_*(D_r)$. Since $\overline{D_r} \sim 2M_0 + \alpha\ell$ with $\alpha > 0$, we have $(\overline{D_r} \cdot M_0) = \alpha > 0$. Since $\overline{D_r}$ is a nonsingular rational curve and since

$$-2 = (\overline{D_r} \cdot \overline{D_r} + K_{\mathbf{F}_0}) = 2(\alpha - 2),$$

we have $\alpha = 1$. Let ℓ' be a fiber of the second $\mathbf{P}^1$-fibration on $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$ through Q. The proper transform $f'(\ell')$ of ℓ' is then a (-1)-curve satisfying

$$(f'(\ell') \cdot D) = (f'(\ell') \cdot D_k) = 1$$

where $k = 1$ or 2. Q.E.D.

Finally, we consider the case where $\mathrm{Supp}\,(\Delta)$ is a linear chain. Analyzing singular fibers of $\mathbf{P}^1$-fibrations and using Lemmas 1.7 and 1.8, we have the following.

Lemma 3.3 *With the notation and assumptions as above, let D_i and D_{i_0} be the terminal components of the linear chain Δ, where $i_0 < i$. Then the following assertions hold:*

(1) $i_0 = 5$ *and* $i = r$.

(2) *There exists a (-1)-curve E $(\neq C)$ such that $(E \cdot D) = (E \cdot D_k) = 1$ where $k = 1$ or 2. The weighted dual graph of $E + D$ is given as in Appendix B.*

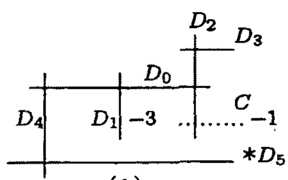
For the proof of Lemma 3.3, see [5, §3].

Appendix

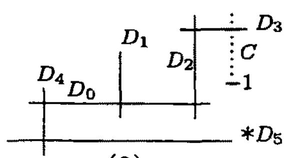
A The case of type E_n

In the following list of configurations, a solid line stands for a component of $\mathrm{Supp}\,(D)$; the self-intersection number -2 of a (-2)-curve of D is omitted; a line with $*$ on it is not contained in any fiber of the vertical $\mathbf{P}^1$-fibration $\Phi : X \to \mathbf{P}^1$.

(Type E_6)

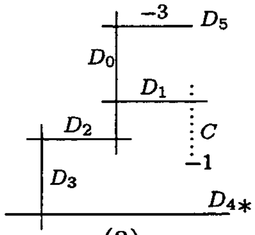


(1)

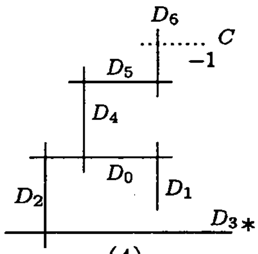


(2)

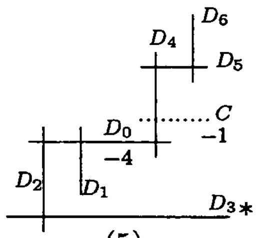
(Type E_7)



(3)

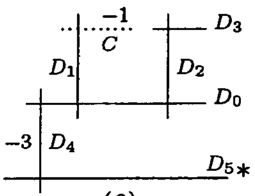


(4)

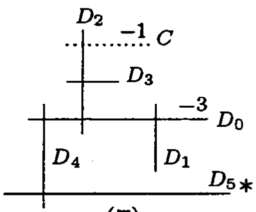


(5)

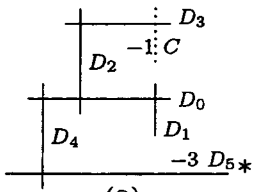
(Type E_8)



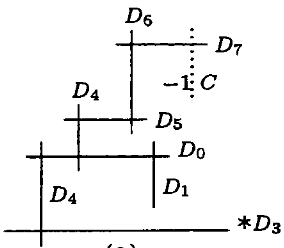
(6)



