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ON THE K_1-GROUPS OF ALGEBRAIC CURVES

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1. Introduction

The K_n-group $K_n(X)$ of a scheme X is defined to be the $(n+1)$-th homotopy group of the geometric realization of Quillen's $\mathcal{Q}$-construction of the category of locally free sheaves on X ([Q], [Sr]): $K_n(X) := \pi_{n+1}(BQP(X), 0)$. There are the pull-backs, transfer maps for proper morphisms, and so on. It is well-known that $K_0(X)$ is the Grothendieck group of locally free sheaves on X, $K_1(A) = K_1(\mathrm{Spec}A) = A^*$ for a local ring A ([Mi], [Sr]), and $K_2(A)$ is the Milnor K_2-group $K_2^M(A)$ when A is a local ring with the residue field which has at least 5 elements ([vdK]).

Let C be a nonsingular projective curve over $\mathbf{C}$. We denote $Z_0(C)$ the free abelian group of closed points of C. By the localization theorem, we have the following exact sequence:

$$\cdots \to K_2^M(\mathbf{C}(C)) \xrightarrow{T} \mathbf{C}^* \otimes Z_0(C) \to K_1(C) \to \mathbf{C}(C)^* \xrightarrow{v} Z_0(C) \to \cdots.$$

Here v is the valuation map, and T is the tame symbol map:

$$v : f \longmapsto \sum_P v_P(f)[P], \quad T : \{f, g\} \longmapsto \sum_P (-1)^{v_P(f)v_P(g)}(f^{v_P(g)}/g^{v_P(f)})(P) \otimes [P].$$

Define $SK_1(C) := \mathrm{Coker}T$. Therefore we have the exact sequence

$$0 \longrightarrow SK_1(C) \longrightarrow K_1(C) \longrightarrow \mathbf{C}^* \longrightarrow 0.$$

There is the norm map $SK_1(C) \to \mathbf{C}^*$ given by $\sum \lambda_i \otimes P_i \mapsto \prod \lambda_i$. To see the well-definedness of the norm map, it suffices to see that it is induced from the transfer map $f_* : K_1(C) \to K_1(\mathbf{C}) = \mathbf{C}^*$ for the structure morphism $f : C \to \mathrm{Spec}\mathbf{C}$. We write the kernel of the norm map by $V(C)$:

$$0 \longrightarrow V(C) \longrightarrow SK_1(C) \xrightarrow{\mathrm{Norm}} \mathbf{C}^* \longrightarrow 0.$$

$V(C)$ is the subgroup generated by the images of $\sum \lambda \otimes [P - P']$.

It is known that $SK_1(C)$ is isomorphic to Bloch's higher Chow group $\mathrm{CH}^2(C, 1)$ ([B2]). We do not make a distinction between them. We also write $V(C)$ by $F^2\mathrm{CH}^2(C, 1)$. When $C = \mathbf{P}^1_{\mathbf{C}}$, it is easy to see that $SK_1(C) = \mathrm{CH}^2(C, 1) = \mathbf{C}^*$. However, if the genus of C is not zero, the group $SK_1(C)$ is known to be enormous (J.Lewis, W.Raskind).

In the previous papers [A1] and [A2], we introduced the notion of *arithmetic Hodge structure* (see also [SaM4]). We denote its category by $\underline{\mathrm{M}}(\mathbf{C})$, which is defined by the inductive limit of variations of mixed Hodge structures (Definition 2.3). $\underline{\mathrm{M}}(\mathbf{C})$ admits an exact and faithful functor (called the realization functor) to the category MHS($\mathbf{C}$) of graded polarizable mixed Hodge structures. There is the cohomology object $H^k(X, \mathbf{Z}(r)) \in \underline{\mathrm{M}}(\mathbf{C})$ for each algebraic variety X over $\mathbf{C}$. The main difference between arithmetic Hodge structures and mixed Hodge structures is that higher Yoneda extension groups in $\underline{\mathrm{M}}(\mathbf{C})$ does not vanish in general, though so does in MHS($\mathbf{C}$) (in particular for degree ≥ 2).

We use arithmetic Hodge structure to study K_1-groups of algebraic curves. Let us explain more precisely. We first construct the cycle map

$$\rho^2 : V(C) = F^2\mathrm{CH}^2(C,1) \longrightarrow \mathrm{Ext}^2_{\underline{\mathrm{M}}(\mathbf{C})}(\mathbf{Z}(0), H^1(C, \mathbf{Z}(2))) \tag{1.1}$$

for a projective nonsingular curve C over $\mathbf{C}$. The cycle map (1.1) tensored with $\mathbf{Q}$ is already constructed in [A1] (or essentially [SaM3]). However, since we treat extension groups of $\mathbf{Z}$-coefficient arithmetic Hodge structures, we give a self-contained construction of the cycle map again. Note that if we replace $\underline{\mathrm{M}}(\mathbf{C})$ by MHS($\mathbf{C}$), the cycle map (1.1) is zero, and entirely useless to the study of $K_1(C)$. Our arithmetic Hodge structures cover the weakness. More strongly, we conjecture:

Conjecture 1.1. *The kernel of the cycle map* (1.1) *is at most torsion.*

This is the K_1 version of the Bloch conjecture (cf. [B1] Chapter I).

Our main theorem supports Conjecture 1.1. Before explaining it, we define the notion of "generic".

Definition 1.2. (1) Let $C \in \mathbf{P}^2_{\mathbf{C}}$ be a nonsingular plane curve defined by the homogeneous equation F. Let (X_0, X_1, X_2) be a homogeneous coordinate of $\mathbf{P}^2_{\mathbf{C}}$, and let $F = \sum a_I X^I$ where $I = (i_0, i_1, i_2)$ is a multi-index. We call C a *generic plane curve* if all a_I are not zero, and the numbers $\{a_I/a_{I_0}\}$ are algebraically independent over $\bar{\mathbf{Q}}$.

