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ON THE EXTENSION PROBLEM OF PLURICANONICAL FORMS

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Dedicated to Professor Friedrich Hirzebruch on his seventieth birthday

ABSTRACT. We review some recent development on the extension problem of pluricanonical forms from a divisor to the ambient space by Siu, Kawamata and Nakayama with simplified proofs.

1. INTRODUCTION

The purpose of this paper is to review some recent development on the extension problem of pluricanonical forms from a divisor to the ambient space. The main tools of the proofs are the multiplier ideal sheaves and the vanishing theorems for them.

Let X be a compact complex manifold. The *m-genus* $P_m(X)$ of X for a positive integer m is defined by $P_m(X) = \dim H^0(X, mK_X)$. The growth order of the plurigenera for large m is called the *Kodaira dimension* $\kappa(X)$: we have $P_m(X) \sim m^{\kappa(X)}$ for any sufficiently large and divisible m. We have the following possibilities: $\kappa(X) = -\infty, 0, 1, \cdots$, or $\dim X$. In particular, if $\kappa(X) = \dim X$, then X is said to be of *general type*. It is important to note that these invariants are independent of the birational models of X.

The plurigenera are fundamental discrete invariants for the classification of algebraic varieties. But they are by definition not topological invariants. However, in order that such classification is reasonable, the following conjecture due to Iitaka should be true:

Conjecture 1.1. *Let S be an algebraic variety, and let $f : \mathcal{X} \to S$ be a smooth algebraic morphism. Then the plurigenus $P_m(X_t)$ is constant on $t \in S$ for any positive integer m.*

A morphism between complex varieties which is birationally equivalent to a projective morphism will be called an *algebraic morphism* in this paper. The algebraicity assumption in the conjecture is slightly weaker than the projectivity.

This conjecture is confirmed by Iitaka [I1, I2] in the case in which $\dim X_t = 2$ by using the classification theory of surfaces. Nakayama [N1] proved that the conjecture follows if the minimal model exists for the family and the abundance

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conjecture holds for the generic fiber. Thus the conjecture is true if dim $X_t = 3$ by [K4] and [KM].

On the other hand, Nakamura [Nm] provided a counterexample for the generalization of the conjecture in the case where the morphism f is not algebraic. In his example, the central fiber X_0 is a quotient of a 3-dimensional simply connected solvable Lie group by a discrete subgroup. We note that X_0 is a non-Kähler manifold which has non-closed holomorphic 1-forms. So we only consider algebraic morphisms in this paper. It is interesting to extend our results to the case in which the fibers are in Fujiki's class $\mathcal{C}$.

The following theorem of Siu was the starting point of the recent progress on this conjecture which we shall review.

Theorem 1.2. *[Si] Let S be a complex variety, and let $f : \mathcal{X} \to S$ be a smooth projective morphism. Assume that the generic fiber X_η of f is a variety of general type. Then the plurigenus $P_m(X_t)$ is constant on $t \in S$ for any positive integer m.*

We have also a slightly stronger version:

Theorem 1.2'. *[K5] Let S be an algebraic variety, let $\mathcal{X}$ be a complex variety, and let $f : \mathcal{X} \to S$ be a proper flat algebraic morphism. Assume that the fibers $X_t = f^{-1}(t)$ have only canonical singularities for any $t \in S$ and that the generic fiber X_η is a variety of general type. Then the plurigenus $P_m(X_t)$ is constant on $t \in S$ for any positive integer m.*

According to Nakayama [N2], we define the *numerical Kodaira dimension* $\nu(X)$ as follows (this is $\kappa_\sigma(X)$ in [N2]; there is another version $\kappa_\nu(X)$ of numerical Kodaira dimension in [N2] which we do not use). Let X be a compact complex manifold and let k be a nonnegative integer. We define $\nu(X) \geq k$ if there exist a divisor H on X and a positive number c such that dim $H^0(X, mK_X + H) \geq cm^k$ for any sufficiently large and divisible m. If there is no such k, then we put $\nu(X) = -\infty$. It is easy to see that $\kappa(X) \leq \nu(X) \leq$ dim X. By the Kodaira lemma, $\kappa(X) =$ dim X if and only if $\nu(X) =$ dim X. The *abundance conjecture* states that the equality $\kappa(X) = \nu(X)$ always holds. Nakayama confirmed this conjecture in the case $\nu(X) = 0$ ([N2]).

By considering $mK_X + H$ instead of mK_X, Nakayama obtained the following:

Theorem 1.3. *[N3] Let S be an algebraic variety, let $\mathcal{X}$ be a complex veriety, and let $f : \mathcal{X} \to S$ be a proper flat algebraic morphism. Assume that the fibers $X_t = f^{-1}(t)$ have only canonical singularities for any $t \in S$. Then the numerical Kodaira dimension $\nu(X_t)$ is constant on $t \in S$. In particular, if one fiber X_0 is of general type, then so are all the fibers.*

For the finer classification of algebraic varieties, it is useful to consider not only the discrete invariants $P_m(X)$ but also the infinite sum of vector spaces

$$R(X) = \bigoplus_{m \geq 0} H^0(X, mK_X)$$

which has a natural graded ring structure over $\mathbb{C} = H^0(X, \mathcal{O}_X)$. This continuous invariant $R(X)$, called the *canonical ring* of X, is also independent of the birational models of X. It is conjectured that R(X) is always finitely generated as a graded $\mathbb{C}$-algebra. If this is the case, then Proj $R(X)$ is called a *canonical model* of X.

A *canonical singularity* (resp. *terminal singularity*) is defined as a singularity which may appear on a canonical model of a variety of general type whose canonical ring is finitely generated (resp. on a minimal model of an algebraic variety). The formal definition by Reid is as follows: a normal variety X is said to have only canonical singularities (resp. terminal singularities) if the canonical divisor K_X is $\mathbb{Q}$-Cartier and, for a resolution of singularities $\mu : Y \to X$ which has exceptional divisors F_j, if we write $\mu^* K_X = K_Y + \sum_j a_j F_j$, then $a_j \leq 0$ (resp. $a_j < 0$) for all j.

For example, the canonical singularities in dimension 2 have been studied extensively. They are called in many names such as du Val singularities, rational double points, simple singularities, or A-D-E singularities. The terminal singularity in dimension 2 is smooth, and the terminal singularities in dimension 3 are classified by Mori and others (cf. [R2]).

