代数幾何学シンポジウム記録
1997年度 pp.48-65

JACOBIAN RINGS OF OPEN VARIETIES

MASANORI ASAKURA

Throughout whole sections, we fix $n, r, s \geq 1$ such that $r+1 \leq n$, and $d_1, \cdots, d_r, e_1, \cdots, e_s \geq 1$ positive integers. We put $\mathbf{d} = \sum_{i=1}^{r} d_i$, $\mathbf{e} = \sum_{j=1}^{s} e_j$, $d_{\min} = \min\{d_i, e_j\}$ and $m = n+r+s-1$. Moreover for a complex vector space V (*resp.* a locally free sheaf $\mathcal{V}$ on a variety X), V^* denotes the dual vector space of V (*resp.* $\mathcal{V}^*$ the dual sheaf of $\mathcal{V}$), and we put

$$\overset{0}{\wedge} V = \mathbf{C} \quad (\textit{resp.} \ \overset{0}{\wedge} \mathcal{V} = \mathcal{O}_X), \qquad \overset{p}{\wedge} V = 0 \quad (\textit{resp.} \ \overset{p}{\wedge} \mathcal{V} = 0) \quad \text{for } p < 0.$$

§1. DEFINITION OF JACOBIAN RINGS AND MAIN THEOREMS

Let S be a polynomial ring over $\mathbf{C}$ generated by indeterminants $X_0, \cdots, X_n$. Denote by S^l the elements of homogeneous polynomials of degree l. Let A be a polynomial ring over S generated by $\mu_1, \cdots, \mu_r, \lambda_1, \cdots, \lambda_s$. We use multiindex $\underline{a} = (a_1, \cdots, a_r) \in \mathbf{Z}_{\geq 0}^{\oplus r}$, $\underline{b} = (b_1, \cdots, b_s) \in \mathbf{Z}_{\geq 0}^{\oplus s}$, and denote $\mu^{\underline{a}} = \mu_1^{a_1} \cdots \mu_r^{a_r}$, $\lambda^{\underline{b}} = \lambda_1^{b_1} \cdots \lambda_r^{b_s}$. Then for $q \in \mathbf{Z}_{\geq 0}$ and $l \in \mathbf{Z}$, we write

$$A_q(l) = \bigoplus_{\sum_{i=1}^{r} a_i + \sum_{j=1}^{s} b_j = q} S^{\sum_{i=1}^{r} a_i d_i + \sum_{j=1}^{s} b_j e_j + l} \cdot \mu^{\underline{a}} \lambda^{\underline{b}}.$$

Definition(**1-1**). For $\underline{F} = (F_1, \cdots, F_r)$, $\underline{G} = (G_1, \cdots, G_s)$ with $F_i \in S^{d_i}$, $G_j \in S^{e_j}$, we define the **Jacobian ideal** $J(\underline{F}, \underline{G})$ the ideal of A generated by $\sum_{i=1}^{r} \frac{\partial F_i}{\partial X_k} \mu_i + \sum_{j=1}^{s} \frac{\partial G_j}{\partial X_k} \lambda_j$, F_i, $G_j \lambda_j$ $(1 \leq i \leq r$, $1 \leq j \leq s$, $0 \leq k \leq n)$. The quotient ring $B = B(\underline{F}, \underline{G}) = A/J(\underline{F}, \underline{G})$, we call the **Jacobian ring of** $(\underline{F}, \underline{G})$. We denote

$$B_q(l) = B_q(l)(\underline{F}, \underline{G}) = A_q(l)/A_q(l) \cap J(\underline{F}, \underline{G}).$$

Let $\mathbb{P}^n = \operatorname{Proj} S$ a complex projective space. Thus we have a variety X defined by equations $F_1, \cdots, F_r$ and subvarieties $Z_j (\subset X)$ defined by $G_j, F_1, \cdots, F_r$ for $1 \leq j \leq s$. We also call the $B(\underline{F}, \underline{G})$ the Jacobian rings of the pair $(X, \bigcup_{j=1}^{s} Z_j)$.

We mension three main theorems. Hereafter we assume that X is a smooth complete intersection of codimension r, and $Z = \cup Z_j$ is a simple normal crossing divisor of X.

The first main theorem is concernings about the geometric meanings of Jacobian rings.

Theorem(I).

(1) There is a natural morphism

$$\varphi : B_p(\mathbf{d} - n - 1) \longrightarrow H^{n-r-p}(X, \Omega_X^p(\log \Sigma Z_j))^*.$$

for $0 \leq p \leq n-r$. Here φ is an isomorphism if $r+s \leq n$ or $p \neq n-r$. If $r+s > n$ and $p = n-r$, then φ is injective, but not surjective, and whose cokernel is pure Hodge type $(-n+r, -n+r)$. More explicitly, the Cokerφ can be descibed as follows:

$$\operatorname{Coker}\varphi = \operatorname{Ker}(\overset{n-r}{\wedge}(e_1, \cdots, e_s) \xrightarrow{d} \overset{n-r-1}{\wedge}(e_1, \cdots, e_s))^*$$

where $e_j := d \log G_j|_X \in H^0(X, \Omega_X^1(\log \Sigma Z_j))$, and $\wedge(e_1, \cdots, e_s)$ denotes the free wedge algebra generated by e_j, and d is defined to be the map $e_{j_1} \wedge \cdots \wedge e_{j_{n-r}} \mapsto \sum_{k=1}^{n-r} (-1)^k e_{j_1} \wedge \cdots \wedge \check{e}_{j_k} \wedge \cdots \wedge e_{j_{n-r}}$.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

(2) There is a natural map

$$\psi : B_1(0) \longrightarrow H^1(X, T_X(-\log \Sigma Z_j))$$

which is an isomorphism, except for X K3-surface or a curve. If X is a $K3$-surface or a curve of genus ≥ 2, ψ is only injective. The cup-product are compatible with the ring multiplication up to scalar:

$$H^1(X, T_X(-\log \Sigma Z_j)) \otimes H^q(X, \Omega_X^p(\log \Sigma Z_j)) \longrightarrow H^{q+1}(X, \Omega_X^{p-1}(\log \Sigma Z_j)).$$

Roughly speaking, the Jacobian rings describe the Hodge structures of open variety $X \setminus \cup Z_j$, and the cup-product with Kodaira-Spencer class coincides with the ring multiplication up to non-zero scalar. This result was originally invented by P.Griffiths, and generalized into several directions by many people([CGGH],[T],[K]). Our result is a further generalization.

The second is the duality theorem, which is fundamental property of Jacobian rings.

Theorem(II). Let $r + s \leq n$. There is an isomorphism

$$B_{n-r}(2(\mathbf{d} - n - 1) + \mathbf{e}) \simeq \mathbf{C}$$

if $\mathbf{d} + \mathbf{e} - n - 1 > 0$, and induced pairing

$$B_p(\mathbf{d} - n - 1 + \ell) \otimes B_{n-r-p}(\mathbf{d} + \mathbf{e} - n - 1 - \ell) \longrightarrow \mathbf{C}$$

is non-degenerate for $0 \leq p \leq n - r$ and $0 \leq \ell \leq \mathbf{e}$.

For example, this will be used to prove Torelli theorem.

The last one was proved firstly by R.Donagi in a special case([D],[DG]).

Theorem(III). Let $W \subset A_1(0)$ is a base point free subspace of codimension c(*i.e.* for any $x \in \mathbb{P}^n(\mathbf{C})$, the evaluation map $W(\subset A_1(0)) \to \underset{i}{\oplus}\mathbf{C}\mu_i \oplus \underset{j}{\oplus}\mathbf{C}\lambda_j$ at x is surjective), then The Koszul complex

$$B_p(\ell) \otimes \overset{q+1}{\wedge} W \to B_{p+1}(\ell) \otimes \overset{q}{\wedge} W \to B_{p+2}(\ell) \otimes \overset{q-1}{\wedge} W$$

is exact if one of the following conditions is satisfied.

(1) $p \geq 0$, $d_{\min}(r+p) + \ell - \mathbf{d} \geq q + c$, $\mathbf{d} - n - 1 \leq \ell \leq \mathbf{d} + \mathbf{e} - n - 1$ and $r + s \leq n + 2$.
(2) $p \geq 0$, $d_{\min}(r+p) + \ell - \mathbf{d} \geq q + c$, $\mathbf{d} - n - 1 \leq \ell \leq \mathbf{d} + \mathbf{e} - n - 1$ and $p \neq n - r - 1, n - r$.
(3) $q = 1$, or 0, $p \geq 0$ and $d_{\min}(r+p) + \ell - \mathbf{d} \geq q + c$.

This theorem has the following important Corollaries.

Let

$$\widetilde{S}^* = \overset{r}{\underset{i=1}{\oplus}} \mathbb{P}_*(H^0(\mathcal{O}_{\mathbb{P}^n}(d_i))) \oplus \overset{s}{\underset{j=1}{\oplus}} \mathbb{P}_*(H^0(\mathcal{O}_{\mathbb{P}^n}(e_j)))$$

We put $\widetilde{S}$ the locus of $\widetilde{S}^*$ such that $(F_1, \cdots, F_r, G_1, \cdots, G_s) \in \widetilde{S}$ if and only if the subvariety

$$F_1 = \cdots = F_r = 0 \quad \text{in } \mathbb{P}^n$$

is a nonsingular complete intersection of codimension r, and

$$F_1 = \cdots = F_r = G_j = 0 \quad \text{in } \mathbb{P}^n$$

is its nonsingular hypersurface section for $\forall j$. The algebraic group $\mathrm{PGL}_{n+1}(\mathbf{C})$ and $Q = \{\sigma \in \mathrm{Aut}(\mathcal{E}) | \sigma(\mathcal{E}_0) \subset \mathcal{E}_0, \quad \sigma(\mathcal{E}_0 \oplus \mathcal{O}(e_j)) \subset \mathcal{E}_0 \oplus \mathcal{O}(e_j)(\forall j)\}$ act on $\widetilde{S}$, and there exists a quotient:

$$S = \widetilde{S}/\mathrm{PGL}_{n+1}(\mathbf{C}) \times Q.$$

Now we have $X \longrightarrow S$ the family of nonsingular complete intersections, and $Z_j \longrightarrow S$ the family of its nonsingular hypersurface sections defined by the equation G_j with a regular embedding $Z_j \hookrightarrow X$. Let $U := X \setminus \cup Z_j \xrightarrow{f} S$.

Corollary(1-2)(the symmetrizer lemma for open varieties). Let $S' \subset S$ a nonsingular Zariski closed subset of codimension c. Let T be any smooth covering of S'. We put $U_T = U \times_S T$ and denote $H^{a,b}(U_T/T) = R^b f_* \Omega^a_{X_T/T}(\log \Sigma Z_j)$. Then the sequence induced by the Gauss-Manin connection

$$\Omega^{q-1}_T \otimes H^{a+2,b-2}(U_T/T) \xrightarrow{\overline{\nabla}_{U_T}} \Omega^q_T \otimes H^{a+1,b-1}(U_T/T) \xrightarrow{\overline{\nabla}_{U_T}} \Omega^{q+1}_T \otimes H^{a,b}(U_T/T) \quad (a+b=n-r)$$

is exact at the middle term for one of the following cases.

(1) $r+s \leq n+2$, $a \geq 0$ and $d_{\min}(r+a) \geq q+c+n+1$.
(2) $0 \leq a \leq n-r-2$, $d_{\min}(r+a) \geq q+c+n+1$ and $a+q \leq n-r$.
(3) $q=0,1$, $a \geq 0$ and $d_{\min}(r+a) \geq q+c+n+1$.