(2) Let C be a generic plane curve. A set $\{P_1, \cdots, P_n\}$ of closed points of C is called to be in a *generic position*, if there is a generic plane curve D defined by $G = \sum b_J X^J$ (whose degree is arbitrary) such that the numbers $\{a_I/a_{I_0}, b_J/b_{J_0}\}$ are algebraically independent over $\bar{\mathbf{Q}}$, and the set $\{P_1, \cdots, P_n\}$ is contained in $C \cap D$.

The following is the main theorem in this paper:

Theorem 1.3 ([A3] Theorem 1.3). *Let $C \subset \mathbf{P}^2_{\mathbf{C}}$ be a generic smooth plane curve of degree $d \geq 4$, and $\{P_1, \cdots, P_n\}$ closed points of C in a generic position. Then the following map induced from* (1.1)

$$\bigoplus_{i=1}^{n-1} \mathbf{C}^* \otimes [P_i - P_{i+1}] \longrightarrow \mathrm{Ext}^2_{\underline{\mathrm{M}}(\mathbf{C})}(\mathbf{Z}(0), H^1(C, \mathbf{Z}(2)))$$

is injective.

From the construction of the map (1.1), we can reduce the proof to some calculation of 1-extension groups of variations of mixed Hodge structures (see Proposition 3.1). We will do it by using the symmetrizer lemma ([AS]) and the Mordell-Weil theorem for function fields due to Lang and Néron ([LN], [La2] Chapter 6).

We finally mention an immediate corollary of Theorem 1.3:

Corollary 1.4 ([A3] Corollary 1.4). *Let C and $\{P_1, \cdots, P_n\}$ be as in Theorem 1.3. Then the natural map*

$$\bigoplus_{i=1}^{n} \mathbf{C}^* \otimes [P_i] \longrightarrow SK_1(C) = \mathrm{CH}^2(C,1)$$

is injective.

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Notation and Conventions

1. A variety means a quasi-projective variety over a field.
2. For a variety X over $\mathbf{C}$, X^{an} denotes the associated analytic space: $X^{an} = X(\mathbf{C})$.
3. For a nonsingular variety over a field k, $Z^r(X)$ (resp. $Z_d(X)$) denotes the free abelian group generated by integral subvarieties of X of codimension r (resp. dimension d). $Z_0(X)_{\mathrm{deg}=0}$ denotes the subgroup of 0-cycles of degree 0. $Z^r(X)_{\mathrm{rat}}$ is the subgroup of cycles which are rationally equivalent to zero (cf. [B1]):
$$Z^r(X)_{\mathrm{rat}} := \mathrm{Image}(\bigoplus_{x \in X^{r-1}} \kappa(x)^* \longrightarrow Z^r(X)).$$
(X^r denotes the set of points of X of codimension r.) We put $\mathrm{CH}^r(X) = Z^r(X)/Z^r(X)_{\mathrm{rat}}$.
4. For a commutative ring A, $K_n^M(A)$ denotes the Milnor K-group, which is defined as follows ([Mi], [Sr]):

$$K_n^M(A) := (A^*)^{\otimes n}/\{\sum x_1 \otimes \cdots \otimes y \otimes \cdots \otimes (1-y) \otimes \cdots \otimes x_n \ ; \ x_i, y, 1-y \in A^*\}.$$

2. Arithmetic Hodge structure and Milnor K-groups

We introduce arithmetic Hodge structure, and construct the cycle map (1.1) from higher Chow groups of algebraic curves to 2-extension groups of arithmetic Hodge structures. To do this, we construct the regulator maps (2.6) and (2.7) (for the latter, we use the theory of mixed Hodge modules), and show Lemma 2.7 which defines the cycle map.

2.1. **Admissible variation of mixed Hodge structure.** We introduce the notion of variation of mixed Hodge structure on a nonsingular variety X whose base field is not necessarily $\mathbf{C}$, but a subfield k of $\mathbf{C}$, though it is a slight modification of [SZ]. We simply write $X^{an} := X^{an}_{\mathbf{C}}$ the associated analytic variety of $X \otimes_k \mathbf{C}$.

Definition 2.1. Let X be a nonsingular variety over k. Then an *admissible variation of mixed Hodge structure* on X is defined to be the data $(H_{\mathbf{Z}}, H_{\mathcal{O}}, W_{\mathbf{Q},\bullet}, W_\bullet, F^\bullet, \nabla, i)$ where:

- $H_{\mathbf{Z}}$ is a local system of finite $\mathbf{Z}$-module on X^{an},
- $H_{\mathcal{O}}$ is a locally free (Zariski) sheaf of $\mathcal{O}_X$-module of the same rank as $H_{\mathbf{Z}}$,
- $W_{\mathbf{Q},\bullet}$ (resp. $W_\bullet$) is a finite increasing filtration of $H_{\mathbf{Q}} := H_{\mathbf{Z}} \otimes \mathbf{Q}$ (resp. $H_{\mathcal{O}}$), called the *weight filtration*,
- $F^\bullet$ is a finite decreasing filtration on $H_{\mathcal{O}}$ by locally free $\mathcal{O}_X$-submodules, called the *Hodge filtration*,
- $\nabla : H_{\mathcal{O}} \to H_{\mathcal{O}} \underset{\mathcal{O}_X}{\otimes} \Omega^1_{X/k}$ an integrable connection (called the (*algebraic*) *Gauss-Manin connection*),
- $i : H_{\mathbf{Z}} \otimes \mathcal{O}^{an}_X \xrightarrow{\sim} H^{an}_{\mathcal{O}}$ (called the *comparison isomorphism*), or equivalently, i induces an isomorphism $H_{\mathbf{Z}} \otimes \mathbf{C}_X \xrightarrow{\sim} \ker\nabla^{an}$,

and these satisfy:

(1) $W_{\mathbf{Q},\bullet}$ and $W_\bullet$ are compatible under the comparison isomorphism i.
(2) For all points $s \in X^{an}$, the fiber $H_{\mathbf{Z},s} \overset{i}{\hookrightarrow} H_{\mathcal{O}} \otimes \mathbf{C}(s)$ with the induced filtrations $W_{\bullet,s}$ on $H_{\mathbf{Q},s}$ and $F^\bullet_s$ on $H_{\mathcal{O}} \otimes \mathbf{C}(s)$ defines a mixed Hodge structure.
(3) (Griffiths transversality) $W_\bullet$ and $F^\bullet$ satisfy the following:

$$\nabla(W_\ell) \subset W_\ell \otimes \Omega^1_{X/\mathbf{C}}, \quad \nabla(F^p) \subset F^{p-1} \otimes \Omega^1_{X/\mathbf{C}} \quad \text{for } \forall \ell, p.$$

(4) (polarizability) For each ℓ, there is a $\mathbf{Q}$-bilinear form $Q : \mathrm{Gr}^W_\ell(H_{\mathbf{Q}}) \otimes \mathrm{Gr}^W_\ell(H_{\mathbf{Q}}) \to \mathbf{Q}(-\ell)$ and $\mathcal{O}_X$-bilinear form $Q_X : \mathrm{Gr}^W_\ell(H_{\mathcal{O}}) \otimes \mathrm{Gr}^W_\ell(H_{\mathcal{O}}) \to \mathcal{O}_X$ satisfying:
   (a) Q and Q_X are compatible under the comparison isomorphism i.
   (b) Q defines a polarization form on the $\mathbf{Q}$-Hodge structure $(\mathrm{Gr}^W_\ell(H_{\mathbf{Q},s}), F^\bullet_s)$ for all $s \in X^{an}$.
   (c) $Q_X(\nabla(x), y) + Q_X(x, \nabla(y)) = dQ_X(x, y)$ for any local sections $x, y \in H_{\mathcal{O}}$.
(5) (admissibility) When the data $(H_{\mathbf{Q}}, H_{\mathcal{O}}, W_\bullet, F^\bullet, \nabla, i)$ is pulled back to a nonsingular complex algebraic curve C, it satisfies the admissibility ([K] 1.9):
   (a) Any local monodromy around $\overline{C} - C$ is quasi-unipotent. Here $\overline{C}$ denotes the smooth completion of C.
   (b) The logarithm N of the unipotent part of the local monodromy admits a weight filtration relative to $W_\bullet$ ([SZ] §2).
   (c) The Hodge filtration $F^\bullet$ can be extended to a locally free subsheaf $\overline{F}^\bullet$ of Deligne's canonical extension $\overline{H}_C$ such that $\mathrm{Gr}^\bullet_{\overline{F}}\mathrm{Gr}^W_\bullet(\overline{H}_C)$ is locally free.

We denote the category of admissible variations of mixed Hodge structures on X by VMHS(X). We write VMHS(k) by MHS(k), and call it the category of (*graded polarizable*) *mixed Hodge structures.*

We denote $\mathbf{Z}_X(r) = ((2\pi i)^r\mathbf{Z}_X, \mathcal{O}_X, W_{\mathbf{Q}}, W, F, \nabla, i)$ the *Tate variation of Hodge structure* on X of weight $-2r$, where $\mathrm{Gr}^W_{-2r}\mathbf{Q}_X(r) = \mathbf{Q}_X(r)$ and $\mathrm{Gr}_F^{-r}\mathcal{O}_X = \mathcal{O}_X$.

2.2. Let X be a nonsingular variety over a subfield k of $\mathbf{C}$. Let f be an invertible algebraic function on X: $f \in \mathcal{O}(X)^*_{\mathrm{Zar}}$. Then we define the variation of mixed Hodge structure $V_f = (V_{f,\mathbf{Z}}, V_{f,\mathcal{O}}, W_{\mathbf{Q}}, W, F, \nabla, i)$ as follows.

$$V_{f,\mathbf{Z}} = \mathbf{Z}_X(1)e_{-2} \oplus \mathbf{Z}_X(0)(e_0 - \log f^{an}\cdot e_{-2}), \quad W_{\mathbf{Q},0} = V_{f,\mathbf{Q}} \supset W_{\mathbf{Q},-1} = W_{\mathbf{Q},-2} = \mathbf{Q}(1)e_{-2},$$

$$V_{f,\mathcal{O}} := \mathcal{O}_X \cdot e_{-2} \oplus \mathcal{O}_X \cdot e_0, \quad W_{\mathcal{O},0} = V_{f,\mathcal{O}} \supset W_{\mathcal{O},-1} = W_{\mathcal{O},-2} = \mathcal{O}_X \cdot e_{-2},$$

$$F^{-1} = V_{f,\mathcal{O}} \supset F^0 = \mathcal{O}_X \cdot e_0,$$

the comparison isomorphism

$$i : V_{\mathbf{Z}} \longrightarrow V_{f,\mathcal{O}^{an}}$$

is defined in the natural way, and the Gauss-Manin connection is given as follows

$$\nabla : V_{f,\mathcal{O}} \longrightarrow V_{f,\mathcal{O}} \otimes \Omega^1_{X/k}, \quad e_{-2} \mapsto 0, \; e_0 \mapsto \frac{df}{f} e_{-2}.$$

Then V_f defines the following extension of variations of mixed Hodge structure:

$$0 \longrightarrow \mathbf{Z}_X(1) \longrightarrow V_f \longrightarrow \mathbf{Z}_X(0) \longrightarrow 0. \tag{2.1}$$

The following lemma is well known to specialists:

Lemma 2.2 (cf. [A3] §2). *Let X be a nonsingular algebraic variety over a subfield k of $\mathbf{C}$. Assume that k is algebraically closed.*

(1) *The natural map*

$$\mathrm{Ext}^1_{\mathrm{VMHS}(X)}(\mathbf{Z}(0), \mathbf{Z}(1)) \longrightarrow \mathrm{Ext}^1_{\mathrm{VMHS}(X_{\mathbf{C}})}(\mathbf{Z}(0), \mathbf{Z}(1))$$

is bijective.

(2) *The regulator map*

$$r^1_{X_{\mathbf{C}}} : \mathcal{O}(X_{\mathbf{C}})^*_{\mathrm{Zar}} \longrightarrow \mathrm{Ext}^1_{\mathrm{VMHS}(X_{\mathbf{C}})}(\mathbf{Z}(0), \mathbf{Z}(1)), \quad f \longmapsto [V_f]$$

is bijective. Here $[V_f]$ denotes the extension class (2.1).