Let us consider the subset of a Hilbert scheme with a given Hilbert polynomial which consists of points corresponding to the canonical models of varieties of general type. This set should be open from the view point of the moduli problem of varieties (cf. [V2]). The following is a local version of Theorem 1.2' and says that this is the case (this result was previously known up to dimension 2):

Theorem 1.4. *[K5] Let $f : \mathcal{X} \to B$ be a flat morphism from a germ of an algebraic variety to a germ of a smooth curve. Assume that the central fiber $X_0 = f^{-1}(P)$ has only canonical singularities. Then so has the total space $\mathcal{X}$ as well as any fiber X_t of f. Moreover, if $\mu : V \to \mathcal{X}$ is a birational morphism from a normal variety with the strict transform X of X_0, then $K_V + X \geq \mu^*(K_{\mathcal{X}} + X_0)$.*

The following theorem answers a similar question for the deformations of minimal models (this was previously known up to dimension 3):

Theorem 1.5. *[N3] Let $f : \mathcal{X} \to B$ be a flat morphism from a germ of an algebraic variety to a germ of a smooth curve. Assume that the central fiber $X_0 = f^{-1}(P)$ has only terminal singularities. Then so has the total space $\mathcal{X}$ as well as any fiber X_t of f. Moreover, if $\mu : V \to \mathcal{X}$ is a birational morphism from a normal variety with the strict transform X of X_0, then the support of $K_V + X - \mu^*(K_{\mathcal{X}} + X_0)$ contains all the exceptional divisors of μ.*

The following theorem, which is stronger than Theorem 1.2', says that only the abundance conjecture for the generic fiber implies the deformation invariance of the plurigenera:

Theorem 1.6. *[N3] Let S be an algebraic variety, let $\mathcal{X}$ be a complex variety, and let $f : \mathcal{X} \to S$ be a proper flat algebraic morphism. Assume that the fibers $X_t = f^{-1}(t)$ have only canonical singularities and that $\kappa(X_\eta) = \nu(X_\eta)$ for the generic fiber X_η of f. Then the plurigenus $P_m(X_t)$ is constant on $t \in S$ for any positive integer m.*

Now we explain the idea of the proofs. Since we assumed the algebraicity of varieties, there exist divisors which are big. Hence we can use the vanishing theorems of Kodaira type as in [K1] and [V1] (cf Theorem 2.6). Indeed, if K_{X_0} is nef and big for the central fiber X_0 in Theorem 1.2, then the extendability of pluricanonical forms follows immediately from the vanishing theorem.

Thus the problem is to extract the nef part from the big divisor K_{X_0}. This is similar to the *Zariski decomposition* problem (cf. [K3]): Let X be a smooth projective variety of general type. If we fix a positive integer m, then there exists a projective birational morphism $\mu_m : Y_m \to X$ such that $\mu_m^*(mK_X)$ is decomposed into the sum of the free part and the fixed part: $\mu_m^*(mK_X) = P_m + M_m$. If there exists one $\mu : Y \to X$ which serves as the μ_m simultaneously for all m, then $P = \sup_{m>0} P_m/m$ is the desired nef part, and the decomposition $\mu^* K_X = P + N$ in $\mathrm{Div}(Y) \otimes \mathbb{R}$ for $N = \inf_{m>0} N_m/m$ gives the Zariski decomposition of K_X. The difficulty arises when we have an infinite tower of blow-ups. It is known that if the Zariski decomposition of the canonical divisor exists, then the canonical ring $R(X)$ is finitely generated ([K3]).

So we use instead the concept of *multiplier ideal sheaf* which was first introduced by Nadel [Nd]. We consider the series of ideal sheaves on X_0 instead of the decompositions on the series of blow-ups. Since the structure sheaf of X_0 is noetherian, we do not have the difficulty of the infinity in this case; we take just the union of the ideals (cf. Definitions 2.5 and 2.10).

The remaining thing to be proved is the compatibility of the multiplier ideal sheaves on X_0 and on the total space $\mathcal{X}$ constructed similarly for $K_{\mathcal{X}}$. This is proved by a tricky induction on m discovered by Siu (cf. Lemma 3.6).

The theorems in the introduction will be reduced to Theorems A, B and C in §2, which will be proved by using vanishing theorems in §3.

We use the following terminology besides those in [KMM]. Let $f : X \to S$ be a morphism of algebraic varieties. A sheaf $\mathcal{F}$ on X is said to be *f-generated* if the natural homomorphism $f^* f_* \mathcal{F} \to \mathcal{F}$ is surjective. A Cartier divisor D on X is called *f-effective* (resp. *f-free*) if $f_* \mathcal{O}_X(D) \neq 0$ (resp. $\mathcal{O}_X(D)$ is f-generated). A $\mathbb{Q}$-Cartier divisor D on X is said to be *f-$\mathbb{Q}$-effective* (resp. *f-semi-ample*) if there exists a positive integer m such that mD is a f-effective (resp. f-free) Cartier divisor. A $\mathbb{Q}$-Cartier divisor D on X is said to be *f-pseudo-effective* if $D + H$ is f-$\mathbb{Q}$-effective for any f-ample $\mathbb{Q}$-Cartier divisor H.

All varieties and morphisms are defined over the complex number field $\mathbb{C}$ in this paper.

2. Main theorems

Setup 2.1. We fix the following notation in Theorems A, B and C below.

(1) V is a smooth algebraic variety.
(2) X is a smooth divisor on V.
(3) S is a germ of an algebraic variety.
(4) $\pi : V \to S$ is a projective morphism:

$$X \subset V \xrightarrow{\pi} S.$$

A divisor D on V will be called *π-effective for the pair (V, X)* if the natural homomorphism $\pi_* \mathcal{O}_V(D) \to \pi_* \mathcal{O}_X(D|_X)$ is not zero. It is called *π-$\mathbb{Q}$-effective for the pair (V, X)* if mD is π-effective for the pair (V, X) for some positive integer m. D is said to be *π-big for the pair (V, X)* if we can write $mD = A + B$ for a positive integer m, π-ample divisor A and a π-effective divisor B for the pair (V, X).

2.2 Theorem A. *Assume that $K_V + X$ is π-big for the pair (V, X). Then the natural homomorphism $\pi_*\mathcal{O}_V(m(K_V + X)) \to \pi_*\mathcal{O}_X(mK_X)$ is surjective for any positive integer m.*

Proof: Theorem A implies Theorem 1.4. By the resolution of singularities, we construct a projective birational morphism $\mu : V \to \mathcal{X}$ from a smooth variety such that the strict transform X of the central fiber X_0 is also smooth. We take $S = \mathcal{X}$ and $\pi = \mu$. Since π and $\pi|_X$ are birational morphisms, we can see that $K_V + X$ is π-big for the pair (V, X).

Let m be a positive integer such that mK_{X_0} is a Cartier divisor. Since X_0 has only canonical singularities, a nowhere vanishing section s_0 of $\mathcal{O}_{X_0}(mK_{X_0})$ lifts to a section of $\mathcal{O}_X(mK_X)$, which in turn extends to a section of $\mathcal{O}_V(m(K_V + X))$ by Theorem A. Therefore, s_0 extends to a nowhere vanishing section of $\mathcal{O}_{\mathcal{X}}(m(K_{\mathcal{X}} + X_0))$ which lifts to a section of $\mathcal{O}_V(m(K_V + X))$. Hence $K_{\mathcal{X}} + X_0$ is a $\mathbb{Q}$-Cartier divisor and $K_V + X \geq \mu^*(K_{\mathcal{X}} + X_0)$ in this case. Since any μ is dominated by some μ as above, we have the last assertion. Since $\mu^* X_0 \geq X$, it follows that $\mathcal{X}$ has only canonical singularities. □

Proof: Theorem A implies Theorem 1.2'. We may assume that S is a germ of a smooth curve. By the resolution of singularities, we construct a proper birational morphism $\mu : V \to \mathcal{X}$ from a smooth variety such that the strict transform X of the central fiber X_0 is also smooth and that $\pi = f \circ \mu$ is a projective morphism.