Proof. When $T=S'$, it is well-known that the Gauss-Manin connection $\overline{\nabla}_U$ is induced by the cup-product

$$R^1 f_* T_{X/S}(\log \Sigma Z_j) \otimes R^a f_* \Omega^b_{X/S}(\log \Sigma Z_j) \longrightarrow R^{a+1} f_* \Omega^{b-1}_{X/S}(\log \Sigma Z_j).$$

Therefore by Theorem(I), it suffices to show that for $B \subset B_1(0)$ codimension c subspace,

$$\text{(1-2-1)} \qquad B_a(\ell_0) \otimes \overset{q+1}{\wedge} B \longrightarrow B_{a+1}(\ell_0) \otimes \overset{q}{\wedge} B \longrightarrow B_{a+2}(\ell_0) \otimes \overset{q-1}{\wedge} B \quad (\ell_0 := \mathbf{d} - n - 1)$$

is exact for $a \geq 0$. Let $W := \mathrm{Ker}(A_1(0) \to B_1(0)/B)$, $J := J(\underline{F}, \underline{G}) \cap A_1(0)$, then we have the Koszul exact sequence

$$0 \to S^{\cdot}(J) \to W \otimes S^{\cdot-1}(J) \to \cdots \to \overset{\cdot-1}{\wedge} W \otimes J \to \overset{\cdot}{\wedge} W \to \overset{\cdot}{\wedge} B \to 0.$$

This complex tensored with $B_*(\ell_0)$ induces the following diagram.

$$\begin{array}{ccccccccc}
\cdots & \to & B_a(\ell_0) \otimes \overset{q+1-i}{\wedge} W \otimes S^i(J) & \to & \cdots & \to & B_a(\ell_0) \otimes \overset{q+1}{\wedge} W & \to B_a(\ell_0) \otimes \overset{q+1}{\wedge} B & \to 0 \\
 & & \downarrow & & & & \downarrow & \downarrow & \\
\cdots & \to & B_{a+1}(\ell_0) \otimes \overset{q-i}{\wedge} W \otimes S^i(J) & \to & \cdots & \to & B_{a+1}(\ell_0) \otimes \overset{q}{\wedge} W & \to B_{a+1}(\ell_0) \otimes \overset{q}{\wedge} B & \to 0 \\
 & & \downarrow & & & & \downarrow & \downarrow & \\
\cdots & \to & B_{a+2}(\ell_0) \otimes \overset{q-1-i}{\wedge} W \otimes S^i(J) & \to & \cdots & \to & B_{a+2}(\ell_0) \otimes \overset{q-1}{\wedge} W & \to B_{a+2}(\ell_0) \otimes \overset{q-1}{\wedge} B & \to 0 \\
 & & \downarrow & & & & \downarrow & \downarrow & \\
 & & \vdots & & & & \vdots & \vdots &
\end{array}$$

where the vertical arrow is the Koszul complex tensored with $S^{\cdot}(J)$. Since J annihilates $B_a(l)$, this diagram is commutative. Therefore to show that (1-2-1) is exact, we only show that

$$B_{a+i}(\ell_0) \otimes \overset{q-i+1}{\wedge} W \longrightarrow B_{a+i+1}(\ell_0) \otimes \overset{q-i}{\wedge} W \longrightarrow B_{a+i+2}(\ell_0) \otimes \overset{q-i-1}{\wedge} W$$

is exact for $0 \leq i \leq q$. But this follows from the first half of Theorem (III).

In general, let $u : T \longrightarrow S'$ be a smooth covering. From the exact sequence

$$0 \longrightarrow u^* \Omega^1_{S'} \longrightarrow \Omega^1_T \longrightarrow \Omega^1_{T/S'} \longrightarrow 0,$$

we have a filtration $F^{\cdot}\Omega^q_T$ such that its subquotient is isomorphic to $u^*\Omega^c_{S'} \otimes \Omega^{q-c}_{T/S'}$. We put $F^{\cdot} H^{a,b}(U_T/T) \otimes \Omega^q_T := H^{a,b}(U_T/T) \otimes F^{\cdot}\Omega^q_T$. Since $H^{a,b}(U_T/T) \cong u^* H^{a,b}(U/S')$, and $\overline{\nabla}_{U_T} : H^{a,b}(U_T/T) \longrightarrow H^{a-1,b+1}(U_T/T) \otimes \Omega^1_T$ is the composition of $u^* H^{a,b}(U/S') \xrightarrow{u^*\overline{\nabla}_U} u^* H^{a-1,b+1}(U/S') \otimes u^*\Omega^1_{S'}$ and $u^* H^{a-1,b+1}(U/S') \otimes u^*\Omega^1_{S'} \xrightarrow{\mathrm{id}\otimes h} u^* H^{a-1,b+1}(U/S') \otimes \Omega^1_T$, where $h : u^*\Omega^1_{S'} \longrightarrow \Omega^1_T$ is the pull-back of Kähler differentials.

Therefore we can easily check that $\overline{\nabla}_{U_T}$ is compatible with the above filtration and its subquotient can be identified with

$$u^*H^{a,b}(U/S')\otimes u^*\Omega^c_{S'}\otimes\Omega^{q-c}_{T/S'}\xrightarrow{u^*\overline{\nabla}_U\otimes\mathrm{id}}u^*H^{a-1,b+1}(U/S')\otimes u^*\Omega^{c+1}_{S'}\otimes\Omega^{q-c}_{T/S'}.$$

Thus we can reduce to the case $T=S'$. □

Corollary(1-3)(the symmetrizer lemma for complete intersections). Let the notation be as in the previous theorem. We put $X_T:=X\times_{S'}T$. Then

$$\Omega^{q-1}_T\otimes H^{a+2,b-2}(X_T/T)_{\mathrm{prim}}\xrightarrow{\overline{\nabla}_{X_T}}\Omega^q_T\otimes H^{a+1,b-1}(X_T/T)_{\mathrm{prim}}\xrightarrow{\overline{\nabla}_{X_T}}\Omega^{q+1}_T\otimes H^{a,b}(X_T/T)_{\mathrm{prim}}$$

is exact if $a\geq 0$ $d_{\min}(r+a)\geq q+c+n+1$.

Proof. We use Corollary(1-2)(1) in the case $s=1$. Put $Z=Z_1$ and $Z_T:=Z\times_{S'}T$. Then we have

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
\cdots\to & H^{a+1,b-1}(X_T)_0\otimes\Omega^{q-1}_T & \xrightarrow{\overline{\nabla}_{X_T}} & H^{a,b}(X_T)_0\otimes\Omega^q_T & \xrightarrow{\overline{\nabla}_{X_T}} & H^{a-1,b+1}(X_T)_0\otimes\Omega^{q+1}_T \\
 & \downarrow & & \downarrow & & \downarrow \\
\cdots\to & H^{a+1,b-1}(U_T)\otimes\Omega^{q-1}_T & \xrightarrow{\overline{\nabla}_{U_T}} & H^{a,b}(U_T)\otimes\Omega^q_T & \xrightarrow{\overline{\nabla}_{U_T}} & H^{a-1,b+1}(U_T)\otimes\Omega^{q+1}_T \\
 & \downarrow & & \downarrow & & \downarrow \\
\cdots\to & H^{a,b-1}(Z_T)_0\otimes\Omega^{q-1}_T & \xrightarrow{\overline{\nabla}_{Z_T}} & H^{a-1,b}(Z_T)_0\otimes\Omega^q_T & \xrightarrow{\overline{\nabla}_{Z_T}} & H^{a-2,b+1}(Z_T)_0\otimes\Omega^{q+1}_T \\
 & \downarrow & & \downarrow & & \downarrow \\
 & 0 & & 0 & & 0 \\
 & \text{(exact)} & & \text{(exact)} & & \text{(exact)}
\end{array}$$

where $H^{\cdot}(\ \ )_0$ denotes the primitive part. Since the middle horizontal sequence is exact by Corollary(1-2), it suffices to show that

$$\Omega^{q-2}_T\otimes H^{a+1,b-2}(Z_T/T)_0\xrightarrow{\overline{\nabla}_{Z_T}}\Omega^{q-1}_T\otimes H^{a,b-1}(Z_T/T)_0\xrightarrow{\overline{\nabla}_{Z_T}}\Omega^q_T\otimes H^{a-1,b}(Z_T/T)_0$$

is exact. Therefore by the induction on the relative dimension of $X_T\longrightarrow T$, we can assume that it is 0. But then the assertion is clear. □

§2. Green's definition of Jacobian rings

So far, we introduced Jacobian rings in explicit forms. However, these forms are not so useful to variuos computations. For example, we don't know how to prove the theorem II(duality), theorem III(the symmetrizer lemma) directry from the computations of polynomial algebras.

Another (and probably excellent) definition of Jacobian rings was introduced by M.Green. This one is rather abstract, but, due to it, we can use the technique of Koszul cohomology. Thus available cohomology theory enables us sharp computation of Jacobian rings.

2.1. Before defining Green's Jacobian ring of open varieties, we review the one in complete case briefly([K] or [T]).

Let $\mathcal{E}=\bigoplus_{i=1}^{t}\mathcal{O}_{\mathbb{P}^n}(l_i)$ be a locally free sheaf of rank t on a complex projective space $\mathbb{P}^n$. We assume all the $l_i\geq 1$, and $1\leq t\leq n$. Let consider the projective space bundle

$$\pi:\mathbb{P}(\mathcal{E}):=\mathrm{Proj}(S^{\cdot}(\mathcal{E}))\longrightarrow\mathbb{P}^n$$

where $S^{\cdot}(\mathcal{E})=\bigoplus_{r\geq 0}\mathrm{Sym}^r\mathcal{E}$ denotes the sheaf of symmetric $\mathcal{O}_{\mathbb{P}^n}$-algebra generated by $\mathcal{E}$. We denote the tautological invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ by $\mathcal{L}$. Then we have the exact sequence of differential sheaves:

(2-1-1) $$0\to\pi^*\Omega^1_{\mathbb{P}^n}\to\Omega^1_{\mathbb{P}(\mathcal{E})}\to\Omega^1_{\mathbb{P}(\mathcal{E})/\mathbb{P}^n}\to 0$$

and the Euler exact sequence

(2-1-2) $$0 \to \Omega^1_{\mathbb{P}(\mathcal{E})/\mathbb{P}^n} \to \pi^*\mathcal{E} \otimes \mathcal{L}^{-1} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to 0.$$

In particular, we have

(2-1-3) $$K_{\mathbb{P}(\mathcal{E})} = \mathcal{L}^{-t} \otimes \pi^*\mathcal{O}(\sum_{i=1}^{t} l_i - n - 1)$$

Moreover, there are following well-known formulas, which will be used to show vanishing lemma, later.

(2-1-4) $$R^i\pi_*\mathcal{L}^\nu \simeq \begin{cases} S^\nu(\mathcal{E}) & \text{if } \nu \geq 0,\ i = 0 \\ \det\mathcal{E}^* \otimes S^{-\nu-t}(\mathcal{E}^*) & \text{if } \nu \leq -t,\ i = t-1 \\ 0 & \text{otherwise} \end{cases}$$

(2-1-5) $$H^q(\mathbb{P}, \mathcal{L}^\nu \otimes \pi^*\mathcal{V}) \simeq \begin{cases} H^q(\mathbb{P}^n, S^\nu(\mathcal{E}) \otimes \mathcal{V}) & \text{if } \nu \geq 0 \\ H^{q-t+1}(\mathbb{P}^n, S^{-\nu-t}(\mathcal{E}^*) \otimes \det\mathcal{E}^* \otimes \mathcal{V}) & \text{if } \nu \leq -t \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{V}$ is a holomorphic vector bundle on $\mathbb{P}^n$.