(3) *The regulator map*

$$r^2_X : K_2^M(\mathcal{O}(X)) \longrightarrow \mathrm{Ext}^2_{\mathrm{VMHS}(X)}(\mathbf{Z}(0), \mathbf{Z}(2)), \quad \{f, g\} \mapsto [V_f] \cup [V_g]$$

is well-defined, that is, the Steinberg symbol $\{f, 1-f\}$ is annihilated in the right hand side.

2.3. Arithmetic Hodge structure.

Definition 2.3. We define the category of *arithmetic Hodge structures* by

$$\underline{\mathrm{M}}(\mathbf{C}) = \varinjlim_{S} \mathrm{VMHS}(S), \tag{2.2}$$

where $S = \mathrm{Spec}A$ runs over all sub-algebras $A \subset \mathbf{C}$ which are regular and finitely generated over $\bar{\mathbf{Q}}$, and the transition morphisms are the pull-backs of variations of mixed Hodge structures (cf. Definition 2.1 $k = \bar{\mathbf{Q}}$).

We first construct the k-th cohomology functor H^k from the category of algebraic varieties over $\mathbf{C}$ to $\underline{\mathrm{M}}(\mathbf{C})$. Let X be an algebraic variety over $\mathbf{C}$. By considering the coefficients of the defining equations of X as parameters, we can obtain a model $f : X_S \to S$ and the Cartesian diagram:

$$\begin{array}{ccc} X_S & \longleftarrow & X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ S & \xleftarrow{\;a\;} & \mathrm{Spec}\mathbf{C}, \end{array} \tag{2.3}$$

where S is a nonsingular variety over $\bar{\mathbf{Q}}$, and the map a factors through the generic point $\mathrm{Spec}\ \bar{\mathbf{Q}}(S) \hookrightarrow S$. If necessary, by replacing S by a sufficiently small open S', we may assume that the higher direct image $R^k f_* \mathbf{Z}_{X_S}(r)$ is a local system, and admits a variation of mixed Hodge structure. Let $[R^k f_* \mathbf{Z}_{X_S}(r)]$ be the arithmetic Hodge structure represented by the variation of mixed Hodge structure $R^k f_* \mathbf{Z}_{X_S}(r)$. Although the model (2.3) is not uniquely determined, any two models X_{S_1} and X_{S_2} can be imbedded into the following diagram:

$$\begin{array}{ccccc} & & X_{S'} & & \\ & \swarrow & & \searrow & \\ X_{S_1} & & & & X_{S_2}. \end{array}$$

Therefore the representative $[R^k f_* \mathbf{Z}_{X_S}(r)]$ does not depend on the choice of the model (2.3). We define the k-th cohomology of X by $H^k(X, \mathbf{Z}(r)) := [R^k f_* \mathbf{Z}_{X_S}(r)] \in \underline{\mathrm{M}}(\mathbf{C})$.

Next we construct the realization functor $\underline{\mathrm{M}}(\mathbf{C}) \to \mathrm{MHS}(\mathbf{C})$ to the category of graded polarizable mixed Hodge structures. The morphism $a : \mathrm{Spec}\mathbf{C} \to S$ in the diagram (2.3) defines the closed point s of $S_{\mathbf{C}} = S \otimes_{\bar{\mathbf{Q}}} \mathbf{C}$. Then, for a variation of mixed Hodge structure $H \in \mathrm{VMHS}(S)$, we associate the fiber $H_s \in \mathrm{MHS}(\mathbf{C})$ over the point s. These are functorial and commute with the transition functors $\mathrm{VMHS}(S) \to \mathrm{VMHS}(S')$. Therefore these define the functor $\underline{\mathrm{M}}(\mathbf{C}) \to \mathrm{MHS}(\mathbf{C})$, which we call the realization functor. It is clear that the realization functor is exact and faithful.

Lemma 2.4. *There is the isomorphism*

$$\mathrm{Ext}^1_{\underline{\mathrm{M}}(\mathbf{C})}(\mathbf{Z}(0),\mathbf{Z}(1)) \simeq (\mathbf{C}\otimes_{\bar{\mathbf{Q}}}\mathbf{C})^*(\simeq \mathbf{C}^* \times_{\bar{\mathbf{Q}}^*} \mathbf{C}^*). \tag{2.4}$$

Under the above isomorphism, the natural map induced from the realization functor

$$\mathrm{Ext}^1_{\underline{\mathrm{M}}(\mathbf{C})}(\mathbf{Z}(0),\mathbf{Z}(1)) \longrightarrow \mathrm{Ext}^1_{\mathrm{MHS}(\mathbf{C})}(\mathbf{Z}(0),\mathbf{Z}(1)) \simeq \mathbf{C}^* \tag{2.5}$$

is given by the multiplication $\lambda\otimes\mu \mapsto \lambda\mu$. (The isomorphism in the right hand side is due to Lemma 2.2 (3) *).*

Proof. By Lemma 2.2 (2) and (3), we have

$$\begin{aligned}\mathrm{Ext}^1_{\underline{\mathrm{M}}(\mathbf{C})}(\mathbf{Z}(0),\mathbf{Z}(1)) &= \varinjlim_S \mathrm{Ext}^1_{\mathrm{VMHS}(S))}(\mathbf{Z}_S(0),\mathbf{Z}_S(1)) \\ &\simeq \varinjlim_S (\mathcal{O}(S)_{\mathrm{Zar}}\otimes_{\bar{\mathbf{Q}}}\mathbf{C})^* \\ &= (\mathbf{C}\otimes_{\bar{\mathbf{Q}}}\mathbf{C})^*.\end{aligned}$$

The latter assertion is clear from the construction of the above isomorphism. □

Let A be a subalgebra of $\mathbf{C}$ which is regular and finitely generated over $\bar{\mathbf{Q}}$. Put $S = \mathrm{Spec}A$. There is the regulator map (cf. §2.2):

$$r^1_S : A^* \longrightarrow \mathrm{Ext}^1_{\mathrm{VMHS}(S)}(\mathbf{Z}_S(0),\mathbf{Z}_S(1)), \quad f \longmapsto [V_f].$$

Taking the inductive limit over $S = \mathrm{Spec}A$, we have the regulator map to the extension group of arithmetic Hodge structures:

$$r^1_{\mathbf{C}} : \mathbf{C}^* \longrightarrow \mathrm{Ext}^1_{\underline{\mathrm{M}}(\mathbf{C})}(\mathbf{Z}(0),\mathbf{Z}(1)). \tag{2.6}$$

Under the isomorphism (2.4), $r^1_{\mathbf{C}}$ is given by $\lambda \mapsto \lambda\otimes 1$ or $1\otimes\lambda$. The ambiguity depends on the choice of the isomorphism (2.4).