Let A be a π-very ample divisor on V. Since $K_{\mathcal{X}}$ is f-big, there exists a positive integer m_1 such that $f_*\mathcal{O}_{\mathcal{X}}(m_1 K_{\mathcal{X}} - \mu_* A) \neq 0$, although $\mu_* A$ is a Weil divisor which may not be a Cartier divisor. Therefore, there exists an effective Weil divisor $\bar{B}$ on $\mathcal{X}$ whose support does not contain X_0 such that $m_1 K_{\mathcal{X}} \sim \mu_* A + \bar{B}$. Then $K_V + X$ is π-big for the pair (V, X) by the last assertion of Theorem 1.4. Since $\mathcal{X}$ has only canonical singularities by Theorem 1.4, Theorem A implies that the natural homomorphism $f_*\mathcal{O}_{\mathcal{X}}(mK_{\mathcal{X}}) \to H^0(X_0, mK_{X_0})$ is surjective for any positive integer m. □

2.3 Theorem B. *Let H be a π-very ample divisor on V. Assume that K_X is π-pseudo-effective. Then $K_V + X$ is also π-pseudo-effective, and the natural homomorphism $\pi_*\mathcal{O}_V(m(K_V + X) + H) \to \pi_*\mathcal{O}_X(mK_X + H|_X)$ is surjective for any positive integer m.*

Proof: Theorem B implies Theorem 1.3. We may assume that S is a germ of a smooth curve. We define V, X and π as in the proof that Theorem A implies Theorem 1.2'. If $\nu(X_0) = -\infty$, then K_{X_0} is not pseudo-effective, and $\nu(X_\eta) = -\infty$ for the generic fiber X_η by the upper semi-continuity theorem. Otherwise, the rest of the proof is similar to the proof that Theorem A implies Theorem 1.2'. □

Proof: Theorem B implies Theorem 1.5. There exist a resolution of singularities $\mu : V \to \mathcal{X}$ and an effective divisor E which is supported on the exceptional locus of μ such that $H = -E$ is μ-very ample. Let $S = \mathcal{X}$ and $\pi = \mu$. Let X be the strict transform of X_0 which is assumed to be smooth. Since X_0 has only terminal singularities, there exists a positive integer m such that mK_{X_0} is a Cartier divisor and that $mK_X - m\mu^* K_{X_0} \geq E|_X$. Thus a nowhere vanishing section s_0 of $\mathcal{O}_{X_0}(mK_{X_0})$ lifts to a section of $\mathcal{O}_X(mK_X - E|_X)$, which in turn extends to a section of $\mathcal{O}_V(m(K_V + X) - E)$ by Theorem B. Therefore, s_0 extends to a nowhere vanishing section of $\mathcal{O}_{\mathcal{X}}(m(K_{\mathcal{X}} + X_0))$ which lifts to a section of $\mathcal{O}_V(m(K_V + X) - E)$. Hence $K_V + X - \mu^*(K_{\mathcal{X}} + X_0) \geq \frac{1}{m}E$ in this case. Since any μ is dominated by some μ as above, we have the last assertion. Since $\mu^* X_0 \geq X$, it follows that $\mathcal{X}$ has only terminal singularities. □

2.4 Theorem C. *Let X_ξ and V_η be the generic fibers of $\pi : X \to \pi(X)$ and $\pi : V \to \pi(V)$, respectively. Assume that $K_V + X$ is π-$\mathbb{Q}$-effective for the pair (V, X), dim X_ξ = dim V_η, and that $\nu(X_\xi) = \nu(V_\eta) = \kappa(V_\eta)$. Then the natural homomorphism $\pi_*\mathcal{O}_V(m(K_V + X)) \to \pi_*\mathcal{O}_X(mK_X)$ is surjective for any positive integer m.*

Proof: Theorem C implies Theorem 1.6. We may assume that S is a germ of a smooth curve. We define V, X and π as in the proof that Theorem A implies Theorem 1.2'. We may assume that K_{X_0} is pseudo-effective. By Theorem 1.3, we have $\nu(X_\xi) = \nu(X_0) \geq 0$, where X_ξ is the generic fiber of f. By the assumption on the abundance, it follows that $K_V + X$ becomes π-$\mathbb{Q}$-effective for the pair (V, X). The rest is similar to the proof that Theorem A implies Theorem 1.2'. □

Definition 2.5. Let X be a smooth complex variety and D a divisor on X. Let $\mu : Y \to X$ be a proper birational morphism from a smooth variety Y. Assume that there exists a decomposition $\mu^* D = P + M$ in $\mathrm{Div}(Y) \otimes \mathbb{R}$ such that P is μ-nef and M is effective having a normal crossing support. The *multiplier ideal sheaf* $\mathcal{I}_M$ is defined by the following formula:

$$\mu_*\mathcal{O}_Y(\ulcorner P \urcorner + K_Y) = \mathcal{I}_M(D + K_X).$$

We note that $\mathcal{I}_M$ is a coherent sheaf of ideals of $\mathcal{O}_X$ which is determined only by M and μ. If $\nu : Y' \to Y$ is another proper birational morphism, then it is easy to see that $\mathcal{I}_M = \mathcal{I}_{\nu^* M}$.

The following vanishing theorem of Kawamata-Viehweg type ([KMM, 1.2.3]) is the main tool for the proof of Theorems A and B. Nadel's vanishing theorem [Nd] and Ohsawa-Takegoshi's extension theorem [OT] played the same role in Siu's proof.

Theorem 2.6. *Let $f : X \to S$ be a proper algebraic morphism from a smooth complex manifold to an algebraic variety. Let L be an $\mathbb{R}$-divisor on X which is f-nef and f-big, and whose fractional part has a normal crossing support. Then*

$$R^p f_*\mathcal{O}_X(\ulcorner L \urcorner + K_X) = 0$$

for any positive integer p. □

Corollary 2.7. *In the situation of Definition 2.5, let S be an algebraic variety, and let $f : X \to S$ be a proper algebraic morphism. If P is $f \circ \mu$-nef and $f \circ \mu$-big, then*

$$R^p f_*\mathcal{I}_M(D + K_X) = 0$$

for any positive integer p.

Proof. We have $R^p(f \circ \mu)_*\mathcal{O}_Y(\ulcorner P \urcorner + K_Y) = 0$ and $R^p\mu_*\mathcal{O}_Y(\ulcorner P \urcorner + K_Y) = 0$ for any positive integer p. □

We need a vanishing theorem of Kollár type ([Ko]) in order to prove Theorem C:

Theorem 2.8. *Let $f : X \to S$ be a proper algebraic morphism from a smooth complex manifold to an algebraic variety. Let L be a $\mathbb{Q}$-divisor on X which is f-semi-ample and whose fractional part has a normal crossing support. Let D be an effective divisor on X such that $mL - D$ is f-effective for a positive integer m. Then a natural homomorphism*

$$R^p f_* \mathcal{O}_X(\ulcorner L \urcorner + K_X) \to R^p f_* \mathcal{O}_X(\ulcorner L \urcorner + D + K_X)$$

is injective for any nonnegative integer p.

Proof. We may assume that S is affine and f is projective by Theorem 2.6. By compactifying S, adding the pull-back of an ample divisor of S to L, and using the Serre vanishing theorem, we reduce the assertion to the injectivity of the homomorphism

$$H^p(X, \mathcal{O}_X(\ulcorner L \urcorner + K_X)) \to H^p(X, \mathcal{O}_X(\ulcorner L \urcorner + D + K_X))$$

in the case in which X is projective and L is semi-ample. This is just [K2, Theorem 3.2]. □

The condition that L is f-semi-ample can be relaxed to that L is f-nef and f-abundant (or f-good) by [K2, Proposition 2.1].

Corollary 2.9. *In the situation of Corollary 2.7, if f is surjective and P is $f \circ \mu$-semi-ample, then*

$$R^p f_* \mathcal{I}_M(D + K_X)$$

is torsion free for any nonnegative integer p.