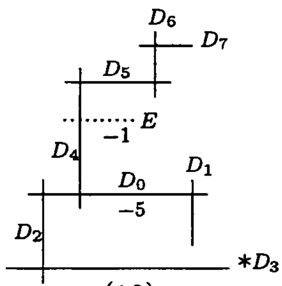
(7)



(8)

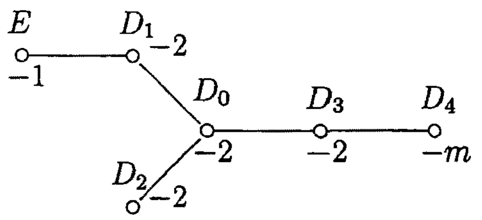


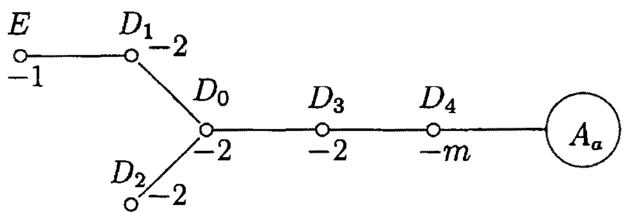
(9)



(10)

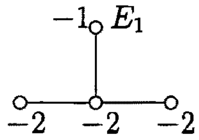
B The case of type D



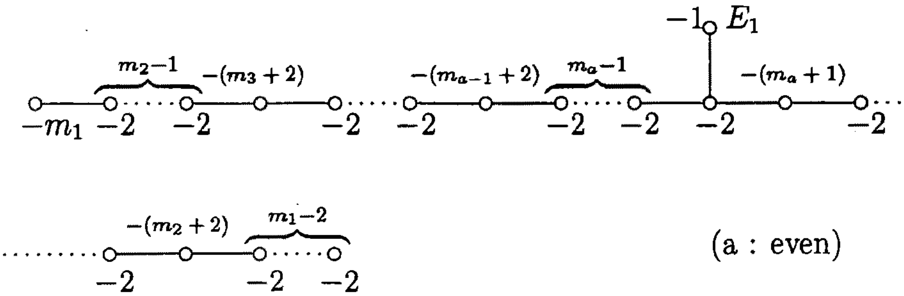


In the above list of the weighted dual graphs, $m \geq 2$ and the subgraph denoted by the encircled A_a ($a \geq 1$) is given as in the following Figure 2, where $m_1 \geq 2$ and $m_i \geq 1$ for $2 \leq i \leq a$.

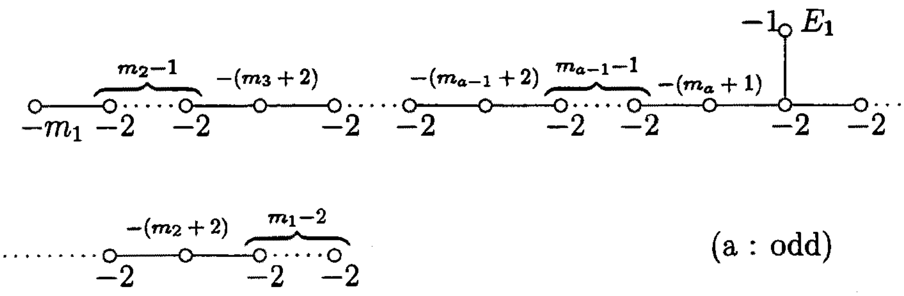
A_1:



$A_a (a \geq 2)$:



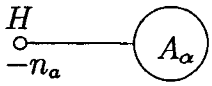
(a : even)



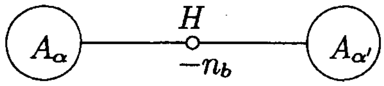
(a : odd)