By (2-1-5), we have the natural isomorphism $H^0(\mathbb{P}(\mathcal{E}), \mathcal{L}) \simeq H^0(\mathbb{P}^n, \mathcal{E})$. Therefore the global sections of $\mathcal{E}$ and $\mathcal{L}$ have one-one correspondance to each other. More explicitely, a global section $(F_i) \in H^0(\mathbb{P}^n, \mathcal{E}) = \bigoplus_{i=1}^{t} H^0(\mathbb{P}^n, \mathcal{O}(l_i))$ corresponds to $\sigma = \sum_{i=1}^t F_i\mu_i \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{L})$, where μ_i is the global section of $\mathcal{L} \otimes \pi^*\mathcal{O}(-l_i)$ associated to the effective divisor $\mathbb{P}(\bigoplus_{j\neq i} \mathcal{O}(l_j)) \hookrightarrow \mathbb{P}(\mathcal{E})$ defined by the natural projection $\mathcal{E} \to \bigoplus_{j\neq i} \mathcal{O}(l_j)$.

Then for a section $\sigma = \sum_{i=1}^t F_i\mu_i \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{L})$, we put

$$X : F_1 = \cdots = F_t = 0 \text{ in } \mathbb{P}^n,$$

$$\mathcal{X} : \sum_{i=1}^{t} F_i\mu_i = 0 \text{ in } \mathbb{P}$$

and assume that X is a nonsingular complete intersection of codimension t (, which implies that $\mathcal{X}$ be a nonsingular divisor).

Now let $\Sigma_{\mathcal{L}} = \mathit{Diff}^{\leq 1}(\mathcal{L}, \mathcal{L})$ be the sheaf of first order differential operators of $\mathcal{L}$. It is a locally free sheaf generated by local sections $e\frac{\partial}{\partial e}$, $\frac{\partial}{\partial x_i}$, when e is a local frame of $\mathcal{L}$ and (x_i) is a local coordinate of $\mathbb{P}(\mathcal{E})$. Then we have two important exact sequence.

(2-1-6) $$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \Sigma_{\mathcal{L}} \to T_{\mathbb{P}(\mathcal{E})} \to 0$$

(2-1-7) $$0 \to T_{\mathbb{P}(\mathcal{E})}(-\log \mathcal{X}) \to \Sigma_{\mathcal{L}} \stackrel{j(\sigma)}{\to} \mathcal{L} \to 0$$

where $j(\sigma)$ is the composite of $\Sigma_{\mathcal{L}} \stackrel{1\otimes\sigma}{\to} \Sigma_{\mathcal{L}} \otimes \mathcal{L} \to \mathcal{L}$, and $T_{\mathbb{P}(\mathcal{E})}(-\log \mathcal{X})$ is the dual sheaf of $\Omega^1_{\mathbb{P}(\mathcal{E})}(\log \mathcal{X})$ the holomorphic differential sheaf with log poles along $\mathcal{X}$. The dual of (2-1-7) gives rise to the Koszul exact sequence

$$0 \to \Omega^p_{\mathbb{P}(\mathcal{E})}(\log \mathcal{X}) \to \mathcal{L} \otimes \overset{p+1}{\wedge} \Sigma^*_{\mathcal{L}} \to \cdots \to \mathcal{L}^{n+1-p} \otimes \overset{n+t}{\wedge} \Sigma^*_{\mathcal{L}} \to 0.$$

By (2-1-6), we have $\overset{n+t-1}{\wedge} \Sigma^*_{\mathcal{L}} = \Sigma_{\mathcal{L}} \otimes K_{\mathbb{P}(\mathcal{E})}$, $\overset{n+t}{\wedge} \Sigma^*_{\mathcal{L}} = K_{\mathbb{P}(\mathcal{E})}$ and it is easy to see that this gives an acyclic resolution of $\Omega^p_{\mathbb{P}(\mathcal{E})}(\log \mathcal{X})$, because we assumed all the $l_i \geq 1$. Therefore we have

$$H^{n+t-p-1}(\mathbb{P}(\mathcal{E}), \Omega^p_{\mathbb{P}(\mathcal{E})}(\log \mathcal{X})) \simeq \mathrm{Coker}(H^0(\mathcal{L}^{n+t-p-1} \otimes \Sigma_{\mathcal{L}} \otimes K_{\mathbb{P}(\mathcal{E})}) \to H^0(\mathcal{L}^{n+t-p} \otimes K_{\mathbb{P}(\mathcal{E})})).$$

The right hand side, we call the *Jacobian ring* of X. The left hand side is isomorphic to

$$F^p/F^{p+1}H^{n+t-1}(\mathbb{P}(\mathcal{E}) \setminus \mathcal{X}, \mathbf{C}) \simeq F^p/F^{p+1}H^{n+t-1}(\mathbb{P}^n \setminus X, \mathbf{C}) \simeq F^{p-t}/F^{p-t+1}H^{n-t}(X, \mathbf{C}),$$

where the first isomorphism follows from that $\mathbb{P}(\mathcal{E}) \setminus \mathcal{X} \xrightarrow{\pi} \mathbb{P}^n \setminus X$ is an affine space bundle. For further argument, see [K].

2.2. Now we define the Green's Jacobian rings in open case. The method is almost along the complete case.

Firstly we prepare the notations, which will be used in whole sections. Let $\mathcal{E}_0 = \bigoplus_{i=1}^{r} \mathcal{O}(d_i)$ and $\mathcal{E}_1 = \bigoplus_{j=1}^{s} \mathcal{O}(e_j)$ the locally free sheaves on $\mathbb{P}^n$. Put $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$, and consider the projective space bundle

$$\pi : \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}^n.$$

Hereafter we write $\mathbb{P}(\mathcal{E})$ by $\mathbb{P}$ simply, and denote the tautological invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ by $\mathcal{L}$. We fix the global section μ_i (resp. λ_j) of $\mathcal{L} \otimes \pi^*\mathcal{O}(-d_i)$ (resp. $\mathcal{L} \otimes \pi^*\mathcal{O}(-e_j)$) associated to the effective divisor $\mathbb{P}(\bigoplus_{k\neq i} \mathcal{O}(d_k) \oplus \mathcal{E}_1) \hookrightarrow \mathbb{P}(\mathcal{E})$ (resp. $\mathbb{P}(\mathcal{E}_0 \oplus \bigoplus_{k\neq j} \mathcal{O}(e_k)) \hookrightarrow \mathbb{P}(\mathcal{E})$) defined by natural projections.

Further we fix a global section $\sigma = \sum_{i=1}^{r} F_i\mu_i + \sum_{j=1}^{s} G_j\lambda_j \in H^0(\mathbb{P}, \mathcal{L})$, and put

$$X : F_1 = \cdots = F_r = 0 \text{ in } \mathbb{P}^n,$$

$$Z_j : F_1 = \cdots = F_r = G_j = 0 \text{ in } \mathbb{P}^n \quad 1 \leq j \leq s,$$

$$\mathbb{P}_j : \lambda_j = 0 \text{ in } \mathbb{P},$$

$$\mathcal{Z} : \sum_{i=1}^{r} F_i\mu_i + \sum_{j=1}^{s} G_j\lambda_j = 0 \text{ in } \mathbb{P}.$$

We assume Z is a nonsingular complete intersection of codimension r and $\bigcup_{j=1}^{s} Z_j$ is a normal crossing divisor in Z. This assumption implies that $\mathcal{Z}$ is a nonsingular divisor in $\mathbb{P}$.

In order to define the Jacobian ring in open case, we only need to replace $\Sigma_{\mathcal{L}}$ by $\Sigma_{\mathcal{L}}(-\log \sum_{j=1}^{s} \mathbb{P}_j)$, which is defined to be a subsheaf of $\Sigma_{\mathcal{L}}$ generated by local sections $e\frac{\partial}{\partial e}$, $\frac{\partial}{\partial x_i}$ $(1 \leq i \leq n+r-1)$, $y_j\frac{\partial}{\partial y_j}$ $(1 \leq j \leq s)$, where e is a local frame of $\mathcal{L}$ and (x_i, y_j) is a local coordinate of $\mathbb{P}(\mathcal{E})$ such that y_j is a local equation of $\mathbb{P}_j$.

Then the exact sequences (2-1-6) and (2-1-7) can be replaced by the followings:

(2-2-1)
$$0 \to \mathcal{O} \to \Sigma_{\mathcal{L}}(-\log \sum_{j=1}^{s} \mathbb{P}_j) \to T_{\mathbb{P}}(-\log \sum_{j=1}^{s} \mathbb{P}_j) \to 0$$

(2-2-2)
$$0 \to T_{\mathbb{P}}(-\log \mathcal{Z} + \sum_{j=1}^{s} \mathbb{P}_j) \to \Sigma_{\mathcal{L}}(-\log \sum_{j=1}^{s} \mathbb{P}_j) \stackrel{j(\sigma)}{\to} \mathcal{L} \to 0$$

From now on, we write $\Sigma_{\mathcal{L}}(-\log \sum_{j=1}^{s} \mathbb{P}_j)$ by Σ simply.

Definition(**2-2-3**). Let q and ℓ be integers. Then we put

$$S_q(\ell) = H^0(\mathcal{L}^q \otimes \pi^*\mathcal{O}(\ell))$$

$$J_q(\ell) = \mathrm{Image}(H^0(\Sigma \otimes \mathcal{L}^{q-1} \otimes \pi^*\mathcal{O}(\ell)) \stackrel{j(\sigma)}{\to} S_q(\ell))$$

$$R_q(\ell) = S_q(\ell)/J_q(\ell).$$

We call $R = \oplus R_q(\ell)$ the **Green's Jacobian ring** of $(X, \bigcup_{j=1}^{s} Z_j)$.

Lemma(2-2-4). Green's Jacobian ring coincides with the one in Definition (1-1):

$$A_q(\ell) = S_q(\ell), \qquad R_q(\ell) = B_q(\ell).$$

□

We will have an analoguos arguement in [K], in the next section.
Before to do this, we show the vanishing lemma.

Proposition(2-2-5)(vanishing lemma). Let p, q, ν, ℓ be integers. Then

$$H^q(\mathbb{P}, \overset{p}{\wedge}\Sigma^* \otimes \mathcal{L}^\nu \otimes \pi^*\mathcal{O}(\ell)) = 0$$

if one of the following conditions is satisfied.