Proof. Suppose the contrary and take an effective Cartier divisor D_1 on S which contains the support of the torsion part. Then the multiplication homomorphism $R^p(f \circ \mu)_* \mathcal{O}_Y(\ulcorner P \urcorner + K_Y) \to R^p(f \circ \mu)_* \mathcal{O}_Y(\ulcorner P \urcorner + (f \circ \mu)^* D_1 + K_Y)$ is not injective. □

Definition 2.10. In the situation of Setup 2.1, we define several kinds of multiplier ideal sheaves $\mathcal{J}_D^0, \mathcal{I}_D^0, \mathcal{J}_D^1, \mathcal{I}_D^1$ on X in the following.

Let D be a divisor on X which is π-$\mathbb{Q}$-effective. For each positive integer m such that mD is π-effective, we construct a proper birational morphism $\mu_m : Y_m \to X$ from a smooth variety such that the following conditions are satisfied: there is a decomposition $\mu_m^*(mD) = P_m + M_m$ in $\mathrm{Div}(Y_m)$ such that P_m is $\pi \circ \mu_m$-free, M_m is effective and has a normal crossing support, and that the natural homomorphism $(\pi \circ \mu_m)_* \mathcal{O}_{Y_m}(P_m) \to \pi_* \mathcal{O}_X(mD)$ is an isomorphism. We define

$$\mathcal{J}_D^0 = \bigcup_m \mathcal{I}_{\frac{1}{m} M_m}$$

where the union is taken for all positive integers m such that mD is π-effective. Since X is noetherian, there exists a positive integer m such that $\mathcal{J}_D^0 = \mathcal{I}_{\frac{1}{m} M_m}$.

In the case in which D itself is π-effective, we define $\mathcal{I}_D^0 = \mathcal{I}_{M_1}$. We have $\mathcal{I}_D^0 \subset \mathcal{J}_D^0$.

Let D be a divisor on V which is π-$\mathbb{Q}$-effective for the pair (V, X). For each positive integer m such that mD is π-effective for the pair (V, X), we construct a proper birational morphism $\mu_m : W_m \to V$ from a smooth variety with the strict transform Y_m of X in W_m being smooth and such that the following conditions are satisfied: there is a decomposition $\mu_m^*(mD) = Q_m + N_m$ in $\mathrm{Div}(W_m)$ such that Q_m is $\pi \circ \mu_m$-free, N_m is effective, $N_m + Y_m$ has a normal crossing support, and that the natural homomorphism $(\pi \circ \mu_m)_* \mathcal{O}_{W_m}(Q_m) \to \pi_* \mathcal{O}_V(mD)$ is an isomorphism. Since mD is π-effective for the pair (V, X), Y_m is not contained in the support of N_m. We define

$$\mathcal{J}_D^1 = \bigcup_m \mathcal{I}_{\frac{1}{m}N_m|_{Y_m}}$$

where the union is taken for all positive integers m such that mD is π-effective for the pair (V, X). Since X is noetherian, there exists a positive integer m such that $\mathcal{J}_D^1 = \mathcal{I}_{\frac{1}{m}N_m|_{Y_m}}$.

In the case in which D itself is π-effective for the pair (V, X), we define $\mathcal{I}_D^1 = \mathcal{I}_{N_1|_{Y_1}}$. We have $\mathcal{I}_D^1 \subset \mathcal{J}_D^1$.

If we define M_m for $D|_X$ as before, then we have $N_m|_{Y_m} \geq M_m$ as divisors on Y_m. Hence $\mathcal{J}_D^1 \subset \mathcal{J}_{D|_X}^0$. We also have $\mathcal{I}_D^1 \subset \mathcal{I}_{D|_X}^0$ if D is π-effective for the pair (V, X).

In the proof of the main results in the next section, the question on the extendability of global sections of the sheaves $\mathcal{O}_X(D|_X)$ to those of $\mathcal{O}_V(D)$ will be reduced to the comparison of the multiplier ideal sheaves such as $\mathcal{J}_{D|_X}^0$ and $\mathcal{J}_D^1$.

The following lemma enables us to deduce the inclusion of the sheaf from the inclusion of the direct image. A theorem of Skoda [Sk] was used in [Si] to prove the corresponding statement in the analytic setting.

Lemma 2.11. *([N3]) Let $f : X \to S$ be a projective morphism of algebraic schemes such that $n = \dim X$. Let $\mathcal{H}$ be an f-very ample invertible sheaves on X, and let $\mathcal{F}$ be a coherent sheaf on X. Assume that $R^p f_*(\mathcal{F} \otimes \mathcal{H}^{\otimes m}) = 0$ for any $p > 0$ and $m \geq 0$. Then the sheaf $\mathcal{F} \otimes \mathcal{H}^{\otimes n}$ is f-generated.*

Proof. We proceed by induction on n. Let us take an arbitrary point $x \in X$. We may assume that $H^0_{\{x\}}(\mathcal{F}) = 0$ if we replace $\mathcal{F}$ by its quotient by the torsion subsheaf supported at x. Let X' be a general member in the linear system $|\mathcal{H}|$ passing through x. Let $\mathcal{H}' = \mathcal{H} \otimes \mathcal{O}_{X'}$ and $\mathcal{F}' = \mathcal{F} \otimes \mathcal{H}'$. We have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{F} \otimes \mathcal{H} \to \mathcal{F}' \to 0.$$

By the induction hypothesis, $\mathcal{F}' \otimes \mathcal{H}'^{\otimes(n-1)}$ is f-generated. We have $R^1 f_*(\mathcal{F} \otimes \mathcal{H}^{\otimes(n-1)}) = 0$, hence our assertion is proved. □

Remark 2.12. We note that Theorems A, B and C also hold in the case in which X is still smooth but reducible. So we can apply them when $f : \mathcal{X} \to S$ has reducible fibers in Theorems 1.2', 1.3 and 1.6 ([N3]).

3. Proof of the main theorems

Notation 3.1. We fix divisors H and A on V such that H is π-very ample and $A = (\dim X + 1)H$.

Lemma 3.2. *Let D_1 and D be π-effective and π-$\mathbb{Q}$-effective divisors on X (resp. π-effective and π-$\mathbb{Q}$-effective divisors on V for the pair (V, X)), respectively. Then the sheaves $\mathcal{I}^0_{D_1}(D_1+A|_X+K_X)$ and $\mathcal{J}^0_D(D+A|_X+K_X)$ (resp. $\mathcal{I}^1_{D_1}(D_1|_X+A|_X+K_X)$ and $\mathcal{J}^1_D(D|_X+A|_X+K_X)$) are π-generated.*

Proof. Since $A-(\dim X)H$ is π-ample, we apply Corollary 2.7 and Lemma 2.11. □

Lemma 3.3. *Let D_1 and D be π-effective and π-$\mathbb{Q}$-effective divisors on X (resp. π-effective and π-$\mathbb{Q}$-effective divisors on V for the pair (V, X)). Then*

(1) $\mathcal{J}^i_{\alpha D} \subset \mathcal{J}^i_D$ for $i = 0, 1$ if $\alpha \in \mathbb{Q}$ and $\alpha > 1$.

(2) $\mathcal{I}^i_{D_1} \subset \mathcal{I}^i_{D_1+L}$ and $\mathcal{J}^i_D \subset \mathcal{J}^i_{D+L}$ for $i = 0, 1$ if L is a π-free divisor on X (resp. V).

