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Arakelov theory with respect to hyperbolic metrics

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§1. Introduction

Over a compact Riemann surface, for any (smooth) Hermitian line bundle, with respect to any (smooth) volume form, we may introduce the Quillen metric ([Qu]) on the corresponding determinant of cohomology. Essentially, this is because there exists only discrete spectrum for the associated Laplacian, so that the Ray-Singer's zeta function formalism ([RS]) can be applied. By using Quillen metrics, we then have the so-called Riemann-Roch and Noether isometries ([De]).

On the other hand, we cannot apply the same strategy to compact Riemann surfaces with respect to singular volume forms, or better, to punctured Riemann surfaces, due to the fact that a certain continuous spectrum exists for the corresponding Laplacian. Even though, with respect to hyperbolic metrics over Riemann surfaces of finite volume, along with the same line as for compact Riemann surfaces, we now have the works done by Efrat ([Ef]), and Takhtajan-Zograf ([TZ1], [TZ2]), among others, on special values of Selberg zeta functions, regularized determinants of Laplacians, and Quillen metrics, previously it remains to be a very challenge problem to deduce a general but natural theory from them.

Nevertheless, in this talk, we use a quite independent approach to offer a reasonable metric theory for punctured Riemann surfaces. Roughly speaking, we take the Riemann-

Roch and Noether isometries as the motivation and hence as the final goal for developing such a theory, since we believe that a good metric theory for punctured Riemann surfaces should ultimately provide us these two isometries in a natural way. As an application to moduli spaces of punctured Riemann surfaces of our metrics, we give some Mumford type fundamental isometries for determinant line bundles equipped with our metrics.

§2. ω-Arakelov metrics and ω-intersection theory

(2.1) Throughout this talk, we always assume that M^0 is a (punctured) Riemann surface of genus q. Denote its smooth compactification by M, and let $M\backslash M^0 =: \{P_1, \ldots, P_N\}$. We will call P_i, $i = 1, \ldots, N$, *cusps* of M^0, and (q, N) the *signature* of M^0.

Recall that a Hermitian metric ds^2 on M^0 is said to be *of hyperbolic growth near the cusps*, if for each $P_i, i = 1, \ldots, N$, there exists a punctured coordinate disc $\Delta^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ centered at P_i such that for some constant $C_1 > 0$,

(i) $ds^2 \leq \dfrac{C_1|dz|^2}{|z|^2(\log|z|)^2}$ on Δ^*, (2.1.1)

and there exists a local potential function ϕ_i on Δ^* satisfying $ds^2 = \frac{\partial^2\phi_i}{\partial z\partial\bar{z}} dz \otimes d\bar{z}$ on Δ^*, and for some constants C_2, $C_3 > 0$,

(ii) $|\phi_i(z)| \leq C_2 \max\{1, \log(-\log|z|)\}$, and (2.1.2)

(iii) $\left|\dfrac{\partial\phi_i}{\partial z}\right|, \left|\dfrac{\partial\phi_i}{\partial\bar{z}}\right| \leq \dfrac{C_3}{|z|\,|\log|z||}$ on Δ^*. (2.1.3)

In this case, we call ds^2 a *quasi-hyperbolic metric*, which is introduced in [TW].

For a quasi-hyperbolic metric ds^2 over a punctured Riemann surface M^0, it follows easily from (2.1.1) that $\mathrm{Vol}(M^0, ds^2) < \infty$. Denote the normalized volume form of ds^2 by ω so that $\mathrm{Vol}(M, \omega) = 1$. In this talk, ω always denotes the normalized volume form on M associated to a smooth metric (on M) or associated to a quasi-hyperbolic metric on M^0.

(2.2) Even ω could be singular, in [TW, Theorem 1], we show that there exists a unique ω-*Green's function* $g_\omega(\cdot,\cdot)$, or *the Green's function with respect to* ω, on $M^0 \times M^0\backslash\{\text{diagonal}\}$ by using the following

Lemma 2.2.1 ([TW]) *With the same notation as above, the function $g_\omega(P, Q)$ defined on $M^0 \times M^0\backslash\{diagonal\}$ by*

$$g_\omega(P, Q) = g(P, Q) + \beta_\omega(P) + \beta_\omega(Q), \tag{2.2.3}$$

satisfies the above conditions (i)$\sim$(vi).

(2.3) Now we are ready to define the ω-Arakelov metrics on $\mathcal{O}_M(P)$ for any point $P \in M$ and on K_M, the canonical line bundle of M.

First of all, for any $P \in M^0$, define a metric $\rho_{\mathrm{Ar};\omega;P}$ on $\mathcal{O}_M(P)$ by setting

$$\log \|1_P\|^2_{\rho_{\mathrm{Ar};\omega;P}}(Q) := -g_\omega(P,Q) + \beta_\omega(P) \quad \text{for } Q \neq P \text{ in } M^0. \tag{2.3.1}$$

Here 1_P denotes the defining section of $\mathcal{O}_M(P)$. (Please note in particular that the constant $\beta_\omega(P)$ is added.)

Secondly, by Lemma (2.2.1) above, we see that

$$-g_\omega(P,Q) + \beta_\omega(P) = -g(P,Q) - \beta_\omega(Q).$$

Thus, **for any point P$\in$M**, we (may) define a Hermitian metric $\rho_{\mathrm{Ar};\omega;P}$ on $\mathcal{O}_M(P)$ by setting

$$\log \|1_P\|^2_{\rho_{\mathrm{Ar};\omega;P}}(Q) := -g(P,Q) - \beta_\omega(Q) \quad \text{for } Q \neq P \text{ in } M^0. \tag{2.3.2}$$

In particular, this works also for cusps P_i, $i = 1, \ldots, N$. Easily, we see that

$$c_1(\mathcal{O}_M(P), \rho_{\mathrm{Ar};\omega;P}) = \omega. \tag{2.3.3}$$

We will call $\rho_{\mathrm{Ar};\omega;P}$ *the ω-Arakelov metric*, or *the Arakelov metric with respect to ω, on* $\mathcal{O}_M(P)$.

(2.4) A Hermitian line bundle (L, ρ) on M is called *ω-admissible*, if $c_1(L,\rho) = d(L) \cdot \omega$. Here $d(L)$ denotes the degree of L. From (2.3.3), we have the following

Lemma 2.4.1. *With the same notaton as above, $(\mathcal{O}_M(P), \rho_{\mathrm{Ar};\omega;P})$ is ω-admissible.*

Furthermore, by extending $\rho_{\mathrm{Ar};\omega;P}$ linearly on P by using tensor products, we know that over any line bundle L on M, there exist ω-admissible Hermitian metrics, which are parametrized by $\mathbb{R}^+$.