(1) $q > 0$, $\nu \geq -s+1$, $\ell \geq 0$ and $(\nu, \ell) \neq (0,0)$
(2) $q < n$, $\nu = 0$, $\ell < 0$
(3) $p - \nu \leq r + s - 1$, $\nu \leq -1$
(4) $p - \nu \leq q < n + r + s - 1$, $\nu \leq -1$
(5) $0 < q \neq n$, $\nu = \ell = 0$
(6) $q > 0$, $p \notin [n+1, s+n]$, $\nu = \ell = 0$

(1)* $q < n + r + s - 1$, $\nu \leq -1$, $\ell \leq \mathbf{e}$ and $(\nu, \ell) \neq (-s, \mathbf{e})$
(2)* $q > r + s - 1$, $\nu = -s$, $\ell > \mathbf{e}$
(3)* $p - \nu \geq n + s + 1$, $\nu \geq -s+1$
(4)* $0 < q < p - \nu - s$, $\nu \geq -s+1$
(5)* $q < n + r + s - 1$, $q \neq r + s - 1$, $\nu = -s$, $\ell = \mathbf{e}$
(6)* $q < n + r + s - 1$, $p \notin [r, r+s-1]$, $\nu = -s$, $\ell = \mathbf{e}$

Proof. We note that

$$0 \to \pi^*\Omega^1_{\mathbb{P}^n} \to \Omega^1_{\mathbb{P}}(\log \Sigma\mathbb{P}_j) \to \Omega^1_{\mathbb{P}/\mathbb{P}^n}(\log \Sigma\mathbb{P}_j) \to 0 \tag{2-2-5-1}$$

and

$$\Omega^1_{\mathbb{P}/\mathbb{P}^n}(\log \Sigma\mathbb{P}_j) \simeq (\pi^*\mathcal{E}_0 \otimes \mathcal{L}^{-1}) \oplus \mathcal{O}_{\mathbb{P}}^{\oplus s-1} \tag{2-2-5-2}$$

In particular, we have $\overset{n+r+s}{\wedge}\Sigma^* =$ det $\Omega^1_{\mathbb{P}}(\log \Sigma\mathbb{P}_j) = \mathcal{L}^{-r} \otimes \pi^*\mathcal{O}(\mathbf{d}-n-1)$. The assertion (n) $(1 \leq n \leq 6)$ is equivalent to $(n)^*$ by Serre duality:

$$H^q(\overset{p}{\wedge}\Sigma^* \otimes \mathcal{L}^\nu \otimes \pi^*\mathcal{O}(\ell))^* \simeq H^{n+r+s-1-q}(\overset{n+r+s-p}{\wedge}\Sigma^* \otimes \mathcal{L}^{-\nu-s} \otimes \pi^*\mathcal{O}(\mathbf{e}-\ell)).$$

Therefore we only need to show the (1),$\cdots$,(6).

By (2-2-1), we have an exact sequence

$$0 \to \Omega^p_{\mathbb{P}}(\log \Sigma\mathbb{P}_j) \to \overset{p}{\wedge}\Sigma^* \to \Omega^{p-1}_{\mathbb{P}}(\log \Sigma\mathbb{P}_j) \to 0. \tag{2-2-5-3}$$

Moreover by (2-2-5-1), we have a finite decreasing filtration $F^\cdot$ of $\Omega^p_{\mathbb{P}}(\log \Sigma\mathbb{P}_j)$ such that

$$\mathrm{Gr}^\mu_F(\Omega^p_{\mathbb{P}}(\log \Sigma\mathbb{P}_j)) = \pi^*\Omega^\mu_{\mathbb{P}^n} \otimes \Omega^{p-\mu}_{\mathbb{P}/\mathbb{P}^n}(\log \Sigma\mathbb{P}_j). \tag{2-2-5-4}$$

Therefore in order to show (1),$\cdots$,(4), it suffices to show the following claim.

Claim. If (q,p,ν,ℓ) satisfies one of the conditions $(1), \cdots, (4)$, then

$$H^q(\mathbb{P}, \Omega^{p'-\mu}_{\mathbb{P}/\mathbb{P}^n}(\log \Sigma\mathbb{P}_j) \otimes \mathcal{L}^\nu \otimes \pi^*\Omega^\mu_{\mathbb{P}^n}(\ell)) = 0.$$

for all p', μ such that $0 \leq \mu \leq p'$, and $p-1 \leq p' \leq p$.

By (2-2-5-2), we have

$$\Omega^a_{\mathbb{P}/\mathbb{P}^n}(\log \Sigma\mathbb{P}_j) \simeq \bigoplus_{i=0}^{a} [\overset{i}{\wedge}\pi^*\mathcal{E}_0 \otimes \mathcal{L}^{-i}]^{\binom{s-1}{a-i}}$$

Therefore the claim follows from

$$H^q(\mathbb{P}, \mathcal{L}^{\nu-i} \otimes \pi^*(\Omega^\mu_{\mathbb{P}^n}(\ell) \otimes \overset{i}{\wedge}\mathcal{E}_0)) = 0 \tag{2-2-5-5}$$

$$\text{for } \forall \mu, p', i \text{ such that } \begin{cases} 0 \le \mu \le p' \\ 0 \le i \le r \\ 0 \le i \le p' - \mu \\ p-1 \le p' \le p \end{cases}$$

Firstly we show (1) and (2). Since we assumed $\nu \ge -s+1$, by (2-1-5), we can assume $\nu \ge i$. Then the above cohomology group is isomorphic to

$$H^q(\mathbb{P}^n, S^{\nu-i}(\mathcal{E}) \otimes \Omega^{\mu}_{\mathbb{P}^n}(\ell) \otimes \overset{i}{\wedge}\mathcal{E}_0).$$

If (q,ν,ℓ) satisfies (1), then $\nu - i > 0$ or $i > 0$ or $\ell > 0$, which implies the vanishing of the above one by Bott vanishing.

If (q,ν,ℓ) satisfies (2), then $\nu = i = 0$ and $q < n$, $\ell < 0$, which also implies the vanishing.

Next we will show (3) and (4).

If (q,p,ν,ℓ) satisfies (3), then $\nu - i \le \nu \le -1$ and $\nu - i \ge \nu - (p'-\mu) \ge \nu - p' \ge \nu - p \ge -r-s+1$. Therefore (2-2-5-5) holds by (2-1-5).

If (q,p,ν,ℓ) satisfies (4), then $\nu - i \le \nu \le -1$. Therefore we can assume $\nu - i \le -r-s$ by (2-1-5). Then the left hand side of is isomorphic to

$$H^{q-r-s+1}(\mathbb{P}^n, S^{-\nu+i-r-s}(\mathcal{E}^*) \otimes \Omega^{\mu}_{\mathbb{P}^n} \otimes \overset{i}{\wedge}\mathcal{E}_0(\ell - \mathbf{d} - \mathbf{e})).$$

But since $\mu \le p' - i \le p - i \le p - \nu - r - s$, by Bott vanishing, the above cohomology group vanishes if $p - \nu - r - s < q - r - s + 1 < n$, that is, $p - \nu \le q < n + r + s - 1$.

This complete the proof of (2-2-5-5), and hence (1),$\cdots$,(4).

Finally we show (5) and (6). We assume $\nu = \ell = 0$. Then by (2-2-5-3), $H^q(\overset{p}{\wedge}\Sigma^*) = 0$ if and only if

(i) $H^{q-1}(\Omega^{p-1}_{\mathbb{P}}(\log \Sigma\mathbb{P}_j)) \xrightarrow{\cup c_1(\mathcal{L})} H^q(\Omega^p_{\mathbb{P}}(\log \Sigma\mathbb{P}_j))$ is surjective, and

(ii) $H^q(\Omega^{p-1}_{\mathbb{P}}(\log \Sigma\mathbb{P}_j)) \xrightarrow{\cup c_1(\mathcal{L})} H^{q+1}(\Omega^p_{\mathbb{P}}(\log \Sigma\mathbb{P}_j))$ is injective.

Let $u_i := d\log\lambda_{i+1} - d\log\lambda_i \in H^1(\mathbb{P} \setminus \cup\mathbb{P}_j, \mathbf{Z}(1))$. Then we can easily see that

- Coker$(H^{p+q-2}(\mathbb{P}\setminus\cup\mathbb{P}_j) \xrightarrow{\cup c_1(\mathcal{L})} H^{p+q}(\mathbb{P}\setminus\cup\mathbb{P}_j)$ is canonically isomorphic to $\overset{p+q}{\wedge}(u_1, \cdots, u_{s-1})$, which is pure Hodge type $(p+q, p+q)$,
- Ker $(H^{p+q-1}(\mathbb{P}\setminus\cup\mathbb{P}_j) \xrightarrow{\cup c_1(\mathcal{L})} H^{p+q+1}(\mathbb{P}\setminus\cup\mathbb{P}_j)$ is canonically isomorphic to $\overset{n}{\wedge}\pi^*\omega \otimes \overset{p+q-2n-1}{\wedge}(u_1, \cdots, u_{s-1})$ where ω is a hyperplane class of $\mathbb{P}^n$, which is pure Hodge type $(p+q-n-1, p+q-n-1)$.

Therefore (i) holds if and only if $q > 0$ or $p+q \ge s$, and (ii) does not hold if and only if $q = n$ and $0 \le p+q-2n-1 \le s-1$, or equivalently $q = n$ and $p \in [n+1, s+n]$, which complete the proof of (5) and (6). □

§3 Proof of Main Theorems

In this section, we will prove the main theorems. Due to Lemma(2-2), we may use only Green's Jacobian ring $R_q(\ell)$.

The proof of Theorem(I) and (II) are similar to the one in [K].

The proof of Theorem(III) depends essentially on the results of M.Green , and we will also use similar techniques to prove it.

3.1. By (2-2-2), we have the Koszul exact sequence

$$0 \to \Omega^p_{\mathbb{P}}(\log \mathcal{Z} + \Sigma\mathbb{P}_j) \to \mathcal{L} \otimes \overset{p+1}{\wedge}\Sigma^* \to \cdots \to \mathcal{L}^{m+1-p} \otimes \overset{m+1}{\wedge}\Sigma^* \to 0 \tag{3-1-1}$$

Since the vanishing lemma(1), this is an acyclic resolution of $\Omega^p_{\mathbb{P}}(\log \mathcal{Z} + \Sigma\mathbb{P}_j)$. Together with $\overset{m+1}{\wedge}\Sigma^* \simeq \mathcal{L}^{-r} \otimes \pi^*\mathcal{O}(\mathbf{d} - n - 1)$, and $\overset{m}{\wedge}\Sigma^* \simeq \Sigma \otimes \mathcal{L}^{-r} \otimes \pi^*\mathcal{O}(\mathbf{d} - n - 1)$, we have

$$R_{n+s-p}(\mathbf{d} - n - 1) \simeq H^{m-p}(\Omega^p_{\mathbb{P}}(\log \mathcal{Z} + \Sigma\mathbb{P}_j)). \tag{3-1-2}$$

Now we need:

Lemma(3-1-3). There is a natural morphism of Hodge structures:

$$\rho : H^{n-r}(X \setminus \cup Z_j) \longrightarrow H_m(\mathbb{P} \setminus (\mathcal{Z} \cup \bigcup \mathbb{P}_j))(-n-s).$$

ρ is an isomorphism if $r+s \leq n$. If $r+s > n$, then ρ is surjective but not injective and the Kerρ is pure Hodge type $(n-r, n-r)$, described as follows:

$$\mathrm{Ker}\rho = \mathrm{Ker}(\overset{n-r}{\wedge}(e_1, \cdots, e_s) \xrightarrow{d} \overset{n-r-1}{\wedge}(e_1, \cdots, e_s))$$

where the notations are as in the Theorem(I)(1).

Proof. Let denote $\mathbb{P}_{j_1 \cdots j_l} = \mathbb{P}_{j_1} \cap \cdots \cap \mathbb{P}_{j_l}$, and $\mathcal{Z}_{j_1 \cdots j_l} = \mathcal{Z} \bigcap \mathbb{P}_{j_1 \cdots j_l}$. Put $Y = \mathbb{P} \setminus \mathcal{Z}$, $Y_j = \mathbb{P}_j \setminus \mathcal{Z}_j$. We consider the following exact sequences:

$$\text{(3-1-3-1)} \qquad 0 \longrightarrow j_! \mathbf{Q}_{Y \setminus \cup Y_j} \longrightarrow \mathbf{Q}_Y \longrightarrow \mathbf{Q}_{\cup Y_j} \longrightarrow 0$$

$$\text{(3-1-3-2)} \qquad 0 \to \mathbf{Q}_{\cup Y_j} \to \underset{j}{\oplus} \mathbf{Q}_{Y_j} \to \underset{j<k}{\oplus} \mathbf{Q}_{Y_j \cap Y_k} \to \cdots \to \mathbf{Q}_{Y_1 \cap \cdots \cap Y_s} \to 0,$$

where we denote the constant sheaf on X with a coefficient gruop $\mathbf{Q}$ by $\mathbf{Q}_X$, and $j : Y \setminus \cup Y_j \hookrightarrow Y$ the open immersion. We will compute $H^{\cdot}(Y, j_! \mathbf{Q}_{Y \setminus \cup Y_j}) \simeq H^{\cdot}(Y, \cup Y_j; \mathbf{Q})$. Firstly we compute the sheaf $\mathbf{Q}_Y$.