(3) $Im(\pi_*\mathcal{O}_V(D_1) \to \pi_*\mathcal{O}_X(D_1|_X)) \subset \pi_*\mathcal{I}^1_{D_1}(D_1|_X)$ (in the latter case only).

Proof. (1) and (2) are clear. (3) follows from $K_{Y_1} \geq \mu_1^* K_X$. □

The vanishing theorem is used to prove the following lemma.

Lemma 3.4. *Let D be a π-$\mathbb{Q}$-effective divisor for the pair (V, X). Then*

$$\pi_*\mathcal{J}^1_D(D|_X + K_X) \subset Im(\pi_*\mathcal{O}_V(D + K_V + X) \to \pi_*\mathcal{O}_X(D|_X + K_X)).$$

Proof. We have an exact sequence

$$0 \to \mathcal{O}_{W_m}(\ulcorner \frac{1}{m} Q_m \urcorner + K_{W_m}) \to \mathcal{O}_{W_m}(\ulcorner \frac{1}{m} Q_m \urcorner + K_{W_m} + Y_m)$$
$$\to \mathcal{O}_{Y_m}(\ulcorner \frac{1}{m} Q_m|_{Y_m} \urcorner + K_{Y_m}) \to 0.$$

If D is π-big for the pair (V, X), then we have $R^1(\pi \circ \mu_m)_*\mathcal{O}_{W_m}(\ulcorner \frac{1}{m} Q_m \urcorner + K_{W_m}) = 0$ by Theorem 2.6. In the general case, since Q_m is $\pi \circ \mu_m$-free, the homomorphism

$$R^1(\pi \circ \mu_m)_*\mathcal{O}_{W_m}(\ulcorner \frac{1}{m} Q_m \urcorner + K_{W_m}) \to R^1(\pi \circ \mu_m)_*\mathcal{O}_{W_m}(\ulcorner \frac{1}{m} Q_m \urcorner + K_{W_m} + Y_m)$$

is injective by Theorem 2.8. Anyway, we have a surjective homomorphism

$$\pi_*\mathcal{O}_V(D + K_V + X) \supset (\pi \circ \mu_m)_*\mathcal{O}_{W_m}(\ulcorner \frac{1}{m} Q_m \urcorner + K_{W_m} + Y_m)$$
$$\twoheadrightarrow (\pi \circ \mu_m)_*\mathcal{O}_{Y_m}(\ulcorner \frac{1}{m} Q_m|_{Y_m} \urcorner + K_{Y_m}) = \pi_*\mathcal{J}^1_D(D|_X + K_X),$$

hence the assertion. □

Corollary 3.5. *(1) Let D be a π-$\mathbb{Q}$-effective divisor for the pair (V, X). Then $D + A + K_V + X$ is π-effective for the pair (V, X).*

(2) If D_1 and $D_1 + K_V + X$ are π-effective for the pair (V, X), and D and $D + K_V + X$ are π-$\mathbb{Q}$-effective for the pair (V, X). Then

$$\pi_*\mathcal{J}^1_{D_1}(D_1|_X + K_X) \subset \pi_*\mathcal{I}^1_{D_1+K_V+X}(D_1|_X + K_X)$$
$$\pi_*\mathcal{J}^1_D(D|_X + K_X) \subset \pi_*\mathcal{J}^1_{D+K_V+X}(D|_X + K_X).$$

Proof. (1) By Lemma 3.2, we have $\pi_*\mathcal{J}^1_D(D|_X + A|_X + K_X) \neq 0$. By Lemma 3.3 (2), $\pi_*\mathcal{J}^1_{D+A}(D|_X + A|_X + K_X) \neq 0$. Then by Lemma 3.4, we obtain our assertion. (2) follows from Lemmas 3.3 (3) and 3.4. □

The following is the key lemma for Theorems A and C.

Lemma 3.6. *Assume that $K_V + X$ is π-$\mathbb{Q}$-effective for the pair (V, X). Then*

$$\mathcal{J}^0_{mK_X} \subset \mathcal{I}^1_{m(K_V+X)+A}$$

for any non-negative integer m.

Proof. We note that $m(K_V + X) + A$ is π-effective for any non-negative integer m by Corollary 3.5 (1). We proceed by induction on m. If $m = 0$, then the assertion is obvious, because A is π-free. Assume that the assertion is true for $m - 1$. By the induction hypothesis and Corollary 3.5 (2),

$$\pi_*\mathcal{J}^0_{(m-1)K_X}(mK_X + A|_X) \subset \pi_*\mathcal{I}^1_{(m-1)(K_V+X)+A}(mK_X + A|_X)$$
$$\subset \pi_*\mathcal{I}^1_{m(K_V+X)+A}(mK_X + A|_X).$$

Since $\pi_*\mathcal{J}^0_{(m-1)K_X}(mK_X + A|_X)$ is π-generated by Lemma 3.2, it follows that

$$\mathcal{J}^0_{(m-1)K_X} \subset \mathcal{I}^1_{m(K_V+X)+A}.$$

Since $\mathcal{J}^0_{mK_X} \subset \mathcal{J}^0_{(m-1)K_X}$ by Lemma 3.3 (1), we are done. □

Proof of Theorem A. Since $K_V + X$ is π-big for the pair (V, X), there exists a positive integer m_0 such that $m_0(K_V + X) \sim A + B$ for an effective divisor B whose support does not contain X. By Lemmas 3.3 (1) and 3.6, we have

$$\mathcal{J}^0_{mK_X}(-B|_X) \subset \mathcal{J}^1_{m(K_V+X)+A}(-B|_X)$$
$$\subset \mathcal{J}^1_{m(K_V+X)+A+B} = \mathcal{J}^1_{(m+m_0)(K_V+X)} \subset \mathcal{J}^1_{m(K_V+X)}.$$

This implies the following: for any positive integer m, there exists a positive integer d such that, if Y is any smooth model of X which dominates Y_m and Y_{dm} by $\mu : Y \to X$, $\nu_m : Y \to Y_m$ and $\nu_{dm} : Y \to Y_{dm}$, then

$$-\nu_m^* M_m - \mu^* B|_X \leq \ulcorner -\frac{1}{d}\nu_{dm}^* N_{dm}|_{Y_{dm}} \urcorner + K_Y - \mu^* K_X.$$

If $m = en$ for positive integers e and n and if Y dominates Y_n by a morphism $\nu_n : Y \to Y_n$, then we have

$$-\nu_n^* M_n \leq -\frac{1}{e}\nu_m^* M_m \leq \frac{1}{e}\mu^* B|_X + \frac{1}{e}C_{m,d} - \frac{1}{de}\nu_{dm}^* N_{dm}|_{Y_{dm}} + \frac{1}{e}(K_Y - \mu^* K_X)$$

where $C_{m,d} = \ulcorner -\frac{1}{d}\nu_{dm}^* N_{dm}|_{Y_{dm}} \urcorner + \frac{1}{d}\nu_{dm}^* N_{dm}|_{Y_{dm}}$. We take a large enough integer e such that $(X, \frac{2}{e}B|_X)$ is log terminal. Then we have $\llcorner \frac{2}{e}\mu^* B|_X - (K_Y - \mu^* K_X) \lrcorner \leq 0$. Hence

$$-\nu_n^* M_n \leq \ulcorner -\frac{1}{de}\nu_{dm}^* N_{dm}|_{Y_{dm}} \urcorner + K_Y - \mu^* K_X.$$

Thus $\pi_*\mathcal{O}_X(nK_X) \subset \pi_*\mathcal{J}^1_{n(K_V+X)}(nK_X) \subset \pi_*\mathcal{J}^1_{(n-1)(K_V+X)}(nK_X)$ by Lemma 3.3 (1). Therefore, we obtain our assertion by Lemma 3.4. □

We modify Lemma 3.6 for Theorem B as follows:

Lemma 3.7. *Assume that K_X is π-pseudo-effective. Then $K_V + X$ is π-pseudo-effective, and*

$$\mathcal{J}^0_{mK_X+eH|_X} \subset \mathcal{I}^1_{m(K_V+X)+eH+A}.$$

for any non-negative integer m and any positive integer e.