For later use, denote $(\mathcal{O}_M(P), \rho_{\mathrm{Ar};\omega;P})$ by $\underline{\mathcal{O}_M(P)}_\omega$, or simply $\underline{\mathcal{O}_M(P)}$ if no confusion arises. If (L, ρ) is an ω-admissible Hermitian line bundle on M, we denote (L, ρ) by $\bar{L}^\omega$ or simply $\bar{L}$ by abuse of notation. Similarly, we use $\bar{L}(\underline{P})$ to denote $\bar{L} \otimes \underline{\mathcal{O}_M(P)}$.

Thus, in particular, on the canonical line bundle K_M of M, there exist ω-admissible Hermitian metrics. But such metrics are far from being unique. We next make a certain normalization.

On K_M, define *the ω-Arakelov metric $\rho_{\mathrm{Ar};\omega}$*, or *the Arakelov metric with respect to ω* by setting

$$\|h(z)\,dz\|^2_{\rho_{\mathrm{Ar};\omega}}(P) := |h(P)|^2 \cdot \lim_{Q\to P} \frac{|z(P)-z(Q)|^2}{e^{-g_\omega(P,Q)}} \cdot e^{-2q\beta_\omega(P)} \text{ for } P \in M^0. \qquad (2.4.1)$$

Here $h(z)\,dz$ denotes a section of K_M. We have the following

Proposition 2.4.2. *With the same notation as above, $(K_M, \rho_{\mathrm{Ar};\omega})$ is ω-admissible.*

For later use, denote $(K_M, \rho_{\mathrm{Ar};\omega})$ by $\underline{K_{M_\omega}}$, or simply by $\underline{K_M}$ if no confusion arises. Also we denote $(K_M, \rho_{\mathrm{Ar};\omega}\cdot e^{\frac{c}{2}})$ (resp. $\underline{K_M}\otimes\underline{\mathcal{O}_M(P)}$) by $\overline{K_M}^c$ (resp. $\underline{K_M(P)}$) for any constant c.

We end this subsection by giving a geometric interpretation for the ω-Arakelov metric $\rho_{\mathrm{Ar};\omega}$. We begin with a preperation.

Let $\bar{L}$ be an ω-admissible Hermitian line bundle, then for any point $P \in M$, on the restriction $L|_P$, we introduce a metric by multiplying the restriction metric from $\bar{L}$ to P an additional factor $\exp[d(L)\cdot\frac{1}{2}\beta_\omega(P)]$, and we will use the symbol $\bar{L}\|_P$ to indicate the vector space $L|_P$ together with this modification of the metric, and sometimes call it the *ω-restriction* of $\bar{L}$ at P. With this, by using (2.4.2), (2.2.1), and the fact that the Arakelov metric induces a natural isometry via the residue map $\mathrm{res} : K_M(P)|_P \to \mathbb{C}$, we see that the Arakelov metric with respect to ω on K_M is the unique metric such that, at each point $P \in M$, the natural residue map res induces the following *ω-adjunction isometry*

$$\mathrm{res} : \underline{K_M(P)}\|_P \to \underline{\mathbb{C}}. \qquad (2.4.3)$$

Here $\underline{\mathbb{C}}$ denotes the complex plane $\mathbb{C}$ equipped with the ordinary flat metric.

(2.5) For any two line bundles L, L' on M, denote by $\langle L, L'\rangle$ the Deligne pairing associated to L and L'. In this subsection, we define an ω-Deligne norm $h_{\mathrm{De},\omega}$ on $\langle L, L'\rangle$ for any two ω-admissible Hermitian line bundles $\bar{L}$ and $\bar{L}'$.

First, let us define the *ω-Deligne norm* for $\langle\mathcal{O}_M(P),\mathcal{O}_M(Q)\rangle$ with $P \neq Q \in M^0$, for

ω-Arakelov metrized line bundles $\underline{\mathcal{O}_M(P)}$ and $\underline{\mathcal{O}_M(Q)}$, by setting

$$\log \|\langle 1_P, 1_Q\rangle\|^2_{h_{\mathrm{De},\omega}} := -g_\omega(P,Q) + \beta_\omega(P) + \beta_\omega(Q). \tag{2.5.1}$$

Secondly, note that the right hand side of (2.5.1) can be written as $-g(P,Q)$, the Arakelov-Green's function for P and Q. Hence, even though (2.5.1) does not make any sense for cusps, but is we change it to

$$\log \|\langle 1_P, 1_Q\rangle\|^2_{h_{\mathrm{De},\omega}} := -g(P,Q), \tag{2.5.2}$$

then we have the metrized ω-Deligne pairing $\langle \underline{\mathcal{O}_M(P)}, \underline{\mathcal{O}_M(Q)}\rangle$ for **all** $P \neq Q \in M$.

Finally extending $h_{\mathrm{De},\omega}$ by linearity, we get a definition for ω-Deligne norm $h_{\mathrm{De},\omega}(\bar{L}, \bar{L}')$ on $\langle L, L'\rangle$ for any two ω-admissible Hermitian line bundles $\bar{L}$ and $\bar{L}'$ on M. By abuse of notation, we denote $\Big(\langle L, L'\rangle, h_{\mathrm{De};\omega}(\bar{L}, \bar{L}')\Big)$ simply by $\langle \bar{L}, \bar{L}'\rangle$.

Remark 2.5.1. Even though we study the ω-intersection, the Arakelov-Green's function is used in an essential way. This is indeed not quite surprising. After all, we only define the ω-intersection for **the** Hermitian line bundles $\underline{\mathcal{O}_M(P)}$ and $\underline{\mathcal{O}_M(Q)}$ by using $-g(P,Q)$. Put this in a more formal manner, we have the following:

Proposition 2.5.1. (Mean Value Lemma I.) *For any two normalized volume forms ω_1 and ω_2 on M, there exists a natural isometry*

$$\langle \underline{\mathcal{O}_M(P)}_{\omega_1}, \underline{\mathcal{O}_M(Q)}_{\omega_1}\rangle \simeq \langle \underline{\mathcal{O}_M(P)}_{\omega_2}, \underline{\mathcal{O}_M(Q)}_{\omega_2}\rangle \quad \textit{for } P \neq Q \in M. \tag{2.5.3}$$

As a driect consequence of the ω-adjunction isometry (2.4.3), by definition, we have the following:

Proposition 2.5.2. (ω-Adjunction Isometry) *With the same notation as above, we have the isometry*

$$\langle \underline{K_M(P)}, \underline{\mathcal{O}_M(P)}\rangle \simeq \underline{\mathbb{C}} \quad \textit{for any } P \in M. \tag{2.5.4}$$

In a similar style, by using (2.2.1) and (2.4.2), we get

Proposition 2.5.3. (Mean Value Lemma II.) *With the same notation as above, for any two normalized volume forms ω_1 and ω_2 on M, there exists a natural isometry*

$$\langle \underline{K_{M}}_{\omega_1}, \underline{K_{M}}_{\omega_1} \rangle \simeq \langle \underline{K_{M}}_{\omega_2}, \underline{K_{M}}_{\omega_2} \rangle. \tag{2.5.5}$$

As an application to arithmetic surfaces, we see that the self-intersection of Arakelov canonical divisor can be understood in any of these ω-admissible theories. (For the detailed discussion, see e.g. [Wel].)