Let denote the singular complex of X by $S^{\cdot}(X)$. Since π induces the fiber bundle $\mathbb{P} \setminus \mathcal{Z} \to \mathbb{P}^n \setminus Z_{1 \cdots s}$ with fibers isomorphic to affine spaces, we have a quasi-isomorphism $S^{\cdot}(Y) = S^{\cdot}(\mathbb{P} \setminus \mathcal{Z}) \simeq S^{\cdot}(\mathbb{P} \setminus Z_{1 \cdots s})$. On the other hand, the Poincaré-Lefschetz duality theorem asserts that $S^{\cdot}(\mathbb{P} \setminus Z_{1 \cdots s})$ is quasi-isomorphic to $S_{2n-\cdot}(\mathbb{P}^n)/S_{2n-\cdot}(Z_{1 \cdots s})$. Therefore we have $\mathbf{Q}_Y \simeq S^{\cdot}(Y) \simeq S_{2n-\cdot}(\mathbb{P}^n)/S_{2n-\cdot}(Z_{1 \cdots s})$. Note that $Y_{j_1} \cap \cdots \cap Y_{j_l} = \mathbb{P}_{j_1 \cdots j_l} \setminus \mathcal{Z}_{j_1 \cdots j_l}$. Thus we have similar results $\mathbf{Q}_{Y_j} \simeq S_{2n-\cdot}(\mathbb{P}^n)/S_{2n-\cdot}(Z_{1 \cdots \check{j} \cdots s})$, $\mathbf{Q}_{Y_j \cap Y_k} \simeq S_{2n-\cdot}(\mathbb{P}^n)/S_{2n-\cdot}(Z_{1 \cdots \check{j} \cdots \check{k} \cdots s})$ and so on. Now from (3-1-3-1) and (3-1-3-2), we can see $j_! \mathbf{Q}_{Y \setminus \cup Y_j}$ is quasi-isomorphic to the following complex:

$$\mathbf{Q}_Y \to \underset{j}{\oplus} \mathbf{Q}_{Y_j} \to \underset{j<k}{\oplus} \mathbf{Q}_{Y_j \cap Y_k} \to \cdots \to \mathbf{Q}_{Y_1 \cap \cdots \cap Y_s}.$$

By the above computation, this is quasi-isomorphic to

$$\text{(3-1-3-3)} \qquad S_{2n-\cdot}(\mathbb{P}^n)/S_{2n-\cdot}(Z_{1 \cdots s}) \to \underset{j}{\oplus} S_{2n-\cdot}(\mathbb{P}^n)/S_{2n-\cdot}(Z_{1 \cdots \check{j} \cdots s}) \to \cdots \to S_{2n-\cdot}(\mathbb{P}^n)/S_{2n-\cdot}(X).$$

From the following exact sequences

$$0 \to S_{2n-\cdot}(\mathbb{P}^n) \to \cdots \to \underset{j}{\oplus} S_{2n-\cdot}(\mathbb{P}^n) \to S_{2n-\cdot}(\mathbb{P}^n) \to 0$$

$$0 \to S_{2n-\cdot}(Z_{1 \cdots s}) \to \cdots \to \underset{j}{\oplus} S_{2n-\cdot}(Z_j) \to S_{2n-\cdot}(\cup Z_j) \to 0,$$

we can see that (3-1-3-3) is quasi-isomorphic to $S_{2n-\cdot}(X)/S_{2n-\cdot}(\cup Z_j)[-s]$, and therefore we have $j_! \mathbf{Q}_{Y \setminus \cup Y_j} \simeq S_{2n-\cdot}(X)/S_{2n-\cdot}(\cup Z_j)[-s]$. In particular, we have isomorphisms $H^{\cdot}(Y, \cup Y_j; \mathbf{Q}) \simeq H_{2n+s-1-\cdot}(X, \cup Z_j; \mathbf{Q}) \simeq H^{\cdot-2r-s+1}(X \setminus \cup Z_j; \mathbf{Q})$ (the last isomorphism is the Poincaré-Lefschetz duality). Therefore we have the following morphism of Hodge structures:

$$\text{(3-1-3-4)} \qquad \rho : H^{\cdot-2r-s+1}(X \setminus \underset{j}{\cup} Z_j) \simeq H^{\cdot}(\mathbb{P} \setminus \mathcal{Z}, \underset{j}{\cup} \mathbb{P}_j \setminus \mathcal{Z}_j) \longrightarrow H_{2m-\cdot}(\mathbb{P} \setminus (\mathcal{Z} \cup (\underset{j}{\cup} \mathbb{P}_j)))(-n-s).$$

We note that

$$H^t(Y_{j_1 \cdots j_q}) \xleftarrow{\simeq} H^t(\mathbb{P}^n \setminus Z_J) \simeq \begin{cases} H^{n-r+q}(Z_J)_{\mathrm{prim}} & \text{if } t = n+r+s-q-1 \\ H^t(\mathbb{P}^n) & \text{if } 0 \leq t \leq 2(r+s-q)-1 \\ 0 & \text{otherwise} \end{cases}$$

where we put $Z_J := Z_{1\cdots\check{j_1}\cdots\check{j_q}\cdots s}$. Moreover the spectral sequence

$$E_2^{pq} = \bigoplus_{j_1<\cdots<j_q} H^p(Y_{j_1\cdots j_q}) \Longrightarrow H^{p+q}(Y \setminus \bigcup_j Y_j) = H^{p+q}(\mathbb{P} \setminus (\mathcal{Z} \cup (\bigcup_j \mathbb{P}_j)))$$

is degenerate at E_2-term. Therefore if $r+s \leq n$ the each graded weight quotient of both sides of (3-1-3-4) is isomorphic to $\oplus H^{n-r+q}(Z_J)_{\text{prim}}$ (when $\cdot = n+r+s-1$). If $r+s > n$, then ρ is surjective and not injective only in the graded quotient of weght $(n-r, n-r)$, whose kernel can be described explicitly as in the Lemma(3-1-4). □

Now Theorem(I)(1) follows from (3-1-2) and the above Lemma. Next we will show Theorem(I)(2).

Lemma(3-1-4). There is a natural map

$$\psi : R_1(0) \longrightarrow H^1(X, T_X(-\log \Sigma Z_j))$$

which is an isomorphism, except for X is a $K3$-surafce or a curve. If X is a $K3$-suface or a curve of genus ≥ 2, ψ is only injective.

Proof. By the dual of (2-2-5-1), there is the exact sequence:

$$0 \to T_{\mathbb{P}/\mathbb{P}^n}(-\log \Sigma \mathbb{P}_j) \to T_{\mathbb{P}}(-\log \Sigma \mathbb{P}_j) \to \pi^* T_{\mathbb{P}^n} \to 0.$$

Together with this sequence and (2-2-1), we have the following commutative diagram.

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & T_{\mathbb{P}/\mathbb{P}^n}(-\log \Sigma \mathbb{P}_j) & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}} & \longrightarrow & \Sigma & \longrightarrow & T_{\mathbb{P}}(-\log \Sigma \mathbb{P}_j) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}a & \longrightarrow & \Sigma & \xrightarrow{a} & \pi^* T_{\mathbb{P}^n} & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

By the snake lemma, $\mathrm{Ker}a \simeq \mathcal{O}_{\mathbb{P}} \oplus T_{\mathbb{P}/\mathbb{P}^n}(-\log \Sigma \mathbb{P}_j) \simeq \pi^* \mathcal{E}_0^* \otimes \mathcal{L} \oplus \mathcal{O}_{\mathbb{P}}^s$. Therefore, by applying π_* to the lower exact sequence in the above diagram, we have

(3-1-4-1) $$0 \to (\mathcal{E}_0^* \otimes \mathcal{E}) \oplus \mathcal{O}_{\mathbb{P}^n}^s \to \pi_* \Sigma \to T_{\mathbb{P}^n} \to 0.$$

Now the section $(F_i, G_j) \in H^0(\mathcal{E}_0) \oplus H^0(\mathcal{E}_1)$ defines the surjective map

$$j_1 : \mathcal{E}_0^* \otimes \mathcal{E} \longrightarrow I_X \otimes \mathcal{E}, \quad \xi_i^* \otimes \cdot \longmapsto f_i \otimes \cdot$$

(I_X denotes the ideal sheaf of X), and $(G_j) \in H^0(\mathcal{E}_1)$ defines the map

$$j_2 : \mathcal{O}_{\mathbb{P}^n}^s \longrightarrow \mathcal{E}, \quad e_j = (0, \cdots, 1, \cdots, 0) \longmapsto g_j \eta_j \quad (1 \leq j \leq s-1)$$
$$e_s \longmapsto \sum_{i=1}^r f_i \xi_i + \sum_{j=1}^s g_j \eta_j$$

where ξ_i (resp. η_j) is a local frame of $\mathcal{E}_0$ (resp. $\mathcal{E}_1$), and $\Sigma f_i \xi_i + \Sigma g_j \eta_j$ is a local description of σ. Put

$$I = \mathrm{Im}(j_1 + j_2 : (\mathcal{E}_0^* \otimes \mathcal{E}) \oplus \mathcal{O}_{\mathbb{P}^n}^s \longrightarrow \mathcal{E}),$$

which is generated by local sections

$$f_i \xi_{i'}, \quad f_i \eta_j, \quad g_j \eta_j \qquad (1 \leq i, i' \leq r, \quad 1 \leq j \leq s).$$

Then, together with (3-1-4-1), we have

$$
\text{(3-1-4-2)}\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & K & \longrightarrow & T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}_0^* \otimes \mathcal{E} \oplus \mathcal{O}_{\mathbb{P}}^s & \longrightarrow & \pi_*\Sigma & \longrightarrow & T_{\mathbb{P}^n} & \longrightarrow & 0 \\
 & & \downarrow j_1 + j_2 & & \downarrow \pi_* j(\sigma) & & \downarrow j_3 & & \\
0 & \longrightarrow & I & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}/I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Here $j_1 + j_2$ induces the following commutative diagram:

$$
\text{(3-1-4-3)}\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_1 & \longrightarrow & L & \longrightarrow & L_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}_0^* \otimes \mathcal{E} & \longrightarrow & \mathcal{E}_0^* \otimes \mathcal{E} \oplus \mathcal{O}_{\mathbb{P}^n}^s & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}^s & \longrightarrow & 0 \\
 & & \downarrow j_1 & & \downarrow j_1 + j_2 & & \downarrow j_2' & & \\
0 & \longrightarrow & I_X \otimes \mathcal{E} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E} \otimes \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

where $j_2' : e_k \longmapsto g_k \eta_k \mod I_X \ (1 \le k \le s)$. Therefore we have

$$L_2 = \operatorname{Ker} j_2' = I_X^{\oplus s}.$$

From the Koszul exact sequence

$$
\text{(3-1-4-4)}\qquad 0 \longrightarrow \overset{r}{\wedge}\mathcal{E}_0^* \longrightarrow \cdots \longrightarrow \overset{2}{\wedge}\mathcal{E}_0^* \longrightarrow \mathcal{E}_0^* \longrightarrow I_X \longrightarrow 0,
$$

we can see that L_i has the following resolution.