Proof. Since K_X is π-pseudo-effective, $mK_X + eH|_X$ is π-big for any non-negative integer m and any positive integer e. Thus the left hand side of the formula is well defined. We shall prove that $m(K_V+X)+H+A$ is π-effective for the pair (V,X) for any non-negative integer m in order to prove that $K_V + X$ is π-pseudo-effective, as well as the inclusion $\mathcal{J}^0_{mK_X+H|_X} \subset \mathcal{I}^1_{m(K_V+X)+H+A}$ in the case $e = 1$ by induction on m. The inclusion for general e is proved similarly for each fixed e by induction on m.

If $m = 0$, then the assertion is obvious. Assume that the assertion is true for $m-1$. By the induction hypothesis and Lemma 3.4, we have

$$\begin{aligned}
&\pi_*\mathcal{J}^0_{(m-1)K_X+H|_X}(mK_X + H|_X + A|_X)\\
&\subset \pi_*\mathcal{I}^1_{(m-1)(K_V+X)+H+A}(mK_X + H|_X + A|_X)\\
&\subset \operatorname{Im}(\pi_*\mathcal{O}_V(m(K_V+X)+H+A) \to \pi_*\mathcal{O}_X(mK_X + H|_X + A|_X)).
\end{aligned}$$

Since $\mathcal{J}^0_{(m-1)K_X+H|_X}(mK_X + H|_X + A|_X)$ is π-generated by Lemma 3.2, it follows that $m(K_V + X) + H + A$ is π-effective for the pair (V, X). Then by Lemma 3.3 (3),

$$\begin{aligned}
&\operatorname{Im}(\pi_*\mathcal{O}_V(m(K_V+X)+H+A) \to \pi_*\mathcal{O}_X(mK_X + H|_X + A|_X))\\
&\subset \pi_*\mathcal{I}^1_{m(K_V+X)+H+A}(mK_X + H|_X + A|_X).
\end{aligned}$$

Hence

$$\mathcal{J}^0_{(m-1)K_X+H|_X} \subset \mathcal{I}^1_{m(K_V+X)+H+A}.$$

Since $\mathcal{J}^0_{mK_X+H|_X} \subset \mathcal{J}^0_{(m-1)K_X+H|_X}$ by Lemma 3.3 (1) and (2), we are done. □

Proof of Theorem B. We fix m. Since $m(K_V + X) + H$ is π-big for the pair (V, X) by Lemma 3.7, there exists a positive integer m_0 such that $m_0(m(K_V+X)+H) \sim H+A+B$ for an effective divisor B whose support does not contain X. By Lemmas 3.3 (1), (2) and 3.7, we have

$$\begin{aligned}
&\mathcal{J}^0_{mK_X+eH|_X}(-B|_X) \subset \mathcal{J}^1_{m(K_V+X)+eH+A}(-B|_X)\\
&\subset \mathcal{J}^1_{m(K_V+X)+eH+A+B} \subset \mathcal{J}^1_{(m_0+1)(m(K_V+X)+eH)}\\
&\subset \mathcal{J}^1_{m(K_V+X)+eH}.
\end{aligned}$$

We have proper birational morphisms $\mu_{m,e} : W_{m,e} \to V$ from a smooth variety, a smooth strict transform $Y_{m,e}$ of X, and decompositions $\mu^*_{m,e}(mK_X + eH|_X) = P_{m,e} + M_{m,e}$ and $\mu^*_{m,e}(m(K_V + X) + eH) = Q_{m,e} + N_{m,e}$ as in Definition 2.10.

It follows that for any positive integers m and e, there exists a positive integer d such that, if Y is any smooth model of X which dominates $Y_{m,e}$ and $Y_{dm,de}$ by $\mu : Y \to X$, $\nu_{m,e} : Y \to Y_{m,e}$ and $\nu_{dm,de} : Y \to Y_{dm,de}$, then

$$-\nu^*_{m,e}M_{m,e} - \mu^*B|_X \leq \ulcorner -\frac{1}{d}\nu^*_{dm,de}N_{dm,de}|_{Y_{dm,de}}\urcorner + K_Y - \mu^*K_X.$$

If $m = en$ for a positive integer n, then we have

$$\begin{aligned} -\nu_{n,1}^* M_{n,1} &\le -\frac{1}{e}\nu_{m,e}^* M_{m,e} \\ &\le \frac{1}{e}\mu^* B|_X + \frac{1}{e}C_{m,d,e} - \frac{1}{de}\nu_{dm,de}^* N_{dm,de}|_{Y_{dm,de}} + \frac{1}{e}(K_Y - \mu^* K_X) \end{aligned}$$

where $C_{m,d,e} = \ulcorner -\frac{1}{d}\nu_{dm,de}^* N_{dm,de}|_{Y_{dm,de}} \urcorner + \frac{1}{d}\nu_{dm,de}^* N_{dm,de}|_{Y_{dm,de}}$. We take a large enough integer e such that $(X, \frac{2}{e}B|_X)$ is log terminal. Then $\llcorner \frac{2}{e}\mu^* B|_X - (K_Y - \mu^* K_X) \lrcorner \le 0$, hence

$$-\nu_{n,1}^* M_{n,1} \le \ulcorner -\frac{1}{de}\nu_{dm,de}^* N_{dm,de}|_{Y_{dm,de}} \urcorner + K_Y - \mu^* K_X.$$

Therefore,

$$\pi_* \mathcal{O}_X(nK_X + H|_X) \subset \pi_* \mathcal{J}^1_{n(K_V+X)+H}(nK_X + H|_X),$$

and the whole of $\pi_* \mathcal{O}_X(nK_X + H|_X)$ is extendable to V. □

Proof of Theorem C. By the flattening and the normalization, we construct a proper birational morphism $\mu : V' \to V$ from a normal variety V', a smooth variety $\bar{V}$ with a structure morphism $\beta : \bar{V} \to S$, and an equidimensional projective morphism $\alpha : V' \to \bar{V}$ which is birationally equivalent to the Iitaka fibration of V over S. We set $\pi' = \beta \circ \alpha = \pi \circ \mu$:

$$\begin{array}{ccc} V & \xleftarrow{\mu} & V' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\alpha} \\ S & \xleftarrow{\beta} & \bar{V}. \end{array}$$

There exists a positive integer m_1 such that $m_1\mu^*(K_V + X) = \alpha^* \bar{L} + E$ for a β-big divisor $\bar{L}$ on $\bar{V}$ and an effective Cartier divisor E on V'. Since the Kodaira dimension of the general fiber of α is zero and α is equidimensional, we may assume that the natural homomorphisms $\beta_* \mathcal{O}_{\bar{V}}(m\bar{L}) \to \pi_* \mathcal{O}_V(mm_1(K_V + X))$ are bijective for any positive integer m. This means in particular that for any prime divisor of $\bar{V}$, there exists a prime divisor of V' lying above it which is not contained in the support of E.