§3. ω-Riemann-Roch metric and its properties

(3.1) With the same notation as in §2, for any line bundle L on M, denote its associated determinant of cohomology, i.e., $\det H^0(M,L)\otimes(\det H^1(M,L))^{\otimes -1}$, by $\lambda(L)$. Then it is well-known that we have the following canonical Deligne-Riemann-Roch isomophism;

$$\lambda(L)^{\otimes 2}\otimes\lambda(\mathcal{O}_M)^{\otimes -2}\simeq\langle L, L\otimes K_M^{\otimes -1}\rangle. \tag{3.1.1}$$

(See e.g., [De], or [Ai].)

For a fixed normalized volume form ω on M associated to a quasi-hyperbolic metric, denote by $\underline{K_M}$ the ω-Arakelov canonical line bundle $(K_M, \rho_{\mathrm{Ar};\omega})$. With respect to $\underline{K_M}$, fix a metric $h_0(\underline{K_M})$ on $\lambda(\mathcal{O}_M)$. Then for any ω-admissible Hermitian line bundle $\bar{L}$ on M, define an ω-determinant metric $h_{\mathrm{RR};\underline{K_M};h_0(\underline{K_M})}(\bar{L})$ on $\lambda(L)$ by the isometry

$$\Big(\lambda(L), h_{\mathrm{RR};\underline{K_M};h_0(\underline{K_M})}(\bar{L})\Big)^{\otimes 2}\otimes\Big(\lambda(\mathcal{O}_M), h_0(\underline{K_M})\Big)^{\otimes -2} :\simeq \langle \bar{L}, \bar{L}\otimes\underline{K_M}^{\otimes -1}\rangle. \tag{3.1.2}$$

We call $h_{\mathrm{RR};\underline{K_M};h_0(\underline{K_M})}(\bar{L})$ on $\lambda(L)$ the *ω-Riemann-Roch metric associated to $\bar{L}$ with respect to $\underline{K_M}$ and $h_0(\underline{K_M})$*. Since for a fixed $\bar{L}$, with respect to $\underline{K_M}$ and $h_0(\underline{K_M})$, both $\Big(\lambda(\mathcal{O}_M), h_0(\underline{K_M})\Big)$ and $\langle \bar{L}, \bar{L}\otimes\underline{K_M}^{\otimes -1}\rangle$ are fixed, $h_{\mathrm{RR};\underline{K_M};h_0(\underline{K_M})}(\bar{L})$ is well-defined. By abuse of notation, we denote $\Big(\lambda(L), h_{\mathrm{RR};\underline{K_M};h_0(\underline{K_M})}(\bar{L})\Big)$ simply by $\underline{\lambda(\bar{L})}$.

The ω-Riemann-Roch metric satisties the following properties, which are very similar to these for Faltings metrics. (See Theorem 4.1.1 below.)

Proposition 3.1.1. *With the same notation as above, we have*
(F1) An isometry of ω-admissible Hermitian line bundles $\bar{L}\to\bar{L}'$ induces an isometry from $\underline{\lambda(\bar{L})}$ to $\underline{\lambda(\bar{L}')}$;

(F2) If the ω-admissible metric on L is changed by a factor $\alpha \in \mathbb{R}^+$, then the metric on $\lambda(L)$ is changed by the factor $\alpha^{\chi(M,L)}$;

(F3) For any point P on M, put the ω-Arakelov metrics on $\mathcal{O}_M(P)$, and take the tensor metric on $L(-P)$. Then the algebraic isomorphism

$$\lambda(L) \simeq \lambda(L(-P)) \otimes L|P$$

induced by the short exact sequence of coherent sheaves

$$0 \to L(-P) \to L \to L|_P \to 0$$

naturally becomes an isometry

$$\underline{\lambda(\bar{L})} \simeq \underline{\lambda(\bar{L} \otimes \underline{\mathcal{O}_M(P)}^{\otimes -1})} \otimes \bar{L}\|P.$$

(F4) (Serre Isometry) $\Big(\lambda(K_M), h_{\mathrm{RR};\underline{K_M};h_0(\underline{K_M})}(\underline{K_M})\Big) \simeq \Big(\lambda(\mathcal{O}_M), h_0(\underline{K_M})\Big).$

Remark 3.1.1. By (F4), we see that giving a normalization for $h_0(\underline{K_M})$ on $\lambda(\mathcal{O}_M)$ is equivalent to normalizing $h_{\mathrm{RR};\underline{K_M};h_0(\underline{K_M})}$ on $\lambda(K_M)$.

(3.2) Similarly, with respect to $\overline{K_M}$, we fix a metric $h_0(\overline{K_M})$ on $\lambda(\mathcal{O}_M)$. Then with respect to $\overline{K_M}'$, i.e., K_M equipped with (possibly) another ω-admissible Hermitian metric, and $h_0(\overline{K_M})$, for any ω-admissible Hermitian line bundle $\bar{L}$, we may define the associated Riemann-Roch metric, denoted by $h_{\mathrm{RR};\overline{K_M}';h_0(\overline{K_M})}(\bar{L})$, by the isometry

$$\Big(\lambda(L), h_{\mathrm{RR};\overline{K_M}';h_0(\overline{K_M})}(\bar{L})\Big)^{\otimes 2} \otimes \Big(\lambda(\mathcal{O}_M), h_0(\overline{K_M})\Big)^{\otimes -2} :\simeq \langle \bar{L}, \bar{L} \otimes (\overline{K_M}')^{\otimes -1}\rangle. \quad (3.2.1)$$

The dependence of $h_{\mathrm{RR};\overline{K_M}';h_0(\overline{K_M})}(\bar{L})$ on $\bar{L}$ and $\overline{K_M}'$ is clear, as it is given by the ω-intersection theory. More precisely, directly from the defintion, we have

Proposition 3.2.1. *The dependence of $h_{\mathrm{RR};\overline{K_M}';h_0(\overline{K_M})}(\bar{L})$ on $\bar{L}$ and $\overline{K_M}'$ is given by the following equality:*

$$h_{\mathrm{RR};\overline{K_M}'\otimes\mathcal{O}_M(e^c);h_0(\overline{K_M})}(\bar{L} \otimes \mathcal{O}_M(e^f)) = h_{\mathrm{RR};\overline{K_M}';h_0(\overline{K_M})}(\bar{L}) \cdot e^{\chi(L)\cdot f - d(L)c/2}. \quad (3.2.2)$$

Here for a constant c, $\mathcal{O}_M(e^c)$ denotes the trivial line bundle equipped with the metric $\|1\|^2 = e^c$.