$$
\text{(3-1-4-5)}\qquad
\begin{array}{l}
0 \longrightarrow (\overset{r}{\wedge}\mathcal{E}_0^*)^{\oplus s} \longrightarrow \cdots \longrightarrow \mathcal{E}_0^{*\oplus s} \longrightarrow L_2 \longrightarrow 0, \\
0 \longrightarrow \overset{r}{\wedge}\mathcal{E}_0^* \otimes \mathcal{E} \longrightarrow \cdots \longrightarrow \overset{2}{\wedge}\mathcal{E}_0^* \otimes \mathcal{E} \longrightarrow L_1 \longrightarrow 0.
\end{array}
$$

There is the exact sequence

$$
\text{(3-1-4-6)}\qquad 0 \longrightarrow I_X \otimes T_{\mathbb{P}^n} \longrightarrow T \longrightarrow T_X(-\log \Sigma Z_j) \longrightarrow 0.
$$

On the other hand, j_3(in the diagram (3-1-4-2)) can be written as follows:

$$
j_3 : \frac{\partial}{\partial x} \longmapsto \sum_{i=1}^{r} \frac{\partial f_i}{\partial x} + \sum_{j=1}^{s} \frac{\partial g_j}{\partial x} \mod I
$$

When we take a local coordinate $x_1, \cdots, x_r, y_1, \cdots, y_s, z_1, \cdots, z_{n-r-s}$ such that $f_i = x_i$, $g_j = y_j$. Then $T = \operatorname{Ker} j_3$ is generated by $I_X \otimes T_{\mathbb{P}^n}$ and local sections $y_j \frac{\partial}{\partial y_j}$, $\frac{\partial}{\partial z_k}$. Therefore the natural map $T/(I_X \otimes T_{\mathbb{P}^n}) \to T_{\mathbb{P}^n}|_X$ factors through $T/(I_X \otimes T_{\mathbb{P}^n}) \to T_X$, and which induces the isomorphism $T/(I_X \otimes T_{\mathbb{P}^n}) \to T_X(-\log \Sigma Z_j)$. This implies the following exact sequence

$$
\text{(3-1-4-6)}\qquad 0 \longrightarrow I_X \otimes T_{\mathbb{P}^n} \longrightarrow T \longrightarrow T_X(-\log \Sigma Z_j) \longrightarrow 0.
$$

Now we complete the proof of Lemma(3-1-4).

$$\begin{aligned} R_{1,0} &:= \operatorname{Coker}(H^0(\mathbb{P},\Sigma) \longrightarrow H^0(\mathbb{P},\mathcal{L})) \\ &\simeq \operatorname{Coker}(H^0(\mathbb{P}^n,\pi_*\Sigma) \longrightarrow H^0(\mathbb{P}^n,\mathcal{E})) \\ &\xrightarrow{a} H^1(\mathbb{P}^n,K) \text{ (from the middle vertical sequence in (3-1-4-2))} \\ &\xrightarrow{b} H^1(\mathbb{P}^n,T) \text{ (from the top horizontal sequence in (3-1-4-2))} \\ &\xrightarrow{c} H^1(X,T_X(-\log \Sigma Z_j)) \text{ (from (3-1-4-6))} \end{aligned}$$

a is isomorphism because $H^1(\pi_*\Sigma)=0$, which follows from (3-1-4-1).
b is injective if $H^1(L)=0$ and surjective $H^2(L)=0$. By (3-1-4-3)(the top horizontal exact sequence) and (3-1-4-5), the former always holds and the latter holds if $\dim X = n-r>1$
c is injective if $H^1(I_X\otimes T_{\mathbb{P}^n})=0$ and is surjective if $H^2(I_X\otimes T_{\mathbb{P}^n})=0$. By the exact sequence (3-1-4-4), we can see that the former holds unless X is a elliptic curve, and the latter holds unless X is a K3-surface or a curve. □

The first half of Theorem(I)(2) follows from Lemma(3-1-4). The compatibility of cup-products and ring multiplications follows from the compatibility of φ and ψ.
We have complete the proof of Theorem(I).

3.2. We will prove the Theorem(II), the duality theorem. We deduce it to the Serre duality theorem.
By (2-2-2), we have the Koszul exact sequence

(3-2-1) $$0\to\mathcal{L}^{-m-1}\to\Sigma^*\otimes\mathcal{L}^{-m}\to\cdots\to\overset{m+1}{\wedge}\Sigma^*\to 0.$$

Tensoring with $\mathcal{L}^{r+q}\otimes\pi^*\mathcal{O}(\ell)$, we have

$$0\to\mathcal{L}^{-m+r+q-1}\otimes\pi^*\mathcal{O}(\ell)\to\cdots\to\overset{m}{\wedge}\Sigma^*\otimes\mathcal{L}^{r+q-1}\otimes\pi^*\mathcal{O}(\ell)\xrightarrow{d}\overset{m+1}{\wedge}\Sigma^*\otimes\mathcal{L}^{r+q}\otimes\pi^*\mathcal{O}(\ell)\to 0.$$

Note that $\overset{m}{\wedge}\Sigma^*\otimes\mathcal{L}^{r+q-1}\otimes\pi^*\mathcal{O}(\ell)=\Sigma\otimes\mathcal{L}^{q-1}\otimes\pi^*\mathcal{O}(\mathbf{d}-n-1+\ell)$, $\overset{m+1}{\wedge}\Sigma^*\otimes\mathcal{L}^{r+q}\otimes\pi^*\mathcal{O}(\ell)=\mathcal{L}^q\otimes\pi^*\mathcal{O}(\mathbf{d}-n-1+\ell)$ and the map d is nothing but $j(\sigma)\otimes 1$. Therefore we have a canonical map

$$h: R_q(\mathbf{d}-n-1+\ell)\to\operatorname{Ker}(H^m(\mathcal{L}^{-m+r+q-1}\otimes\pi^*\mathcal{O}(\ell))\to H^m(\Sigma^*\otimes\mathcal{L}^{-m+r+q}\otimes\pi^*\mathcal{O}(\ell)))$$

By the Serre duality, the right hand side is isomorphic to the dual of

$$\begin{aligned}\operatorname{Coker}(H^0(\Sigma\otimes\mathcal{L}^{n-r-q-1}\otimes\pi^*\mathcal{O}(\mathbf{d}+\mathbf{e}-n-1-\ell))\to S_{n-r-q}(\mathbf{d}+\mathbf{e}-n-1-\ell))\\ = R_{n-r-q}(\mathbf{d}+\mathbf{e}-n-1-\ell).\end{aligned}$$

Lemma(3-2-2). h is an isomorphism if one of the following conditions is satisfied.
(1) $0\le\ell\le\mathbf{e}$, $r+s\le n$.
(2) $q=n-r$, $\ell=\mathbf{d}+\mathbf{e}-n-1>0$.

Remark(3-2-3). Note that (1) holds for all $q\in\mathbf{Z}$. Since $R_q(l)=0$ for $q<0$ clearly, we have also $R_q(l)=0$ for $q>n-r$ if $r+s\le n$. However if $r+s\ge n+2$, this does not hold. In fact, in this case, $R_{n-r+1}(\mathbf{d}-n-1)\ne 0$. Hence the duality also fails.

Proof. By the exact sequence (3-2-1), h is surjective if

$$\text{(a) } H^a(\overset{m+1-a}{\wedge}\Sigma^*\otimes\mathcal{L}^{r+q-a}\otimes\pi^*\mathcal{O}(\ell))=0 \text{ for } 1\le a\le m-1,$$

and is injective if

$$\text{(b) } H^b(\overset{m-b}{\wedge}\Sigma^*\otimes\mathcal{L}^{r+q-b-1}\otimes\pi^*\mathcal{O}(\ell))=0 \text{ for } 1\le b\le m-1.$$

Firstly we show the case (1).

(a). Since $1 \le a \le m-1$ and $0 \le \ell \le \mathbf{e}$, by the vanishing lemma (1) and (1)*, we only need to check the assertion (a) in the case $(r+q-a, \ell) = (0,0)$ and $(r+q-a, \ell) = (-s, \mathbf{e})$. If $(r+q-a, \ell) = (0,0)$, by vanishing lemma (5), it suffices to consider only the case $a = n$. Then since $m+1-a = r+s < n+1$, the assertion follows from vanishing lemma (6). If $(r+q-a, \ell) = (-s, \mathbf{e})$, then by vanishing lemma (5)*, we can assume $a = r+s-1$. Then we have $m+1-a = n+1 > r+s-1$. Therefore the assertion follows from vanishing lemma (6)*.

(b). Similarly we only need to check (b) in the case $(r+q-b-1, \ell) = (0,0)$ and $(r+q-b-1, \ell) = (-s, \mathbf{e})$. If $(r+q-b-1, \ell) = (0,0)$, it suffices to consider only the case $b = n$. Then since $m-b = r+s-1 < n+1$, the assertion follows from vanishing lemma (6). If $(r+q-b-1, \ell) = (-s, \mathbf{e})$, it suffices to consider only the case $b = r+s-1$. Then since $m-b = n > r+s-1$, the assertion follows from vanishing lemma (6)*.

Next we show the case (2).

(a). Since $\ell > 0$, by the vanishing lemma (1), we can assume $r+q-a = n-a \le -s$. Then we have $(m+1-a)-(n-a) = r+s$ and $r+s < n+s \le a \le m-1$. Therefore the assertion follows from the vanishing lemma (4).

(b). Similarly we can assume $n-1-b \le -s$. Then since $(m-b)-(n-b-1) = r+s$ and $r+s \le n+s-1 \le b < m$, the assertion follows from the vanishing lemma (4), also.

This completes the proof. □

By Lemma(3-2-2)(2), we have a canonical isomorphism $R_{n-r}(2\mathbf{d}+2n-2+\mathbf{e}) \to R_0(0)^* \simeq \mathbb{C}$. Since the compatibility of the map h with the cup-products, we have the duality theorem.

3.3. Finally we prove Theorem(III), the symmetrizer lemma.

Before to do this, we recall the regularity of sheaves ([G3]).

A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is called *m-regular* if

$$H^i(\mathbb{P}^n, \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(m-i)) = 0 \quad \text{for } \forall i > 0.$$

We use the following properties of the regularity of sheaves, whose proof can be seen in [G3].

(1) If $\mathcal{F}$ is m-regular, then also $(m+1)$-regular.

(2) If $\mathcal{F}$ and $\mathcal{F}'$ are m-regular and m'-regular respectively, then $\mathcal{F} \otimes \mathcal{F}'$ is $(m+m')$-regular.

In particular, if E is a m-regular locally free sheaf on $\mathbb{P}^n$, then $\overset{p}{\wedge} E$ is (mp)-regular because this one is a direct summand of $E^{\otimes p}$.

For example, let $\ell \ge 0$ be an integer, and define a locally free sheaf E on a projective space $\mathbb{P}^n$ by the exact sequence

$$0 \longrightarrow E \longrightarrow H^0(\mathbb{P}^n, \mathcal{O}(\ell)) \underset{\mathbb{C}}{\otimes} \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}(\ell) \longrightarrow 0.$$

Then clearly E is 1-regular, therefore $\overset{p}{\wedge} E$ is p-regular. In [G3], there is a further result. That is, if we replace $H^0(\mathcal{O}(\ell))$ by V a base point free linear subspace of $H^0(\mathcal{O}(\ell))$ of codimension c, and define E' similarly:

$$0 \longrightarrow E' \longrightarrow V \underset{\mathbb{C}}{\otimes} \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}(\ell) \longrightarrow 0,$$

then $\overset{p}{\wedge} E'$ is $(p+c)$-regular.