Since $\kappa(V_\eta) = \nu(V_\eta)$, the numerical Kodaira dimension of the generic fiber of α is zero by [N2, 7.4.3]. Hence there exists a positive integer m_2 such that the sheaf $\mathcal{F} = \alpha_* \mathcal{O}_{V'}(\mu^* A + mE)$ on $\bar{V}$ is independent of the integer m if $m \ge m_2$.

Let X' be the strict transform of X by μ. Since $\dim X_\xi = \dim V_\eta$, we have $\alpha(X') = \bar{X} \ne \bar{V}$. We may assume that $\bar{X}$ is a smooth divisor on $\bar{V}$. Since $\mathcal{F}$ is torsion free, there exists a β-very ample divisor $\bar{A}$ on $\bar{V}$ such that there are injective homomorphisms

$$\mathcal{O}_{\bar{V}}(-\bar{A})^{\oplus k} \subset \mathcal{F} \subset \mathcal{O}_{\bar{V}}(\bar{A})^{\oplus k}$$

for $k = \operatorname{rank} \mathcal{F}$ which are bijective at the generic point of $\bar{X}$.

By Theorem B, the natural homomorphism $\beta_* \mathcal{F}(m\bar{L}) = \pi'_* \mathcal{O}_{V'}(\mu^* A + m\alpha^* \bar{L} + mE) \to \pi'_* \mathcal{O}_{X'}((\mu^* A + m\alpha^* \bar{L} + mE)|_{X'})$ is surjective for any positive integer $m \ge m_2$. Hence X' is not contained in the support of E.

For each positive integer m such that $mm_1(K_V+X)$ is π-effective for the pair (V,X), we construct a projective birational morphism $\bar{\mu}_m : \bar{V}_m \to \bar{V}$ from a smooth variety such that $\bar{\mu}_m^*(m\bar{L}) = \bar{Q}_m + \bar{N}_m$ where $\bar{Q}_m$ is $\beta\circ\bar{\mu}_m$-free, $\bar{N}_m$ is effective, and that the natural homomorphism $(\beta\circ\bar{\mu}_m)_*\mathcal{O}_{\bar{V}_m}(\bar{Q}_m) \to \beta_*\mathcal{O}_{\bar{V}}(m\bar{L})$ is an isomorphism. By taking the fiber product and the normalization, we construct morphisms $\mu'_m : V_m \to V'$ and $\alpha_m : V_m \to \bar{V}_m$. Set $\mu_m = \mu\circ\mu'_m : V_m \to V$. Let N'_m be the $\pi\circ\mu_m$-fixed part of $\mu_m^*(mm_1(K_V+X))$. Since $\beta_*\mathcal{O}_{\bar{V}}(m\bar{L}) \to \pi_*\mathcal{O}_V(mm_1(K_V+X))$ is bijective, we have $N'_m = \alpha_m^*\bar{N}_m + mE$. Thus $m\bar{L}$ is β-effective for the pair $(\bar{V},\bar{X})$,

Since the numerical Kodaira dimension of the generic fiber of α is zero, the numerical Kodaira dimension of the generic fiber of $\alpha|_{X'}$ is also zero by Theorem B Since dim X_ξ = dim V_η and $\nu(X_\xi) = \nu(V_\eta)$, $\bar{L}|_{\bar{X}}$ is β-big. Moreover, by Theorem B again, there exist a sufficiently large integer m_0 and a global section of $\beta_*\mathcal{F}(m_0\bar{L} - 2\bar{A})$ which induces a non-zero section over $\bar{X}$. Thus there exists a global section of $\mathcal{O}_{\bar{V}}(m_0\bar{L}-\bar{A})$ which does not vanish identically on $\bar{X}$. Hence $\bar{L}$ is β-big for the pair $(\bar{V},\bar{X})$; there exists an effective divisor $\bar{B}$ whose support does not contain $\bar{X}$ and such that $m_0\bar{L} \sim \bar{A}+\bar{B}$.

Let N''_m be the $\pi\circ\mu_m$-fixed part of $\mu_m^*(mm_1(K_V+X)+A)$. Let $\mathcal{F}_m = \alpha_{m*}\mathcal{O}_{V_m}(\mu_m^*A + m\mu_m'^*E)$. Although $\mathcal{F}_m$ may be different from $\bar{\mu}_m^*\mathcal{F}$/torsion, the natural homomorphism $\beta_*\mathcal{F}(m\bar{L}) \to (\beta\circ\bar{\mu}_m)_*\mathcal{F}_m(m\bar{\mu}_m^*\bar{L})$ is bijective. We have

$$\begin{aligned}&\mathcal{O}_{\bar{V}_m}(\bar{\mu}_m^*(m\bar{L}-\bar{A}))^{\oplus k} \subset \bar{\mu}_m^*\mathcal{F}(m\bar{L})/\text{torsion}\\ &\subset \mathcal{O}_{\bar{V}_m}(\bar{\mu}_m^*(m\bar{L}+\bar{A}))^{\oplus k} \subset \mathcal{O}_{\bar{V}_m}((m+m_0)\bar{\mu}_m^*\bar{L})^{\oplus k}\end{aligned}$$

where the last inclusion is defined by $\bar{B}$. Thus for $m \geq m_2$, we have

$$\begin{aligned}N''_m &\geq \alpha_m^*(\bar{N}_{m+m_0} - \bar{\mu}_m^*((m+m_0)\bar{L} - (m\bar{L}-\bar{A}))) + (m-m_2)E\\ &= N'_{m+m_0} - \bar{\mu}_m'^*\alpha^*(2\bar{A}+\bar{B}) - (m_0+m_2)E\\ &\geq N'_{m+m_0} - \mu_m^*C\end{aligned}$$

for $C = (2m_0+m_2)m_1(K_V+X)$. Therefore, we have

$$\mathcal{I}^1_{mm_1(K_V+X)+A}(-C|_X) \subset \mathcal{I}^1_{(m+m_0)m_1(K_V+X)} \subset \mathcal{J}^1_{mm_1(K_V+X)}.$$

On the other hand, since K_V+X is π-$\mathbb{Q}$-effective for the pair (V,X), we have

$$\mathcal{J}^0_{mK_X} \subset \mathcal{I}^1_{m(K_V+X)+A}$$

for any non-negative integer m by Lemma 3.6. The rest is the same as in the proof of Theorem A. □

4. Concluding remarks

By Theorem 1.5 and by the base point free theorem, small deformations of a minimal model are always (not necessarily $\mathbb{Q}$-factorial) minimal models (cf. [KMM]). We might ask whether a similar statement holds for global deformations. Proposition 4.1 is on the affirmative side, but we have also a counterexample (Example 4.2).