On the other hand, the dependence of $h_{\mathrm{RR};\overline{K_M}';h_0(\overline{K_M})}(\bar{L})$ on $\overline{K_M}$ is not so easy to determined. We have then

$$h_0(\overline{K_M}^c) := h_0(\underline{K_M}) \cdot e^{\frac{2q-2}{12}\cdot c}. \tag{3.2.3}$$

Here, as before, $\overline{K_M}^c = \underline{K_M} \otimes \mathcal{O}_M(e^c)$.

That is, we have the following

Proposition-Definition 3.2.2. (Polyakov Variation Formula I) *With the same notation as above, we have the following equality*

$$h_{\mathrm{RR};\overline{K_M}';h_0(\overline{K_M}\otimes\mathcal{O}_M(e^c))}(\bar{L}) = h_{\mathrm{RR};\overline{K_M}';h_0(\overline{K_M})}(\bar{L}) \cdot e^{\frac{2q-2}{12}\cdot c}. \tag{3.2.4}$$

Easily we get the following

Proposition 3.2.3. (Serre Isometry) *With the same notation as above, we get the isometry:*

$$\Big(\lambda(L), h_{\mathrm{RR};\overline{K_M}';h_0(\overline{K_M})}(\bar{L})\Big) \simeq \Big(\lambda(K_M \otimes L^{\otimes -1}), h_{\mathrm{RR};\overline{K_M}';h_0(\overline{K_M})}(\overline{K_M}' \otimes \bar{L}^{\otimes -1})\Big). \tag{3.2.5}$$

(3.3) In (3.1) and (3.2), for a **fixed** normalized volume form ω on M, we introduce $h_{\mathrm{RR};\overline{K_M}';h_0(\overline{K_M})}(\bar{L})$ in such a way that if one of $h_0(\overline{K_M}'')$ is fixed, then all other determinant metrics $h_{\mathrm{RR};\overline{K_M}';h_0(\overline{K_M})}(\bar{L})$ are fixed, by using (3.2.2) and (3.2.4), or better Proposition 3.2.1 and Proposition 3.2.2.

Now we explain how the ω-Riemann-Roch metrics depend on ω.

Proposition 3.3.1. (Mean Value Lemma III) *With the same notation and normalization as above, for any two normalized volume forms ω_1 and ω_2 on M, we get the following isometries:*

(a) (Polyakov Variation Formula II)

$$\underline{\lambda(\underline{K_{M_{\omega_1}}})}_{\omega_1} \simeq \underline{\lambda(\underline{K_{M_{\omega_2}}})}_{\omega_2}; \tag{3.3.2}$$

(b) For all $n_j \in \mathbb{Z}$ and $Q_j \in M$,

$$\underline{\lambda(\underline{\mathcal{O}_M(\Sigma_j n_j Q_j)}_{\omega_1})}_{\omega_1} \simeq \underline{\lambda(\underline{\mathcal{O}_M(\Sigma_j n_j Q_j)}_{\omega_2})}_{\omega_2}. \tag{3.3.3}$$

§4. ω-Faltings metric

(4.1) This approach begins with the following condition:

(F0) With respect to the normalized volume ω associated to a quasi-hyperbolic metric $d\mu$ on a compact Riemann surface M, the metric $h_{\mathrm{RR};\underline{K_M};h_0(\underline{K_M})}$ on $\lambda(K_M)$ is defined to be the determinant of the Hermitian metric on $H^0(M, K_M)$ induced from the following natural pairing

$$(\phi, \psi) \mapsto \frac{\sqrt{-1}}{2}\int_M \phi \wedge \bar{\psi}. \tag{4.1.1}$$

Now we may improve Proposition 3.1.1 as follows.

Theorem 4.1.1. *With respect to the normalized volume ω associated to a quasi-hyperbolic metric on a compact Riemann surface M, for any ω-admissible Hermitian line bundle $\bar{L}$, there exists a unique metric $h_{\mathrm{RR};\underline{K_M};h_0(\underline{K_M})}(\bar{L})$, denoted also by $h_{F;\omega}(\bar{L})$ and called the ω-Faltings metric, on $\lambda(L)$ such that conditions (F0) $\sim$ (F5) are satisfied. Moreover, we have the following Riemann-Roch isometry:*

$$\Big(\lambda(L), h_{F;\omega}(\bar{L})\Big)^{\otimes 2} \otimes \Big(\lambda(\mathcal{O}_M), h_{F;\omega}(\underline{\mathcal{O}_M})\Big)^{\otimes -2} \simeq \langle \bar{L}, \bar{L} \otimes \underline{K_M}^{\otimes -1}\rangle. \tag{4.1.2}$$

(4.2) In this section, we give further properties for the ω-Faltings metrics.

First of all, by definition, we have the following;

Fact 4.2.1. *With the same notation as above, there exists a natural isometry*

$$\Big(\lambda(K_M), h_{F;\omega}(\underline{K_M}_{\omega})\Big) \simeq \Big(\lambda(K_M), h_{F;\omega_{can}}(\underline{K_M}_{\omega_{can}})\Big). \tag{4.2.1}$$

On the other hand, for general points $(Q_1, \ldots, Q_q, Q) \in M^{q+1}$ such that $H^0(M, \mathcal{O}_M(Q_1 + \cdots + Q_q - Q)) = H^1(M, \mathcal{O}_M(Q_1 + \cdots + Q_q - Q)) = \{0\}$, $\lambda(\mathcal{O}_M(Q_1 + \cdots + Q_q - Q))$ is simply $\mathbb{C}$, and the norm 1 in $\mathbb{C}$ is propositional to $\|\theta(Q_1 + \cdots + Q_r - Q)\|$, so that the ratio is independent of $(Q_1, \ldots, Q_q, Q)$. Such a ratio gives an invariant associated to (M, ω). Following Faltings, we define the *ω-Faltings delta function* $\delta(M, \omega)$ by

$$\|1\|_{h_{F;\omega}(\underline{\mathcal{O}_M(Q_1+\cdots+Q_q-Q)})} = e^{-\delta(M;\omega)/8}\|\theta(Q_1 + \cdots + Q_q - Q)\|. \tag{4.2.2}$$

Proposition 4.2.2. *With the same notation as above, we have*

$$\delta(M;\omega) = \delta(M;\omega_{\mathrm{can}})(= \delta(M)). \tag{4.2.3}$$

That is, ω-Faltings delta function $\delta(M;\omega)$ is the same as the original Faltings delta function $\delta(M)$.

Remark 4.2.1. We sometimes call Fact 4.2.1 and Proposition 4.2.2 Mean Value Lemmas too.