These arguments are applicable to not only the case $\mathcal{O}(\ell)$ but also any locally free sheaf satisfying some conditions. We need it later.

Lemma(3-3-1). Let $\mathcal{N}$ be a locally free sheaf on $\mathbb{P}^n$ generated by global sections. We assume that $\mathcal{N}$ satisfies $H^p(\mathcal{N}(-p)) = 0$ for $0 < p < n$ (*e.g.* $\mathcal{N} = \mathcal{E}$). Let V be a linear subspace of $H^0(\mathcal{N})$ of codimension c, such that $V \underset{\mathbb{C}}{\otimes} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{N}$ is surjective (*i.e.* base point free). Define the locally free sheaf N by the exact sequence

$$0 \longrightarrow N \longrightarrow V \underset{\mathbb{C}}{\otimes} \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{N} \longrightarrow 0.$$

Then $\overset{p}{\wedge} N$ is $(p+c)$-regular. □

Now we go back to our situation.

Let $W \subset A_1(0) = H^0(\mathcal{L})$ be a linear subspace of codimension c. We assume that W is base point free, *i.e.* $W \underset{\mathbf{C}}{\otimes} \mathcal{O}_{\mathbb{P}} \to \mathcal{L}$ is surjective, which is equivalent to that $W' \underset{\mathbf{C}}{\otimes} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{E}$ is surjective. (Here W' is a subspace of $H^0(\mathcal{E})$ corresponding to W under the natural isomorphism $H^0(\mathcal{L}) \simeq H^0(\mathcal{E})$.)

The following lemma is a generalization of [G3] Theorem 4.1 (see also [G4] Theorem 2.2).

Lemma(3-3-2). Let $q \geq 0$, $\nu \geq 0$, ℓ integers. Then the Koszul complex

$$A_\nu(\ell) \otimes \overset{q+1}{\wedge} W \longrightarrow A_{\nu+1}(\ell) \otimes \overset{q}{\wedge} W \longrightarrow A_{\nu+2}(\ell) \otimes \overset{q-1}{\wedge} W$$

is exact if $d_{\min}\nu + \ell \geq c + q$.

Proof. We define a locally free sheaf M by the exact sequence

$$0 \longrightarrow M \longrightarrow W \underset{\mathbf{C}}{\otimes} \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{L} \longrightarrow 0. \tag{3-3-2-1}$$

Then we obtain a Koszul exact sequence

$$0 \to \overset{q+1}{\wedge} M \to \overset{q+1}{\wedge} W \underset{\mathbf{C}}{\otimes} \mathcal{O}_{\mathbb{P}} \to \overset{q}{\wedge} W \underset{\mathbf{C}}{\otimes} \mathcal{L} \to \cdots \to \mathcal{L}^{q+1} \to 0.$$

Tensoring with $\mathcal{L}^\nu \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}(\ell)$, this gives an acyclic resolution of $\overset{q+1}{\wedge} M \otimes \mathcal{L}^\nu \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}(\ell)$. In fact if $\nu' \geq \nu$, $H^i(\mathbb{P}, \mathcal{L}^{\nu'} \otimes \pi^*\mathcal{O}(\ell)) \simeq H^i(\mathbb{P}^n, S^{\nu'}(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^n}(\ell)) = 0$ for $\forall i > 0$, because the each degree of the direct summand of $S^{\nu'}(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^n}(\ell)$ is not less than $d_{\min}\nu' + \ell \geq d_{\min}\nu + \ell \geq c + q \geq 0$. Therefore the Koszul cohomology group above is isomorphic to $H^1(\overset{q+1}{\wedge} M \otimes \mathcal{L}^\nu \otimes \pi^*\mathcal{O}_{\mathbb{P}^n}(\ell))$.

We will prove that this cohomology vanishes.

Applying π_* to the exact sequence, we have the exact sequence

$$0 \longrightarrow \pi_* M \longrightarrow W' \underset{\mathbf{C}}{\otimes} \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{E} \longrightarrow 0.$$

(The surjectivity of the right map is due to the base point freeness of W'.) Put $N = \pi_* M$. Then by Lemma(3-3-1), $\overset{i}{\wedge} N$ is $(c+i)$-regular. On the other hand, the natural map $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ induces the commutative diagram:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \pi^* N & \longrightarrow & W \underset{\mathbf{C}}{\otimes} \mathcal{O}_{\mathbb{P}} & \longrightarrow & \pi^*\mathcal{E} & \longrightarrow 0 \\ & \downarrow g & & \downarrow = & & \downarrow g' & \\ 0 \longrightarrow & M & \longrightarrow & W \underset{\mathbf{C}}{\otimes} \mathcal{O}_{\mathbb{P}} & \longrightarrow & \mathcal{L} & \longrightarrow 0. \end{array}$$

By the snake lemma, g is injective and $\mathrm{Coker} g \simeq \mathrm{Ker} g'$. From the Euler exact sequence(2-1-2), we have $\mathrm{Ker} g' \simeq \Omega^1_{\mathbb{P}/\mathbb{P}^n} \otimes \mathcal{L}$. Hence we have

$$0 \longrightarrow \pi^* N \longrightarrow M \longrightarrow \Omega^1_{\mathbb{P}/\mathbb{P}^n} \otimes \mathcal{L} \longrightarrow 0. \tag{3-3-2-2}$$

The exact sequence (3-3-2-2) induces the filtration

$$\overset{q+1}{\wedge} M = F^0 \supset F^1 \supset \cdots \supset F^{q+1} \supset F^{q+2} = 0$$

such that $\mathrm{Gr}^i_F(\overset{q+1}{\wedge} M) = F^i/F^{i+1} \simeq \pi^*(\overset{i}{\wedge} N) \otimes \Omega^{q-i+1}_{\mathbb{P}/\mathbb{P}^n} \otimes \mathcal{L}^{q-i+1}$. So it suffices to show that

$$H^1(\mathbb{P}, \mathcal{L}^{\nu+q-i+1} \otimes \Omega^{q-i+1}_{\mathbb{P}/\mathbb{P}^n} \otimes \pi^*(\overset{i}{\wedge} N \otimes \mathcal{O}_{\mathbb{P}^n}(\ell))) = 0 \quad \text{for } 0 \leq \forall i \leq q+1$$

Moreover, from (2-1-2), we get

$$0 \to \overset{r+s}{\wedge}\pi^*\mathcal{E}\otimes\mathcal{L}^{-r-s} \to \cdots \to \overset{p+1}{\wedge}\pi^*\mathcal{E}\otimes\mathcal{L}^{-p-1} \to \Omega^p_{\mathbb{P}/\mathbb{P}^n} \to 0.$$

Therefore it suffices to show that

$$H^j(\mathbb{P}, \mathcal{L}^{\nu-j}\otimes\pi^*(\overset{q+1-i+j}{\wedge}\mathcal{E}\otimes\overset{i}{\wedge}N\otimes\mathcal{O}_{\mathbb{P}^n}(\ell))) = 0 \quad \text{for } 1 \le \forall j \le r+s-(q+1-i).$$

Case $1 \le j \le \nu$
The above cohomology is isomorphic to

$$H^j(\mathbb{P}^n, S^{\nu-j}(\mathcal{E})\otimes\overset{q+1-i+j}{\wedge}\mathcal{E}\otimes\overset{i}{\wedge}N\otimes\mathcal{O}_{\mathbb{P}^n}(\ell)) \simeq \bigoplus_k H^j(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\alpha_k)\otimes\overset{i}{\wedge}N)$$

where $\alpha_k \ge d_{\min}(\nu-j) + d_{\min}(q+1-i+j) + \ell \ge d_{\min}\nu + \ell$. Since $\overset{i}{\wedge}N$ is $(i+c)$-regular, this vanishes if $\alpha_k + j \ge i + c$. But since $\alpha_k + j - i - c \ge \alpha_k + 1 - (q+1) - c \ge d_{\min}\nu + \ell - q - c \ge 0$, this hold.
Case $j > \nu$
If $\nu - j > -r - s$, the cohomology vanishes always. So we only consider the case $\nu - j \le -r - s \iff j \ge r + s + \nu$. But since $j \le r + s - (q + 1 - i)$, we only consider the case $\nu = 0$, $j = r + s$, $i = q + 1$. Then the cohomology is isomorphic to $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell)\otimes\overset{q+1}{\wedge}N)$. Since $\overset{q+1}{\wedge}N$ is $(q+1+c)$-regular and $\ell + 1 = d_{\min}\nu + \ell + 1 \ge q + 1 + c$, this one vanishes. □

Now we prove Theorem(III). Let (p, q, ℓ) satisfies one of the conditions in Theorem(III). We put $\ell' := \ell - \mathbf{d} + n + 1$.

Put $\mathcal{C}_{k,h}(\ell) = \overset{m+1-h}{\wedge}\Sigma^*\otimes\mathcal{L}^{r+k-h}\otimes\pi^*\mathcal{O}(\ell')$, and $C_{k,h}(\ell) = H^0(\mathcal{C}_{k,h}(\ell))$. Then from the exact sequence (3-2-1), we obtain

$$0 \to \mathcal{C}_{p,m+1}(\ell) \to \cdots \to \mathcal{C}_{p,1}(\ell) \to \mathcal{C}_{p,0}(\ell) \to 0, \tag{3-3-4}$$

$$0 \to C_{p,m+1}(\ell) \to \cdots \to C_{p,1}(\ell) \xrightarrow{\phi} C_{p,0}(\ell) \to 0, \tag{3-3-5}$$

We note that (3-3-4) is exact, but (3-3-5) not necessarily exact. In particular, Coker$\phi = R_p(\ell)$ by definition. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots & \to & C_{p,1}(\ell)\otimes\overset{q+1}{\wedge}W & \to & C_{p,0}(\ell)\otimes\overset{q+1}{\wedge}W & \to & R_p(\ell)\otimes\overset{q+1}{\wedge}W & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & C_{p+1,1}(\ell)\otimes\overset{q}{\wedge}W & \to & C_{p+1,0}(\ell)\otimes\overset{q}{\wedge}W & \to & R_{p+1}(\ell)\otimes\overset{q}{\wedge}W & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & C_{p+2,1}(\ell)\otimes\overset{q-1}{\wedge}W & \to & C_{p+2,0}(\ell)\otimes\overset{q-1}{\wedge}W & \to & R_{p+2}(\ell)\otimes\overset{q-1}{\wedge}W & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Therefore by an easy diagram chase, it suffices to show the followings.
Step1

(1) $C_{p+1+a,a+1}(\ell) \longrightarrow C_{p+1+a,a}(\ell) \longrightarrow C_{p+1+a,a-1}(\ell)$ is exact for $1 \le \forall a \le q-1$,
(2) $C_{p+2+a,a+1}(\ell) \longrightarrow C_{p+2+a,a}(\ell) \longrightarrow C_{p+2+a,a-1}(\ell)$ is exact for $1 \le \forall a \le q-1$.

Step2 $C_{p+b,b}(\ell)\otimes\overset{q+1-b}{\wedge}W \longrightarrow C_{p+b+1,b}(\ell)\otimes\overset{q-b}{\wedge}W \longrightarrow C_{p+b+2,b}(\ell)\otimes\overset{q-1-b}{\wedge}W$ is exact for $\forall b \ge 0$.

Proof of Step1. We may assume $q \ge 2$ (*i.e.* ommit the case Theorem(III)(3)). The exact sequence (3-3-4) induces a specral sequence

$$E_1^{\alpha,\beta} = H^\beta(\mathcal{C}_{k,m+1-\alpha}(\ell)) \Longrightarrow E^{\alpha+\beta} = 0.$$

We want to show that $E_2^{\alpha,0} = 0$ if (α, k) runs over any one of the following cases:

(1)′ $p + 2 \le k \le p + q$ and $k - (m + 1 - \alpha) = p + 1$ ($\iff \alpha = p - k + m + 2$),
(2)′ $p + 3 \le k \le p + q + 1$ and $k - (m + 1 - \alpha) = p + 2$ ($\iff \alpha = p - k + m + 3$).