Proposition 4.1. *Let $f : \mathcal{X} \to S$ be a proper flat algebraic morphism from a complex variety to a germ of a smooth curve. Assume that the fibers $X_t = f^{-1}(t)$ have only canonical singularities for any $t \in S$. Let $\phi : \mathcal{X} \to \mathcal{Z}$ be a projective birational morphism over S which is not an isomorphism and such that $-K_{\mathcal{X}}$ is ϕ-ample. Assume that the fibers of ϕ are at most 1-dimensional. Then the morphism restricted to the generic fiber $\phi|_{X_\eta}$ is not an isomorphism.*

Proof. Let X_0 be the central fiber. For sufficiently large positive integer m, we have $R^1\phi_*\mathcal{O}_{X_0}(mK_{X_0}) \neq 0$, while $R^p\phi_*\mathcal{O}_{X_0}(mK_{X_0}) = 0$ for $p \geq 2$. Thus the proposition follows from the upper semi-continuity theorem combined with Theorem 1.2'. □

Example 4.2. Let $E = \mathbb{P}^d$ for an integer $d \geq 2$, and let $\mathcal{X}$ be the total space of the vector bundle $\mathcal{O}_E(-1)^{\oplus d}$. Let $x_0, \dots, x_d$ be homogeneous coordinates on E, and let $\xi_1, \dots, \xi_d$ be fiber coordinates for $\mathcal{X}$. Then $t = \sum_{i=1}^d x_i\xi_i$ gives a morphism $f : \mathcal{X} \to S = \mathbb{C}$. The central fiber $X_0 = \{t = 0\}$ contains E and has only one ordinary double point as singularity, which is $\mathbb{Q}$-factorial if $d \geq 3$. We have $K_{\mathcal{X}}|_E = \mathcal{O}_E(-1)$. There exists a birational contraction $\phi : \mathcal{X} \to \mathcal{Z}$ whose exceptional locus coincides with E.

The following Example 4.3 shows that the generalizations of Theorems 1.2' and 1.4 to the case of varieties with log terminal singularities are false.

Example 4.3. Let us consider a flip of 3-folds with terminal singularities over a germ $\mathcal{Z}$:

$$\mathcal{X} \xrightarrow{\phi} \mathcal{Z} \xleftarrow{\phi^+} \mathcal{X}^+,$$

where $-K_{\mathcal{X}}$ is ϕ-ample and $K_{\mathcal{X}^+}$ is ϕ^+-ample. Let $g : \mathcal{Z} \to S$ be a generic projection to a germ of a smooth curve so that the central fiber Z_0 is a generic hyperplane section of $\mathcal{Z}$ through the singular point. Let $f : \mathcal{X} \to S$ be the induced morphism, and let X_0 be the central fiber which coincides with the strict transform of Z_0 on $\mathcal{X}$.

Assume that X_0 has only log terminal singularities. Then so has Z_0 because K_{X_0} is negative for ϕ. For example, this is the case for Francia's flip: (1) $\mathcal{X}$ has only one singularity of type $\frac{1}{2}(1,1,1)$ and $\mathcal{X}^+$ is smooth, (2) the exceptional loci C and C^+ of ϕ and ϕ^+, respectively, are isomorphic to $\mathbb{P}^1$, (3) the normal bundle of C^+ in $\mathcal{X}^+$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, (4) Z_0 has a singularity of type $\frac{1}{3}(1,1)$, and (5) X_0 has a singularity of type $\frac{1}{4}(1,1)$.

Since $K_{\mathcal{Z}}$ is not $\mathbb{Q}$-Cartier, $\mathcal{Z}$ is not log terminal. Let m be a positive integer such that mK_{Z_0} is a Cartier divisor. We take $m = 3$ for Francia's flip. Then the natural homomorphism $\phi_*\mathcal{O}_{\mathcal{X}}(mK_{\mathcal{X}}) = \mathcal{O}_{\mathcal{Z}}(mK_{\mathcal{Z}}) \to \phi_*\mathcal{O}_{X_0}(mK_{X_0}) = \mathcal{O}_{Z_0}(mK_{Z_0})$ is not surjective. Therefore, if we compactify $\mathcal{X}$ suitably over S, then we obtain a counterexample to the generalization of Theorem 1.2' for the log terminal case.

In the situation of Theorem 1.4 with X_0 having log terminal singularities, one might still ask whether the general fibers X_t of f have only log terminal singularities. This is also false. The following example is kindly communicated by Professor Shihoko Ishii. By [R1, Lemma 2.7], one can construct from the above example a flat deformation $f : \mathcal{V} \to B$ over a germ of a smooth curve such that $f^{-1}(0) \cong Z_0 \times B$ and $f^{-1}(t) \cong \mathcal{Z}$ for $t \neq 0$. On the other hand, [Is] proved that small deformations of a log terminal singularity have the same lifting property for pluricanonical forms as log terminal singularities although they may not be $\mathbb{Q}$-Gorenstein.

REFERENCES

[I1] S. Iitaka, *Deformation of compact complex surfaces I*, Global Analysis, papers in honor of K. Kodaira, Princeton Univ. Press, 1969, pp. 267–272.

[I2] ———, *Deformation of compact complex surfaces II, III*, J. Math. Soc. Japan **22, 23** (1970, 1971), 247–261, 692–705.

[Is] S. Ishii, *Small deformations of normal singularities*, Math. Ann. **275** (1986), 139–148.

[K1] Y. Kawamata, *A generalization of Kodaira-Ramanujam's vanishing theorem*, Math. Ann. **261** (1982), 43–46.

[K2] ———, *Pluricanonical systems on minimal algebraic varieties*, Invent. Math. **79** (1985), 567–588.

[K3] ———, *The Zariski decomposition of log-canonical divisors*, Proc. Symp. Pure Math. **46** (1987), 425–433.

[K4] ———, *Abundance theorem for minimal threefolds*, Invent. Math. **108** (1992), 229–246.

[K5] ———, *Deformations of canonical singularities*, alg-geom/9712018 (1997), to appear in J. Amer. Math. Soc.

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, Adv. St. Pure Math. **10** (1987), 283–360.

[Ko] J. Kollár, *Higher direct images of dualizing sheaves I*, Ann. of Math. **123** (1986), 11–42.

[KM] J. Kollár and S. Mori, *Classification of three dimensional flips*, J. Amer. Math. Soc. **5** (1992), 533–702.

[Nd] A. Nadel, *Multiplier ideal sheaves and the existence of Kähler-Einstein metrics of positive scalar curvature*, Ann. of Math. **132** (1989), 549–596.

[Nm] I. Nakamura, *Complex parallelisable manifolds and their small deformations*, J. Diff. Geom. **10** (1975), 85–112.

[N1] N. Nakayama, *Invariance of the plurigenera of algebraic varieties under minimal model conjectures*, Topology **25** (1986), 237–251.

[N2] ———, *Zariski decomposition and abundance*, RIMS preprint (June 1997).

[N3] ———, *Invariance of the plurigenera of algebraic varieties*, RIMS preprint (March 1998).

[OT] T. Ohsawa and K. Takegoshi, *On the existence of L^2-holomorphic functions*, Math. Z. **195** (1987), 197–204.

[R1] M. Reid, *Canonical 3-folds*, Géométrie algébrique Angers 1979, Sijthoff & Noordhoff, 1980, pp. 273–310.

[R2] ———, *Young person's guide to canonical singularities*, Proc. Symp. Pure Math. **46-1** (1987), 345–414.

[Si] Y.-T. Siu, *Invariance of plurigenera*, alg-geom/9712016 (1997).

[Sk] H. Skoda, *Application des techniques L^2 à la théorie des ideaux d'un algèbre de fonctions holomorphes avec poids*, Ann. Sci. Ec. Norm. Sup. **5** (1972), 548–580.

[V1] E. Viehweg, *Vanishing theorems*, J. reine angew. Math. **335** (1982), 1–8.

[V2] ———, *Weak positivity and the stability of certain Hilbert points*, Invent. Math. **96** (1989), 639–667.

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