In particular, we have:

Lemma 2.5. *The regulator map* (2.6) *is injective.*

Remark 2.6. Only to show Lemma 2.5, we do not need Lemma 2.4. In fact, it follows from the fact that the composition of (2.6) and (2.5) is bijective.

2.4. **Cycle maps.** Let C be a nonsingular projective curve over $\mathbf{C}$. Let U be a Zariski open set of C. We take a model $f : U_S \to S$ of U. By Lemma 2.2, we have

$$r^2_{U_S} : K^M_2(\mathcal{O}(U_S)) \longrightarrow \mathrm{Ext}^2_{\mathrm{VMHS}(U_S)}(\mathbf{Z}_{U_S}(0),\mathbf{Z}_{U_S}(2)), \quad \{f,g\} \longmapsto [V_f]\cup[V_g].$$

Moreover, if $U \neq C$, we have

$$\mathrm{Ext}^2_{\mathrm{VMHS}(U_S)}(\mathbf{Z}_{U_S}(0),\mathbf{Z}_{U_S}(2)) \longrightarrow \mathrm{Ext}^1_{\mathrm{VMHS}(S)}(\mathbf{Z}_S(0),R^1f_*\mathbf{Z}_{U_S}(2))$$

as follows. For a 2-extension of variation of mixed Hodge structure on U_S

$$0 \longrightarrow \mathbf{Z}_{U_S}(2) \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \mathbf{Z}_{U_S}(0) \longrightarrow 0,$$

we associate the extension of variation of mixed Hodge structure on S

$$0 \longrightarrow R^1 f_* \mathbf{Z}_{U_S}(2) \longrightarrow \mathbb{R}^1 f_*[V_1 \to V_2] \longrightarrow \mathbf{Z}_S(0) \longrightarrow 0.$$

Note that the middle term admits the variation of mixed Hodge structure by the theory of mixed Hodge modules ([SaM2]).

We thus have the map

$$K_2^M(\mathcal{O}(U_S)) \longrightarrow \mathrm{Ext}^1_{\mathrm{VMHS}(S)}(\mathbf{Z}_S(0), R^1 f_* \mathbf{Z}_{U_S}(2)).$$

Taking the inductive limit over S and U, we have

$$r^2_{\mathbf{C}(C)} : K_2^M(\mathbf{C}(C)) \longrightarrow \mathrm{Ext}^1_{\underline{\mathrm{M}}(\mathbf{C})}(\mathbf{Z}(0), H^1(\mathbf{C}(C), \mathbf{Z}(2))). \tag{2.7}$$

Here we put $\mathrm{Ext}^1_{\underline{\mathrm{M}}(\mathbf{C})}(\mathbf{Z}(0), H^1(\mathbf{C}(C), \mathbf{Z}(2))) := \varinjlim_U \mathrm{Ext}^1_{\underline{\mathrm{M}}(\mathbf{C})}(\mathbf{Z}(0), H^1(U, \mathbf{Z}(2)))$.

Lemma 2.7. *The following diagram is commutative:*

$$\begin{array}{ccc} K_2^M(\mathbf{C}(C)) & \xrightarrow{T} & K_1^M(\mathbf{C}) \otimes Z_0(C)_{\mathbf{deg}=0} \\ {\scriptstyle r^2_{\mathbf{C}(C)}} \downarrow & & \downarrow {\scriptstyle r^1_{\mathbf{C}} \otimes \mathrm{id}} \\ \mathrm{Ext}^1_{\underline{\mathrm{M}}(\mathbf{C})}(\mathbf{Z}(0), H^1(\mathbf{C}(C), \mathbf{Z}(2))) & \longrightarrow & \mathrm{Ext}^1_{\underline{\mathrm{M}}(\mathbf{C})}(\mathbf{Z}(0), \mathbf{Z}(1)) \otimes Z_0(C)_{\mathbf{deg}=0}. \end{array}$$

Here T is the tame symbol (cf. §1), and the below map is the one induced from the residue map $H^1(U, \mathbf{Z}(2)) \to \mathbf{Z}(1) \otimes Z_0(C)_{\mathbf{deg}=0}$.

Proof. See [A3] Lemma 3.5. □

By Lemma 2.7 and the long exact sequence

$$\begin{aligned} \cdots \longrightarrow \mathrm{Ext}^1_{\underline{\mathrm{M}}(\mathbf{C})}(\mathbf{Z}(0), H^1(\mathbf{C}(C), \mathbf{Z}(2))) &\longrightarrow \mathrm{Ext}^1_{\underline{\mathrm{M}}(\mathbf{C})}(\mathbf{Z}(0), \mathbf{Z}(1)) \otimes Z_0(C)_{\mathbf{deg}=0} \\ &\longrightarrow \mathrm{Ext}^2_{\underline{\mathrm{M}}(\mathbf{C})}(\mathbf{Z}(0), H^1(C, \mathbf{Z}(2)) \longrightarrow \cdots, \end{aligned}$$

we obtain the cycle map (cf.(1.1)):

$$\rho^2 : F^2\mathrm{CH}^2(C, 1)(= V(C)) \longrightarrow \mathrm{Ext}^2_{\underline{\mathrm{M}}(\mathbf{C})}(\mathbf{Z}(0), H^1(C, \mathbf{Z}(2))). \tag{2.8}$$

3. Proof of the Main Theorem

We prove Theorem 1.3. By Lemma 2.5 and the construction of the map (1.1) (cf. (2.8)), we can reduce it to the following:

Proposition 3.1. *Let C and $\{P_1, \cdots, P_n\}$ be as in Theorem 1.3. Put $U := C - \{P_1, \cdots, P_n\}$. Then the image of the residue map*

$$\mathrm{Ext}^1_{\underline{\mathrm{M}}(\mathbf{C})}(\mathbf{Z}(0), H^1(U, \mathbf{Z}(2))) \longrightarrow \bigoplus_{i=1}^{n-1} \mathrm{Ext}^1_{\underline{\mathrm{M}}(\mathbf{C})}(\mathbf{Z}(0), \mathbf{Z}(1)) \otimes [P_i - P_{i+1}]$$

is zero.