(4.3) With the above definition of ω-Faltings metric, we also have the Noether isometry without any further difficulty. Following Faltings [Fa] and Moret-Bailly [MB], with arithmetic applications in mind, we then have the following

Theorem 4.3.1. (ω-Noether isometry) *With respect to the normalized volume ω (associated to a quasi-hyperbolic metric) on a compact Riemann surface M, for any ω-admissible Hermitian line bundle $\bar{L}$, we have the following isometry:*

$$\Big(\lambda(L), h_{F;\omega}(\bar{L})\Big)^{\otimes 12} \simeq \langle \bar{L}, \bar{L} \otimes \underline{K_M}^{\otimes -1}\rangle^{\otimes 6} \otimes \langle \underline{K_M}, \underline{K_M}\rangle \otimes \mathcal{O}(e^{\delta(M)} \cdot (2\pi)^{-4q}). \tag{4.3.1}$$

§5. New metrics on determinants of cohomology for singular metrics

(5.1) For any normalized volume form ω on M, by §4, there exists an ω-Faltings metric $h_{F,\omega}(\bar{L})$ on $\lambda(L)$ for any ω-admissible metric $\bar{L}$ on M. In particular, we have the following ω-Noether isometry:

$$\Big(\lambda(L), h_{F;\omega}(\bar{L})\Big)^{\otimes 12} \simeq \langle \bar{L}, \bar{L} \otimes \underline{K_M}^{\otimes -1}\rangle^{\otimes 6} \otimes \langle \underline{K_M}, \underline{K_M}\rangle \otimes \mathcal{O}(e^{\delta(M)} \cdot (2\pi)^{-4q}). \tag{5.1.1}$$

Motivated by the arithmetic Deligne-Riemann-Roch and (5.1.1), for $\bar{L}$, with respect to any ω-admissible $\overline{K_M}$, define a new metric $h_{\overline{K_M}}(\bar{L})$ on $\lambda(L)$ by the Noether isometry

$$\Big(\lambda(L), h_{\overline{K_M}}(\bar{L})\Big)^{\otimes 12} \simeq \langle \bar{L}, \bar{L} \otimes \overline{K_M}^{\otimes -1} \rangle^{\otimes 6} \otimes \langle \overline{K_M}, \overline{K_M} \rangle \otimes \mathcal{O}(e^{a(q)}). \qquad (5.1.2)$$

Here $a(q)$ denotes the Deligne constant which is known to be $a(0)(1-q)$ with $a(0) = 24\zeta'_{\mathbb{Q}}(-1) - 1$. (See e.g. [De] and [We2].) Easily, one sees that such a definition is compactible with the normalization process given in §4 and the results for smooth volume forms. That is to say, we have the Polyakov variation formula, the Mean Value Lemma and

$$h_{\underline{K_M}}(\underline{K_M}) = h_{F,\omega}(\underline{K_M}) \cdot e^{-\delta(M,\omega)/12} \cdot (2\pi)^{4q/12} \cdot e^{a(q)/12}. \qquad (5.1.3)$$

(5.2) By the Noether isomorphism, which is equivalent to the Mumford isomorphism and the Riemann-Roch isomorphism, and by the adjunction isomorphism induced from the adjunction formula, we have the followng isomorphism;

$$\lambda(\mathcal{O}_M)^{\otimes 12} \simeq \langle K_M(P_1 + \cdots + P_N), K_M(P_1 + \cdots + P_N) \rangle \otimes \Delta_1 \otimes \Delta_2^{\otimes -2}. \qquad (5.2.1)$$

Here,

$$\Delta_1 := \otimes_{k=1}^N \langle \mathcal{O}_M(P_k), \mathcal{O}_M(P_k) \rangle (= \otimes_{k=1}^N \langle K_M, \mathcal{O}_M(P_k) \rangle^{\otimes -1}), \qquad (5.2.2)$$

and

$$\Delta_2 := \otimes_{1 \leq i < j \leq N} \langle \mathcal{O}_M(P_i), \mathcal{O}_M(P_j) \rangle. \qquad (5.2.3)$$

For our own convenience, we also let

$$\Delta_0 := \langle K_M(P_1 + \cdots + P_N), K_M(P_1 + \cdots + P_N) \rangle. \qquad (5.2.4)$$

Then, we get

Proposition 5.2.1. (Noether Isomorphism) *With the same notation as above, for all line bundles L on M, we have*

$$\lambda(L)^{\otimes 12} \simeq \Delta_0 \otimes \Delta_1 \otimes \Delta_2^{\otimes -2} \otimes \langle L, L \otimes K_M^{\otimes -1} \rangle^{\otimes 6}. \qquad (5.2.5)$$

Thus, if we define the Mumford line bundle (for punctures Riemann surface M^0) by

$$\begin{aligned}\lambda_n :=& \lambda(K_M^{\otimes n} \otimes (\mathcal{O}_M(P_1+\cdots+P_N))^{\otimes n-1}), \quad \text{if } n>0;\\ & \lambda(\mathcal{O}_M), \quad \text{if } n=0;\\ & \lambda((K_M(P_1+\cdots+P_N))^{\otimes n}), \quad \text{if } n<0,\end{aligned} \tag{5.2.6}$$

then by a tedious calculation, we have the following

Theorem 5.2.2. (Generalized Mumford Relations) *With the same notation as above, for all positive integers n, we have the following isomorphisms:*

(a) $\lambda_n \simeq \lambda_{1-n}$;

(b) $\lambda_n^{\otimes 12} \simeq \Delta_0^{\otimes(6n^2-6n+1)} \otimes \Delta_1 \otimes \Delta_2^{\otimes 10-12n}$, *and*

(c) $\lambda_n \simeq \lambda_0^{\otimes(6n^2-6n+1)} \otimes \Delta_1^{\otimes -\frac{n(n-1)}{2}} \otimes \Delta_2^{\otimes(n-1)^2}$.

In particular, if $N=1$, we have $\Delta_2 = \mathcal{O}$, hence in this case we get

$$\lambda_n^{\otimes 12} \simeq \Delta_0^{\otimes(6n^2-6n+1)} \otimes \Delta_1, \tag{5.2.7}$$

and

$$\lambda_n \simeq \lambda_0^{\otimes(6n^2-6n+1)} \otimes \Delta_1^{\otimes -\frac{n(n-1)}{2}}, \tag{5.2.8}$$

for all positive integer n. Moreover, it is well-known that the moduli space $\mathcal{M}_{q,1}$ of punctured Riemann surfaces with signature $(q,1)$ can be viewed as the universal curve over the moduli space $\mathcal{M}_q$ of compact Riemann surfaces of genus q. Hence we have a natural geometric interpretation for $\Delta_1 \otimes \Delta_2^{-1} (= \Delta_1)$, i.e., Δ_1 is the relative tangent bundle of the universal curve over $\mathcal{M}_q$. (See e.g., [TZ2].)

(5.3) Now we give the counter part of the metric theory for the discussion in (5.2). We start with some preperations.