Figure 2

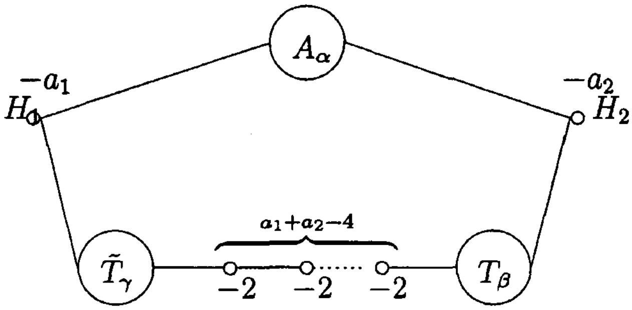
C Cyclic quotient singularities



(a)

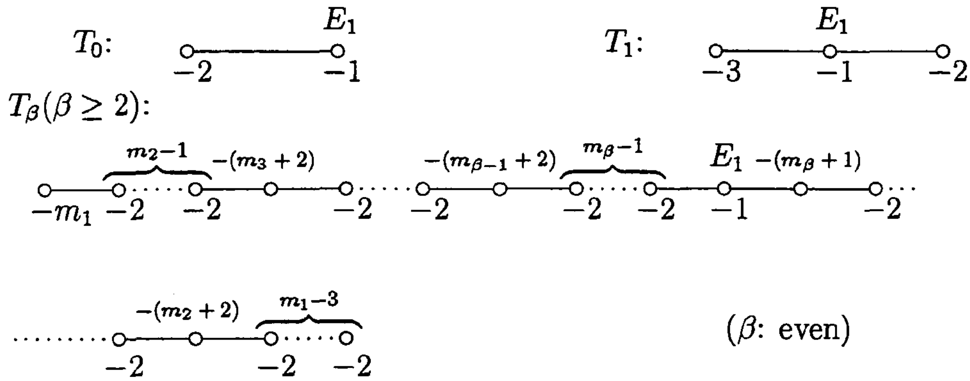


(b)



(c)

We omitt the case where $(X, D) = (\mathbf{F}_n, M_n)$. In the above list of the weighted dual graphs, the subgraph denoted by the encircled A_α ($\alpha \geq 1$) is given as in Figure 2, where E_1 is to be read as E, and the subgraphs denoted by the encircled T_β ($\beta \geq 0$) and $\tilde{T}_\gamma$ ($\gamma \geq 0$) are given as in the following Figure 3, where $m_1, \tilde{m}_1 \geq 3$ and $m_i, \tilde{m}_{i'} \geq 1$ for $2 \leq i \leq \beta$, $2 \leq i' \leq \gamma$.