3.1. **Reduction of Proposition 3.1.** In order to prove proposition 3.1, we reduce it to three parts (3.3), (3.4) and (3.5) below.

Let $\overline{f} : C_S \to S$ and $f : U_S \to S$ be models of C and U respectively, with the embedding $\bar{\mathbf{Q}}(S) \hookrightarrow \mathbf{C}$. Let $p_i : P_{i,S} \to S$ be a model of P_i. By replacing S by S' which is generically finite over a sufficiently small Zariski open set of S, we may assume that each p_i is an isomorphism and that $\{P_{i,S}\}$ are disjoint.

Recall that C and $\{P_1, \cdots, P_n\}$ are generic (Definition 1.2). By definition, $\{P_1, \cdots, P_n\}$ is contained in $C \cap D$ for a generic plane curve D such that the coefficients of the defining equations of C and D are algebraically independent over $\bar{\mathbf{Q}}$. Let e be the degree of the curve D. Let $\mathbb{S} := \bar{\mathbf{Q}}[X_0, X_1, X_2]$ and $\mathbb{S}^r$ denotes the homogeneous part of degree r. There are the versal families $\mathcal{C} \to \mathbb{S}^d$ and $\mathcal{D} \to \mathbb{S}^e$ of plane curves of degree d and e respectively. We also have the versal family $\mathcal{U} = \mathcal{C} - \mathcal{C} \cap \mathcal{D} \to \mathfrak{M}$ where $\mathfrak{M}$ is an open set of $\mathbb{S}^d \times \mathbb{S}^e$. Since C and $\{P_1, \cdots, P_n\}$ are generic, we may assume that the model $U_S \to S$ is a pull back of $\mathcal{U} \to \mathfrak{M}$ by a dominant morphism $S \to \mathfrak{M}$.

We prepare some categories. Let $\mathrm{LS}(X^{an})$ be the category of local systems of finite $\mathbf{Z}$-modules on an analytic site X^{an}. Let $\mathrm{MF}(D_X)$ be the category of good filtered D_X-modules. Note that this is not abelian but exact. For example, $H^1_{\mathrm{dR}}(U_S/S) := R^1 f_* \Omega^{\cdot}_{U_S/S} \simeq R^1 \overline{f}_* \Omega^{\cdot}_{C_S/S}(\log \Sigma P_{i,S})$ is a D_S-module with the Hodge filtration $F^p := R^1 \overline{f}_* \Omega^{\cdot \geq p}_{C_S/S}(\log \Sigma P_{i,S})$ as a good filtration. We write the filtered D_S-module $(H^1_{\mathrm{dR}}(U_S/S), F^{\bullet + r})$ by $H^1_{\mathrm{dR}}(U_S/S)(r)$ simply. Let $\mathrm{VMHS}(X)_{\mathbf{Q}}$ be the category of $\mathbf{Q}$-variations of mixed Hodge structures on X which is defined by replacing $H_{\mathbf{Z}}$ by a $\mathbf{Q}$-local system $H_{\mathbf{Q}}$ in Definition 2.1. We define $\underline{\mathrm{M}}(\mathrm{C})_{\mathbf{Q}} := \varinjlim_S \mathrm{VMHS}(S)_{\mathbf{Q}}$.

There are the natural exact functors $\mathrm{VMHS}(X) \to \mathrm{VMHS}(X)_{\mathbf{Q}}$, $\mathrm{VMHS}(X) \to \mathrm{LS}(X^{an})$ and $\mathrm{VMHS}(X)_{\mathbf{Q}} \to \mathrm{MF}(D_X)$ which induce the maps between each Yoneda extension groups.

Lemma 3.2 ([A3] Lemma 4.2). *There are the following exact sequences:*

(1)

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{LS}(S^{an})}(\mathbf{Z}_S(0), R^1 f_* \mathbf{Z}_{U_S}(2))_{\mathrm{tor}} \xrightarrow{a} \mathrm{Ext}^1_{\mathrm{VMHS}(S)}(\mathbf{Z}_S(0), R^1 f_* \mathbf{Z}_{U_S}(2)) \longrightarrow \mathrm{Ext}^1_{\mathrm{VMHS}(S)_{\mathbf{Q}}}(\mathbf{Q}_S(0), R^1 f_* \mathbf{Q}_{U_S}(2)), \quad (3.1)$$

where M_{tor} denotes the torsion subgroup of an abelian group M,

(2)

$$0 \longrightarrow \mathbf{C}^* \otimes \Gamma(S, R^1 f_* \mathbf{Q}_{U_S}(1)) \longrightarrow \mathrm{Ext}^1_{\mathrm{VMHS}(S)_{\mathbf{Q}}}(\mathbf{Q}_S(0), R^1 f_* \mathbf{Q}_{U_S}(2)) \longrightarrow \mathrm{Ext}^1_{\mathrm{MF}(D_S)}(\mathcal{O}_S, H^1_{\mathrm{dR}}(U_S/S)(2)). \quad (3.2)$$