For a normalized volume form ω on M, define the following metrized lines:

$$\begin{aligned}\underline{\lambda_n} :=& \Big(\lambda_n, h_{\underline{K_M}}(\underline{K_M}^{\otimes n} \otimes \mathcal{O}_M(\underline{P_1+\cdots+P_N})^{\otimes n-1})\Big), \text{ if } n>0;\\ :=& \Big(\lambda_0, h_{\underline{K_M}}(\underline{\mathcal{O}_M}), \text{ if } n=0;\\ :=& \Big(\lambda_n, h_{\underline{K_M}}((\underline{K_M}(\underline{P_1+\cdots+P_N}))^{\otimes n}\Big), \text{ if } n<0.\end{aligned} \tag{5.3.1}$$

$$
\begin{aligned}
\underline{\Delta_n} :=& \langle \underline{K_M(P_1+\cdots+P_N)}, \underline{K_M(P_1+\cdots+P_N)} \rangle, \text{ if } n=0;\\
:=& \otimes_{k=1}^{N} \langle \underline{\mathcal{O}_M(P_k)}, \underline{\mathcal{O}_M(P_k)} \rangle, \text{ if } n=1;\\
:=& \otimes_{1\leq i<j\leq N} \langle \underline{\mathcal{O}_M(P_i)}, \underline{\mathcal{O}_M(P_j)} \rangle, \text{ if } n=2.
\end{aligned}
\tag{5.3.2}
$$

Then we get the following

Theorem 5.3.1. *With the same notation as above, for any positive integer n, we have the following isometries:*

(a) (Serre isometry)

$$\underline{\lambda_n} \simeq \underline{\lambda_{1-n}};$$

(b) (Generalized Mumford isometry)

$$\underline{\lambda_n}^{\otimes 12} \simeq \underline{\Delta_0}^{\otimes 6n^2-6n+1} \otimes \underline{\Delta_1} \otimes \underline{\Delta_2}^{\otimes -12+10} \otimes \mathcal{O}(e^{a(g)});$$

(c) (Generalized Mumford isometry)

$$\underline{\lambda_n} \simeq \underline{\lambda_1}^{\otimes 6n^2-6n+1} \otimes \underline{\Delta_1}^{\otimes -\frac{n(n-1)}{2}} \otimes \underline{\Delta_2}^{\otimes (n-1)^2} \otimes \mathcal{O}(e^{-\frac{n(n-1)}{2}\cdot a(g)}).$$

(5.4) More generally, with the application to the moduli problems in mind, we in this subsection give a generalization for (5.3). As in (5.3), we always fix a normalized volume form ω on M.

For an $n+1$-tuple of real numbers $(\alpha;\beta_1,\ldots,\beta_N)$, define the associated metrized lines as follows:

$$
\begin{aligned}
\overline{\lambda_n}^{\alpha;\beta} :=& \Big(\lambda_n, h_{\overline{K_M}^{\alpha}}((\overline{K_M}^{\alpha})^{\otimes n} \otimes \mathcal{O}_M(\overline{P_1}^{\beta_1}+\cdots+\overline{P_N}^{\beta_N})^{\otimes n-1})\Big), \text{ if } n>0;\\
:=& \Big(\lambda_0, h_{\overline{K_M}^{\alpha}}(\underline{\mathcal{O}_M})\Big), \text{ if } n=0;\\
:=& \Big(\lambda_n, h_{\overline{K_M}^{\alpha}}((\overline{K_M}^{\alpha}(\overline{P_1}^{\beta_1}+\cdots+\overline{P_N}^{\beta_N}))^{\otimes n})\Big), \text{ if } n<0;
\end{aligned}
\tag{5.4.1}
$$

and

$$
\begin{aligned}
\overline{\Delta_n}^{\alpha;\beta} :=& \langle \overline{K_M}^{\alpha}(\overline{P_1}^{\beta_1}+\cdots+\overline{P_N}^{\beta_N}), \overline{K_M}^{\alpha}(\overline{P_1}^{\beta_1}+\cdots+\overline{P_N}^{\beta_N}) \rangle, \text{ if } n=0;\\
:=& \langle \overline{K_M}^{\alpha}, \mathcal{O}_M(\overline{P_1}^{\beta_1}+\cdots+\overline{P_N}^{\beta_N}) \rangle^{\otimes -1}, \text{ if } n=1;\\
:=& \langle \overline{K_M}^{\alpha}(\overline{P_1}^{\beta_1}+\cdots+\overline{P_N}^{\beta_N}), \mathcal{O}_M(\overline{P_1}^{\beta_1}+\cdots+\overline{P_N}^{\beta_N}) \rangle^{\otimes \frac{1}{2}}, \text{ if } n=2.
\end{aligned}
\tag{5.4.2}
$$

Then we get the following

Theorem 5.4.1. *With the same notation as above, for any positive integer n, we have the following isometries:*

(a) (Serre isometry)

$$\overline{\lambda_n}^{\alpha;\beta} \simeq \overline{\lambda_{1-n}}^{\alpha;\beta};$$

(b) (Generalized Mumford isometry)

$$(\overline{\lambda_n}^{\alpha;\beta})^{\otimes 12} \simeq (\overline{\Delta_0}^{\alpha;\beta})^{\otimes 6n^2-6n+1} \otimes (\overline{\Delta_1}^{\alpha;\beta}) \otimes (\overline{\Delta_2}^{\alpha;\beta})^{\otimes -12+10} \otimes \mathcal{O}(e^{a(q)});$$

(c) (Generalized Mumford isometry)

$$\overline{\lambda_n}^{\alpha;\beta} \simeq (\overline{\lambda_1}^{\alpha;\beta})^{\otimes 6n^2-6n+1} \otimes (\overline{\Delta_1}^{\alpha;\beta})^{\otimes -\frac{n(n-1)}{2}} \otimes (\overline{\Delta_2}^{\alpha;\beta})^{\otimes (n-1)^2} \otimes \mathcal{O}(e^{-\frac{n(n-1)}{2}\cdot a(q)}).$$

§6. A geometric interpretation of our new metrics

(6.1) In this chapter, we will give a geometric interpretation for our new metrics on determinants of cohomology. We start with a discussion on hyperbolic metrics on punctured Riemann surfaces.

As before, denote by $\omega_{\rm hyp}$ the normalized volume form associated to the standard hyperbolic metric $\tau^0_{\rm hyp}$ on a punctured Riemann surface M^0 of signature (q,N). Thus, in particular, if we denote the corresponding volume form (with respect to $\tau^0_{\rm hyp}$) by $d\mu_{\rm hyp}$, then $\int_{M^0} d\mu_{\rm hyp} = 2\pi(2q-2+N)$, and $2\pi(2q-2+N)\omega_{\rm hyp} = d\mu_{\rm hyp}$.