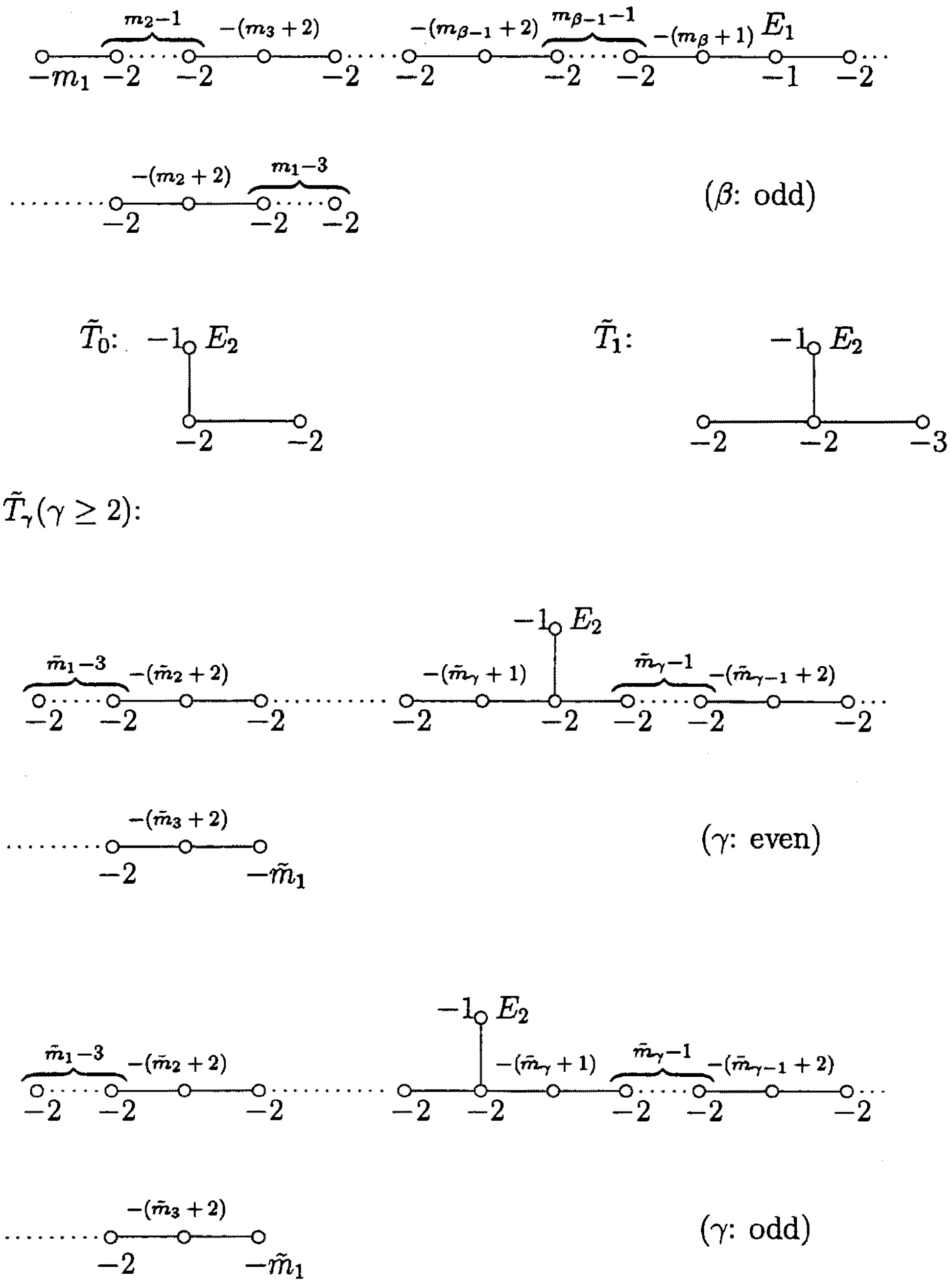


Figure 3

References

[1] E. Brieskorn, Rationale Singularitäten komplexer Flächen, Invent. Math. **4** (1968), 336–358.

[2] R. V. Gurjar and D.-Q. Zhang, π_1 of smooth points of a log del Pezzo surface is finite: I, J. Math. Sci. Tokyo, **1** (1994), 137–180.

[3] S. Keel and J. McKernan, Rational curves on quasi-projective surfaces, preprint, alg-geom/9707016.

[4] H. Kojima, Almost minimal embeddings of quotient singular points into rational surfaces, to appear in J. Math. Kyoto Univ.

[5] H. Kojima, Logarithmic del Pezzo surfaces of rank one with unique singular points, preprint.

[6] M. Miyanishi, Non-complete algebraic surfaces, Lecture Notes in Mathematics, No. **857**, Berlin-Heiderberg-New York, Springer, 1981.

[7] M. Miyanishi and S. Tsunoda, Non-complete algebraic surfaces with logarithmic Kodaira dimension $-\infty$ and with non-connected boundaries at infinity, Japanese J. Math., **10** (1984), 195–242.

[8] M. Miyanishi and S. Tsunoda, Logarithmic del Pezzo surfaces of rank one with non-connected boundaries at infinity, Japanese J. Math., **10** (1984), 271–319

[9] M. Miyanishi and D.-Q. Zhang, Gorenstein log del Pezzo surfaces of rank one, J. Algebra **118** (1988), 63–84.

[10] D.-Q. Zhang, Logarithmic del Pezzo suraces of rank one with contractible boundaries, Osaka J. Math. **25** (1988), 461–497.

[11] D.-Q. Zhang, Logarithmic del Pezzo surfaces with rational double and triple singular points, Tohoku Math. J. **41** (1989), 399–452.