By Lemma 3.2, we have reduced Proposition 3.1 to the followings:

$$\mathrm{Ext}^1_{\mathrm{MF}(D_S)}(\mathcal{O}_S, H^1_{\mathrm{dR}}(U_S/S)(2)) = 0, \quad (3.3)$$

$$\Gamma(S, R^1 f_* \mathbf{Q}_{U_S}(1)) = 0, \tag{3.4}$$

$$\mathrm{Image}(\mathrm{Ext}^1_{\mathrm{LS}(S)}(\mathbf{Z}_S(0), R^1 f_* \mathbf{Z}_{U_S}(2))_{\mathrm{tor}} \to \mathrm{Ext}^1_{\mathrm{VMHS}(S)}(\mathbf{Z}_S(0), \mathbf{Z}_S(1))) = 0. \tag{3.5}$$

Remark 3.3. Due to (3.3) and (3.4), we have

$$\mathrm{Ext}^1_{\mathrm{VMHS}(S)_{\mathbf{Q}}}(\mathbf{Q}_S(0), R^1 f_* \mathbf{Q}_{U_S}(2)) = 0.$$

This shows the following vanishing

$$\mathrm{Ext}^1_{\underline{\mathrm{M}}(\mathbf{C})_{\mathbf{Q}}}(\mathbf{Q}(0), H^1(U, \mathbf{Q}(2))) = 0.$$

However, I do not know whether the above vanishing holds even for $\mathbf{Z}$-coefficients.

3.2. **Proof of** (3.3). We can easily see that $\mathrm{Ext}^1_{\mathrm{MF}(D_S)}(\mathcal{O}_S, H^1_{\mathrm{dR}}(U_S/S)(2))$ is isomorphic to the kernel of the Gauss-Manin connection $\nabla : F^1 H^1_{\mathrm{dR}}(U_S/S) \otimes \Omega^1_{S/\bar{\mathbf{Q}}} \to H^1_{\mathrm{dR}}(U_S/S) \otimes \Omega^2_{S/\bar{\mathbf{Q}}}$ (see [A1] §3). Therefore it suffices to show the following:

Theorem 3.4 (Symmetrizer lemma for open curves). *The following map*

$$F^1 H^1_{\mathrm{dR}}(U_S/S) \otimes \Omega^1_{S/\bar{\mathbf{Q}}} \longrightarrow F^0/F^1 H^1_{\mathrm{dR}}(U_S/S) \otimes \Omega^2_{S/\bar{\mathbf{Q}}}, \quad x \otimes \omega \mapsto \nabla(x) \wedge \omega \tag{3.6}$$

is injective.

Proof. See [AS] Theorem (III) or [A3] Theorem 5.4. □

Corollary 3.5. *Let $r = 0$, or 1. Then the following map*

$$H^0(C, \Omega^1_{C/\mathbf{C}}(\Sigma P_i)) \otimes \Omega^r_{\mathbf{C}/\bar{\mathbf{Q}}} \longrightarrow H^1(C, \mathcal{O}_C) \otimes \Omega^{r+1}_{\mathbf{C}/\bar{\mathbf{Q}}}, \quad x \otimes \omega \mapsto \nabla(x) \wedge \omega$$

is injective.

Proof. In case $r = 1$, it follows from the fact that the exactness of (3.6) also holds when we take a base change by any dominant morphism $S' \to S$. The case $r = 0$ is straightforward from the case $r = 1$. □

3.3. **Proof of** (3.4). Let $K := \bar{\mathbf{Q}}(S)$ be the function field of S, with the embedding $K \hookrightarrow \mathbf{C}$. Put $C_K := C_S \otimes K$, $U_K := U_S \otimes K$ the generic fibers and $C_{\overline{K}} := C_S \otimes \overline{K}$, $U_{\overline{K}} := U_S \otimes \overline{K}$ the geometric generic fibers. Let $G_K := \mathrm{Gal}(\overline{K}/K)$ be the Galois group.

Let $t \in S^{an}$ be the associated point of the embedding $K \hookrightarrow \mathbf{C}$. Put $U_t := f^{-1}(t)$. There is the isomorphism $\Gamma(S^{an}, R^1 f_* \mathbf{Q}_{U_S}(1)) \simeq H^1(U_t, \mathbf{Q}(1))^{\pi_1(S^{an},t)}$, and a (non-canonical) injection $H^1(U_t, \mathbf{Q}(1)) \hookrightarrow H^1_{\text{ét}}(U_t, \mathbf{Q}_\ell(1)) \xleftarrow{\sim} H^1_{\text{ét}}(U_{\overline{K}}, \mathbf{Q}_\ell(1))$. Under the natural inclusion $\pi_1(S^{an}, t) \hookrightarrow \pi_1^{\text{ét}}(S, \overline{K})$, we may assume that the action of those on the étcohomology is compatible. Note that the image of $\pi_1(S^{an}, t)$ is dense (with respect to the profinite topology). Since the action of $\pi_1^{\text{ét}}(S, \overline{K})$ on the étcohomology is ℓ-adically continuous, the invariant part is the same as that by the action of any dense subgroup, in particular, $\pi_1(S^{an},t)$. Moreover the natural map $G_K \to \pi_1^{\text{ét}}(S,\overline{K})$ is surjective. Thus we have

$$\Gamma(S^{an}, R^1 f_* \mathbf{Q}_{U_S}(1)) \simeq H^1(U_t, \mathbf{Q}(1))^{\pi_1(S^{an},t)} \hookrightarrow H^1_{\text{ét}}(U_{\overline{K}}, \mathbf{Q}_\ell(1))^{\pi_1^{\text{ét}}(S,\overline{K})} = H^1_{\text{ét}}(U_{\overline{K}}, \mathbf{Q}_\ell(1))^{G_K}.$$

Therefore, in order to prove (3.4), it suffices to show

$$H^1_{\text{ét}}(U_{\overline{K}}, \mathbf{Q}_\ell(1))^{G_K} = 0,$$

or equivalently,

$$H^1_{\text{ét}}(U_{\overline{K}}, \mathbf{Z}_\ell(1))^{G_K} = 0. \tag{3.7}$$

By the localization sequence and the smooth purity, we have the exact sequence

$$0 \longrightarrow H^1_{\text{ét}}(C_{\overline{K}}, \mathbf{Z}/\ell^m(1)) \longrightarrow H^1_{\text{ét}}(U_{\overline{K}}, \mathbf{Z}/\ell^m(1)) \longrightarrow \bigoplus_{i=1}^{n-1} \mathbf{Z}/\ell^m[P_{i+1} - P_i] \longrightarrow 0.$$