For $\tau^0_{\rm hyp}$, or equivalently for $d\mu_{\rm hyp}$ on M^0, if we view them as a singular metric on M, the compactification of M^0, then the natural line bundle we should attach to it is the so-called logarithmic tangent bundle $T_M\langle\log D\rangle$. Here D denotes the divisr at infinity, i.e., $P_1+\cdots+P_N$. (See e.g., [Mu] or [Fu]). Over the compact Riemann surface M, we see that $T_M\langle\log D\rangle$ is nothing but the dual of the line bundle $K_M(P_1+\cdots+P_N)$. Here as before K_M denotes the canonical line bundle of M. So if we denote the induced Hermitian metric from $\tau^0_{\rm hyp}$ on $K_M(P_1+\cdots+P_N)$ by $\tau^\vee_{{\rm hyp};K_M(D)}$, we get the following Einstein equation

$$c_1\Big(K_M(P_1+\cdots+P_N),\tau^\vee_{{\rm hyp};K_M(D)}\Big) = d\mu_{\rm hyp} = (2q-2+N)\omega_{\rm hyp}. \qquad (6.1.1)$$

We are not quite satisfied with this, as the metric discussed above only has its nice meaning on the logarithmic tangent bundle. We believe that there should have a natural metric $\rho_{\mathrm{hyp};K_M}$ on K_M and natural metrics $\rho_{\mathrm{hyp};P_i}$ on $\mathcal{O}_M(P_i)$, $i=1,\dots,N$, associated to punctures, for the hyperbolic metric. More precisely, the picture we have in mind is that these metrics should be very natural in the following sense:

(i) they are ω_{hyp}-admissible;

(ii) they give the following identity of metrics

$$\rho_{\mathrm{hyp};K_M} \otimes \rho_{\mathrm{hyp};P_1} \otimes \cdots \otimes \rho_{\mathrm{hyp};P_N} = \tau^{\vee}_{\mathrm{hyp};K_M(D)} \tag{6.1.2}$$

on $K_M(P_1+\cdots+P_N)$;

(iii) they should obey the residue isometry, i.e., we have the isometry

$$(K_M(P_i), \rho_{\mathrm{hyp};K_M} \otimes \rho_{\mathrm{hyp};P_i})|_{P_i} \simeq \underline{\mathbb{C}} \tag{6.1.3}$$

for all $i=1,\dots,N$.

Before defining the above metrics on K_M and on $\mathcal{O}_M(P_i)$, $i=1,\dots N$, respectively, motivated by our work for admissible theory for smooth volume forms in [We1], we now introduce an invariant $A_{\mathrm{Ar,hyp}}(M^0)$, the Arakelov-Poincaré volume, associated to a punctured Riemann surface M^0 as follows.

First of all, following Selberg, define the so-called Selberg zeta function $Z_{M^0}(s)$ of M^0 for $\mathrm{Re}(s)>1$ by the absolutely convergent product

$$Z_{M^0}(s) := \prod_{\{l\}} \prod_{m=0}^{\infty} (1-e^{-(s+m)|l|}), \tag{6.1.4}$$

where l runs over the set of all simple closed geodesics on M^0 with respect to the hyperbolic metric $d\mu_{\mathrm{hyp}}$ on M^0, and $|l|$ denotes the length of l. It is known that by using Selberg trace formula for weight zero forms the function $Z_{M^0}(s)$ admits a meromorphic continuation to the whole complex s-plane which has a simple zero at $s=1$. Secondly, motivated by the work of D'Hoker-Phong and Sanark in [D'HP] and [Sa], we introduce the following factorization for the Selberg zeta function:

$$Z_{M^0}(s) =: \det(\Delta_{\mathrm{hyp}} + s(s-1)) \cdot \mathbb{N}(s)^{2q-2+N}. \tag{6.1.5}$$

Here $\Delta_{\rm hyp}$ denotes the hyperbolic Laplacian on M^0, $\mathbb{N}(s)$ denotes the function

$$\mathbb{N}(s) := \frac{e^{-E+s(s-1)}}{2\pi^s} \cdot \frac{\Gamma(s)}{(\Gamma_2(s))^2} \tag{6.1.6}$$

with $E = -\frac{1}{4} - \frac{1}{2}\log 2\pi + 2\zeta_{\mathbb{Q}}'(-1)$, $\Gamma(s)$ the ordinary gamma function, and $\Gamma_2(s)$ the Barnes double gamma funtion. Thirdly, define the regularized determinant for the Laplacian $\Delta_{\rm hyp}$ by

$$\det{}^*(\Delta_{\rm hyp}) := \frac{d}{ds}\Big(\det(\Delta_{\rm hyp} + s(s-1))\Big)\Big|_{s=1}. \tag{6.1.7}$$

(Please carefully compare this definition of the regularized determinant for the Laplacian with the one proposed by Efrat in the one page correction of [Ef].) Finally, following [We1], define the *Arakelov-Poinceré volume* $A_{\rm Ar,hyp}(M^0)$ for M^0 via the formula:

$$\log A_{\rm Ar,hyp}(M^0) := a_{\rm hyp} := \frac{12}{2} \cdot \frac{1}{2q-2} \cdot \Big(\log \frac{\det^*\Delta_{\rm Ar}}{A_{\rm Ar}(M)} - \log \frac{\det^*\Delta_{\rm hyp}}{2\pi(2q-2)}\Big). \tag{6.1.8}$$

Here $\Delta_{\rm Ar}$ denotes the Laplacian for the Arakelov metric on M, $A_{\rm Ar}(M)$ denotes the volume of M with respect to the Arakelov metric.

Remark 6.1.2. Obviously, the Arakelov-Poincaré volume is a very natural invariant for the punctured Riemann surface M^0, hence can be viewed as a certain interesting function on the Teichmüller space $T_{q,N}$ of punctured Riemann surfaces of signature (q, N).

(6.2) With the Arakelov-Poincaré volume for M^0, now we are ready to introduce the above mentioned metrics on K_M and $\mathcal{O}_M(P_i)$, $i = 1, \ldots, N$.

First of all by the $\omega_{\rm hyp}$-admissible condition 6.1.(i), we see that these metrics on K_M and on $\mathcal{O}_M(P_i)$, $i = 1, \ldots, N$ should be propotional to the corresponding $\omega_{\rm hyp}$-Arakelov metrics on K_M and on $\mathcal{O}_M(P_i)$, $i = 1, \ldots, N$, respectively. With this in mind, we define the proposed metric on K_M by multiplying the $\omega_{\rm hyp}$-Arakelov canonical line bundle $\underline{K_M}_{\omega_{\rm hyp}}$ the factor $A_{\rm Ar,hyp}(M^0)$. Denote the resulting Hermitian line bundle by $\underline{K_M}_{\rm hyp}$. Then, we have

$$\underline{K_M}_{\rm hyp} = \underline{K_M}_{\omega_{\rm hyp}} \cdot A_{\rm Ar,hyp}(M^0), \tag{6.2.1}$$

or equivalently,

$$\rho_{{\rm hyp};K_M} = \rho_{\omega_{\rm hyp};K_M} \cdot A_{\rm Ar,hyp}(M^0). \tag{6.2.2}$$