Since $E_\infty^{\alpha,0}=0$, in order to show $E_2^{\alpha,0}=0$, it suffices to show that $E_1^{\alpha-h-1,h}=0$ for $1\le h\le m$, and $0\le \alpha-h-1\le m+1$.

We first show Step1 (1).

If (α,k) runs over the case $(1)'$, then

$$E_1^{\alpha-h-1,h}=H^h(\mathcal{C}_{k,k-p+h}(\ell))=H^h(\overset{m+1-(k-p+h)}{\wedge}\Sigma^*\otimes\mathcal{L}^{p+r-h}\otimes\pi^*\mathcal{O}(\ell')),$$

and (k,h) satisfies $k\ge p+2$, $h\ge 1$, $h+k\le m+1+p$.

Case $p+r-h\ge -s+1$

Since $\ell'=\ell-\mathbf{d}+n+1\ge 0$, by the vanishing lemma(1), we have only to check in the case $p+r=h$ and $\ell'=0$. Then by vanishing lemma(5), we can assume $h=p+r=n$. Hence we only need to consider the case Theorem(III)(1). Then $E_1^{\alpha-h-1,h}=H^n(\overset{n+s-k}{\wedge}\Sigma^*)$ and k satisfies $n-r+2\le k\le n+s$. In this case the assertion follows from vanishing lemma(6) , because we assumed $r+s\le n+2$.

Case $p+r-h\le -s$

Since $h\le m+1+(p-k)\le m-1$ and $\ell'=\ell-\mathbf{d}+n+1\le \mathbf{e}$, by the vanishing lemma $(1)^*$, we have only to check in the case $p+r-h=-s$, $\ell'=\mathbf{e}$. But since $h\ne r+s-1$, the assertion follows from vanishing lemma$(5)^*$.

Next we show Step1 (2).

If (α,k) runs over the case $(2)'$, then

$$E_1^{\alpha-h-1,h}=H^h(\mathcal{C}_{k,k-p+h-1}(\ell))=H^h(\overset{m+1-(k-p+h-1)}{\wedge}\Sigma^*\otimes\mathcal{L}^{p+r-h+1}\otimes\pi^*\mathcal{O}(\ell')),$$

and (k,h) satisfies $k\ge p+3$, $h\ge 1$, $h+k\le m+2+p$.

Case $p+r-h+1\ge -s+1$

Since $\ell'\ge 0$, by the vanishing lemma(1), we have only to check in the case $p+r-h+1=0$ and $\ell'=0$. Then by vanishing lemma(5), we can assume $h=p+r+1=n$. Hence we only need to consider the case Theorem(III)(1). Thus $E_1^{\alpha-h-1,h}=H^n(\overset{n+s-k}{\wedge}\Sigma^*)$ and k satisfies $n-r+2\le k\le n+s$. Therefore the assertion follows from vanishing lemma(6), because we assumed $r+s\le n+2$.

Case $p+r-h+1\le -s$

Since $h\le m+2+(p-k)\le m-1$ and $\ell'\le \mathbf{e}$, by the vanishing lemma $(1)^*$, we have only to check in the case $p+r-h+1=-s$, $\ell'=\mathbf{e}$. But since $h=r+s+p+1\ne r+s-1$, the assertion follows from vanishing lemma$(5)^*$.

We have completed the proof of Step1. □

Proof of Step2. We write $\Omega^1_{\mathbb{P}}(\log\Sigma\mathbb{P}_j)$ by Ω simply. We want to show that

$$\begin{aligned}H^0(\overset{m+1-b}{\wedge}\Sigma^*\otimes\mathcal{L}^{r+p}\otimes\pi^*\mathcal{O}(\ell'))\otimes\overset{q+1-b}{\wedge}W&\\ \to H^0(\overset{m+1-b}{\wedge}\Sigma^*\otimes\mathcal{L}^{r+p+1}\otimes\pi^*\mathcal{O}(\ell'))\otimes\overset{q-b}{\wedge}W&\\ \to H^0(\overset{m+1-b}{\wedge}\Sigma^*\otimes\mathcal{L}^{r+p+2}\otimes\pi^*\mathcal{O}(\ell'))\otimes\overset{q-1-b}{\wedge}W&\end{aligned}$$

is exact for $0\le\forall b\le q$.

The exact sequence

$$0\to\Omega^{\cdot}\otimes\mathcal{L}^{\sharp}\otimes\pi^*\mathcal{O}(\ell')\to\overset{\cdot}{\wedge}\Sigma^*\otimes\mathcal{L}^{\sharp}\otimes\pi^*\mathcal{O}(\ell')\to\Omega^{\cdot-1}\otimes\mathcal{L}^{\sharp}\otimes\pi^*\mathcal{O}(\ell')\to 0$$

remains exact for $\sharp>0$ when taking $H^0(\ \ )$, because $H^1(\mathcal{L}^{\sharp}\otimes\Omega^{\cdot}\otimes\pi^*\mathcal{O}(\ell'))=0$. Thus it suffices to show that the following sequence is exact for all t, b such that $m-b\le t\le m-b+1$ and $0\le b\le q$:

$$\begin{aligned}H^0(\Omega^t\otimes\mathcal{L}^{r+p}\otimes\pi^*\mathcal{O}(\ell'))\otimes\overset{q+1-b}{\wedge}W&\\ \to H^0(\Omega^t\otimes\mathcal{L}^{r+p+1}\otimes\pi^*\mathcal{O}(\ell'))\otimes\overset{q-b}{\wedge}W&\\ \to H^0(\Omega^t\otimes\mathcal{L}^{r+p+2}\otimes\pi^*\mathcal{O}(\ell'))\otimes\overset{q-1-b}{\wedge}W.&\end{aligned}$$

From the exact sequence (2-2-5-1), there is a filtration $F^{\cdot}$ of Ω^t such that $\mathrm{Gr}^u_F(\Omega^t) = \pi^*\Omega^u_{\mathbb{P}^n} \otimes \Omega^{t-u}_{\mathbb{P}/\mathbb{P}^n}(\log \Sigma \mathbb{P}_j) = \pi^*\Omega^u_{\mathbb{P}^n} \otimes (\bigoplus_{i=0}^{t-u} [\overset{i}{\wedge}\pi^*\mathcal{E}_0 \otimes \mathcal{L}^{-i}]^{\binom{s-1}{t-u-i}})$, where (u,i) runs over $0 \le u \le n$, $0 \le t-u \le r+s-1$ and $\max\{0, t-u-s+1\} \le i \le \min\{t-u, r\}$. Since $H^1(\mathcal{L}^{r+\sharp} \otimes \mathrm{Gr}^{\cdot}_F\Omega^t) = 0$ for $\sharp \ge 0$, it suffices to show that

$$H^0(\mathcal{L}^{r+p-i} \otimes \pi^*(\Omega^u_{\mathbb{P}^n} \otimes \overset{i}{\wedge}\mathcal{E}_0(\ell')) \otimes \overset{q+1-b}{\wedge} W \to H^0(\mathcal{L}^{r+p-i+1} \otimes \pi^*(\Omega^u_{\mathbb{P}^n} \otimes \overset{i}{\wedge}\mathcal{E}_0(\ell')) \otimes \overset{q-b}{\wedge} W \to H^0(\mathcal{L}^{r+p-i+2} \otimes \pi^*(\Omega^u_{\mathbb{P}^n} \otimes \overset{i}{\wedge}\mathcal{E}_0(\ell')) \otimes \overset{q-1-b}{\wedge} W.$$

$$\text{is exact for } \forall b,t,u,i \text{ such that } \begin{cases} 0 \le b \le q \\ 0 \le u \le n \\ 0 \le t-u \le r+s-1 \\ \max\{0, t-u-s+1\} \le i \le \min\{t-u, r\} \\ m-b \le t \le m-b+1 \end{cases}$$

Finally, by the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-n-1) \to \mathcal{O}_{\mathbb{P}^n}(-n)^{\oplus n+1} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(-u-1)^{\oplus\binom{n+1}{n-u}} \to \Omega^u_{\mathbb{P}^n} \to 0,$$

we can deduce the assertion to show that

$$H^0(\mathcal{L}^{r+p-i+j} \otimes \pi^*(\overset{i}{\wedge}\mathcal{E}_0(\ell'-u-j-1)) \otimes \overset{q+1-b-j}{\wedge} W \to H^0(\mathcal{L}^{r+p-i+j+1} \otimes \pi^*(\overset{i}{\wedge}\mathcal{E}_0(\ell'-u-j-1)) \otimes \overset{q-b-j}{\wedge} W \to H^0(\mathcal{L}^{r+p-i+j+2} \otimes \pi^*(\overset{i}{\wedge}\mathcal{E}_0(\ell'-u-j-1)) \otimes \overset{q-1-b-j}{\wedge} W$$

$$\text{is exact for } \forall b,t,u,i,j \text{ such that } \begin{cases} 0 \le b \le q \\ 0 \le u \le n \\ 0 \le t-u \le r+s-1 \\ 0 \le j \le n-u \\ \max\{0, t-u-s+1\} \le i \le \min\{t-u, r\} \\ m-b \le t \le m-b+1 \end{cases}$$

Let δ_i be the minimal degree of each direct summand of $\overset{i}{\wedge}\mathcal{E}_0$. Then by Lemma(3-3-2), the above holds if $p \ge 0$ and

$$d_{\min}(r+p-i+j) + (\delta_i - u - j - 1 + \ell') \ge q - b - j + c$$

$$\text{for } \forall b,t,u,i,j \text{ such that } \begin{cases} 0 \le b \le q \\ 0 \le u \le n \\ 0 \le t-u \le r+s-1 \\ 0 \le j \le n-u \\ \max\{0, t-u-s+1\} \le i \le \min\{t-u, r\} \\ m-b \le t \le m-b+1 \end{cases}$$

This holds if $p \ge 0$ and $d_{\min}(r+p) + \ell - \mathbf{d} \ge q + c$. This completes the proof. □

[CGGH] J.Carlson, M.Green, P.A.Griffiths and J.Harris, *Infinitesimal variations of Hodge structures I*, Compositio.Math. **50** (1983), 109-205.

[D] R.Donagi, *Generic Torelli for projective hypersurfaces*, Compositio.Math. **50** (1983), 325-353.

[DG] R.Donagi and M.Green, *A new proof of the symmetrizer lemma and a stronger weak Torelli theorem for projective hypersurfaces*, J.Diff.Geom. **20** (1984), 459-461.

[G1] M. Green, *The period map for hypersurface sections of high degree of an arbitrary variety*, Compositio.Math. **55** (1985), 135-156.

[G2] ———, *Infinitesimal methods in Hodge theory*, Lecture Notes in Math. **1594** (1993), springer., 1-92.

[G3] ———, *Koszul cohomology and Geometry*, in Cornalba, Gomez-Mont, and Verjovsky, Lectures on Riemann surfaces, ICTP, Trieste, Italy., 177-200.

[G4] ———, *Koszul cohomology and the geometry of projective varieties II*, J.Diff.Geom. **20** (1984), 279-289.

[K] K. Konno, *On the variational Torelli problem for complete intersections*, Compositio Math. **78** (1991), 271-296.

[T] T. Terasoma, *Infinitesimal variation of Hodge structures and the weak global Torelli theorem for complete intersections*, Ann. Math. **132** (1990), 213-235.