Secondly, by (6.1.2), we only need to indicate how the metrics are defined on the line bundles $\mathcal{O}_M(P_i)$ for punctures P_i, $i = 1, \ldots, N$. Since we now believe that for our theory of metrics, the punctures should have equal contributions. Hence we assume that the (resulting constant) ratio

$$C_{\mathrm{hyp}}^i := e^{c_{\mathrm{hyp}}^i} := \rho_{\mathrm{hyp};P_i} / \rho_{\mathrm{Ar};\omega_{\mathrm{hyp}};P_i} \tag{6.2.3}$$

does not depend on i. Thus condition (6.1.2), which says that $\underline{K_M(P_1 + \cdots + P_N)}$ multiplying by $e^{a_{\mathrm{hyp}}+c_{\mathrm{hyp}}^1+\cdots+c_{\mathrm{hyp}}^N}$ is isometric to $K(P_1 + \cdots + P_N)$ together with the natural metric $\tau_{\mathrm{hyp};K_M(P_1+\cdots+P_N)}^\vee$ induced from τ_{hyp} on M^0, determines the constant $c_{\mathrm{hyp}} := c_{\mathrm{hyp}}^i$, $i = 1, \ldots, N$ and hence the metrics on $\mathcal{O}_M(P_i)$, $i = 1, \ldots, N$, uniquely. From now on, we always assume that the constants c_{hyp}^i, $i = 1, \ldots, N$, are defined in this way.

(6.3) Before finally giving the geometric interpretation for our metric on the determinant of cohomology, we in this subsection using the result in (5.4) give the Mumford type isometry for hyperbolic metrics, by setting $(\alpha; \beta_1, \ldots, \beta_N)$ to be $(a_{\mathrm{hyp}}; c_{\mathrm{hyp}}^1, \ldots, c_{\mathrm{hyp}}^N)$. We will denote the corresponding Hermitian line bundles by the underline with the lower index hyp, e.g., $\underline{\lambda_n}_{\mathrm{hyp}}$, $\underline{\Delta_n}_{\mathrm{hyp}}$, etc..

Theorem 6.3.1. *With the same notation as above, for any positive integer n, we have the following isometries:*

(a) (Serre isometry)

$$\underline{\lambda_n}_{\mathrm{hyp}} \simeq \underline{\lambda_{1-n}}_{\mathrm{hyp}};$$

(b) (Generalized Mumford isometry)

$$\underline{\lambda_n}_{\mathrm{hyp}}^{\otimes 12} \simeq \underline{\Delta_0}_{\mathrm{hyp}}^{\otimes 6n^2-6n+1} \otimes \underline{\Delta_1}_{\mathrm{hyp}} \otimes \underline{\Delta_2}_{\mathrm{hyp}}^{\otimes -12n+10} \otimes \mathcal{O}(e^{a(q)});$$

(c) (Generalized Mumford isometry)

$$\underline{\lambda_n}_{\mathrm{hyp}} \simeq \underline{\lambda_1}_{\mathrm{hyp}}^{\otimes 6n^2-6n+1} \otimes \underline{\Delta_1}_{\mathrm{hyp}}^{\otimes -\frac{n(n-1)}{2}} \otimes \underline{\Delta_2}_{\mathrm{hyp}}^{\otimes (n-1)^2} \otimes \mathcal{O}(e^{-\frac{n(n-1)}{2}\cdot a(q)}).$$

Obviously, even though we only discuss our metrics for a single curve, but the technique can be globalized so that we get metrized holomorphic line bundles on the base, the Teichmüller space $T_{q,N}$ of punctured Riemann surfaces of signature (q, N), which may naturally decend to the moduli space $\mathcal{M}_{q,N}$ of punctured Riemann surfaces of signature (q, N). Moreover, as

$$\underline{K_M(P_1+\cdots+P_N)}_{\mathrm{hyp}} \simeq (K_M(D), \tau^\vee_{\mathrm{hyp};K_M(D)}), \tag{6.3.1}$$

by a work of Wolpert [Wo], we know that

$$c_1(\underline{\Delta_0}_{\mathrm{hyp}}) = \frac{\omega_{\mathrm{WP}}}{\pi^2}. \tag{6.3.2}$$

Here ω_{WP} denotes the Weil-Petersson Kähler form. Thus in particular, we have the following:

Corollary 6.3.2. *With the same notation as above, for all positive integers n, we have the following identities of (1,1)-forms on $T_{q,N}$ and hence on $\mathcal{M}_{q,N}$:*

$$12\, c_1(\underline{\lambda_n}_{\mathrm{hyp}}) = (6n^2-6n+1)\frac{\omega_{\mathrm{WP}}}{\pi^2} + c_1(\underline{\Delta_1}_{\mathrm{hyp}}) - (12n-10)c_1(\underline{\Delta_2}_{\mathrm{hyp}}). \tag{6.3.3}$$

(6.4) The geometric interpretation of our metrics on determinants of cohomology is given in terms of the new metric on $\lambda(K_M)$ with respect to the hyperbolic metric.

Realize M^0 as a quotient $\Gamma\backslash\mathcal{H}$ of the upper half-plane by the action of a torsion free finitely generated Fuchsian group Γ. Then it is well-known that $\Gamma \subset PSL(2,\mathbb{R})$ is generated by $2q$ hyperbolic transformations $A_1, B_1, \ldots, A_q, B_q$ and N parabolic transformtions $S_1, \ldots, S_N$ satisfying the single relation

$$A_1B_1A_1^{-1}B_1^{-1}\ldots A_qB_qA_q^{-1}B_q^{-1}S_1\ldots S_N = 1.$$

Choose a normalized basis of abelian differentials $\psi_1, \ldots, \psi_q$, i.e., a basis of the vector space $H^0(M, K_M)$ so that

$$\int_z^{A_i z} \psi_j(w)dw = \delta_{ij}, \qquad \int_z^{B_i z} \psi_j(w)dw =: \tau_{ij}, \quad i,j = 1,\ldots,q,$$

with δ_{ij} the Kronecker symbol and $\tau = (\tau_{ij})$ the period matrix of M.

On $\lambda(K_M)$, choose the section $(\psi_1\wedge\cdots\wedge\psi_q)\otimes 1^\vee$, with 1 the canonical section of $H^1(M, K_M) \simeq \mathbb{C}$. Then we have the following

Theorem 6.4.1. *With the same notation as above, as the metric on $\lambda(K_M)$,*

$$\begin{aligned}&\langle(\psi_1\wedge\cdots\wedge\psi_q)\otimes 1^\vee,(\psi_1\wedge\cdots\wedge\psi_q)\otimes 1^\vee\rangle_{h_{\underline{K_M}_{\mathrm{hyp}}}(\underline{K_M}_{\mathrm{hyp}})}\\ =&\Big(\det(\mathrm{Im}\tau)\cdot 2\pi(2q-2)\Big)\cdot(\det{}^*(\Delta_{\mathrm{hyp}}))^{-1}.\end{aligned}$$

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