By applying the continuous Galois cohomology functor $\mathbb{R}\mathrm{Hom}_{G_K\text{-cont}}(\mathbf{Z}/\ell^m, -)$, we obtain the *algebraic Abel-Jacobi map*

$$\rho_m : \bigoplus_{i=1}^{n-1} \mathbf{Z}/\ell^m[P_{i+1} - P_i] \longrightarrow H^1_{\text{cont}}(G_K, H^1_{\text{ét}}(C_{\overline{K}}, \mathbf{Z}/\ell^m(1))).$$

By the Picard-Lefschetz theory, we can easily see that $H^1_{\text{ét}}(C_{\overline{K}}, \mathbf{Z}_\ell(1))^{G_K} = 0$. Therefore, in order to show (3.7), it suffices to show that the following map is injective:

$$\varprojlim_m \rho_m : \bigoplus_{i=1}^{n-1} \mathbf{Z}_\ell[P_{i+1} - P_i] \longrightarrow \varprojlim_m H^1_{\text{cont}}(G_K, H^1_{\text{ét}}(C_{\overline{K}}, \mathbf{Z}/\ell^m(1))).$$

Due to the Hochschild-Serre spectral sequence (cf, [Et] Chapter III, 2.11.), the above is equivalent to that the following map is injective:

$$\varprojlim_m c_m : \bigoplus_{i=1}^{n} \mathbf{Z}_\ell[P_i] \longrightarrow \varprojlim_m H^2_{\text{ét}}(C_K, \mathbf{Z}/\ell^m(1)) (= H^2_{\text{ét}}(C_K, \mathbf{Z}_\ell(1))). \tag{3.8}$$

Note that c_m is given by the Kummer theory, that is, by applying $\mathbb{R}\Gamma(C_K, -)$ to the exact sequence

$$0 \longrightarrow \mathbf{Z}/\ell^m(1) \longrightarrow \mathbb{G}_m \xrightarrow{\ell^m} \mathbb{G}_m \longrightarrow 0,$$

we obtain

$$c_m : H^1_{\text{ét}}(C_K, \mathbb{G}_m) (\simeq \mathrm{CH}^1(C_K)) \longrightarrow H^2_{\text{ét}}(C_K, \mathbf{Z}/\ell^m(1)).$$

(The isomorphism in the left hand side is due to Hilbert 90 ([Et] Chapter III, 4.9.)). In particular, $\mathrm{CH}^1(C_K)/\ell^m \to H^2_{\text{ét}}(C_K, \mathbf{Z}/\ell^m(1))$ is injective. Therefore, the map (3.8) is injective if and only if the natural map

$$\bigoplus_{i=1}^{n} \mathbf{Z}_\ell[P_i] \longrightarrow \varprojlim_m (\mathrm{CH}^1(C_K)/\ell^m) \tag{3.9}$$

is injective. We can reduce it to the following two parts:

Lemma 3.6. (1) *The natural map*

$$\bigoplus_{i=1}^{n} \mathbf{Z}[P_i] \longrightarrow \mathrm{CH}^1(C)$$

is injective.

(2) $\mathrm{CH}^1(C_K)$ *is a finitely generated abelian group, in particular, we have*

$$\varprojlim_m (\mathrm{CH}^1(C_K)/\ell^m) \simeq \mathrm{CH}^1(C_K) \otimes \mathbf{Z}_\ell.$$

Proof of (1). Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\Omega^1_{C/\mathbf{C}}) & \longrightarrow & H^0(\Omega^1_{C/\mathbf{C}}(\Sigma P_i)) & \longrightarrow & \bigoplus_{i=1}^{n-1} \mathbf{C}[P_i - P_{i+1}] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow d & & & & \\ & & H^1(\mathcal{O}_C) \otimes \Omega^1_{\mathbf{C}/\bar{\mathbf{Q}}} & \xrightarrow{=} & H^1(\mathcal{O}_C) \otimes \Omega^1_{\mathbf{C}/\bar{\mathbf{Q}}} & & & & \end{array}$$

We obtain the coboundary map $\bigoplus_{i=1}^{n-1} \mathbf{Z}[P_i - P_{i+1}] \subset \bigoplus_{i=1}^{n-1} \mathbf{C}[P_i - P_{i+1}] \to H^1(\mathcal{O}_C) \otimes \Omega^1_{\mathbf{C}/\bar{\mathbf{Q}}}/H^0(\Omega^1_{C/\mathbf{C}})$, which factors through the linear equivalence class. By Corollary 3.5 (i.e. the symmetrizer lemma), the middle vertical arrow d is injective. Thus we have $\bigoplus_{i=1}^{n-1} \mathbf{Z}[P_i - P_{i+1}] \cap Z_0(C)_{\mathrm{rat}} = 0$.

Proof of (2). We use the Mordell-Weil theorem for function fields. Before to do this, we recall it briefly.

Let $K \supset k$ be a finitely generated extension of fields. We assume K/k is a regular extension, that is, k is algebraically closed in K. Let A be an abelian variety defined over K. Then there is an abelian variety B over k, and a homomorphism $\tau : B \otimes K \to A$ over K which satisfies the following universality: "Given an abelian variety B' over k and a homomorphism $\tau' : B' \otimes K \to A$ over K, there is a unique homomorphism $h : B' \to B$ over k such that $\tau \cdot h = \tau'$." We call the pair (B, τ) the *K/k-trace of A*. There are always K/k-traces for any abelian varieties over K ([La1] Chapter VIII §3). We also note that the kernel of τ is a finite group scheme (in case $\mathrm{char}(k) = 0$, τ is a closed immersion).

The following theorem is the Mordell-Weil theorem for function fields, due to Lang and Néron ([LN], [La2] Chapter 6).

Theorem 3.7 (Lang-Néron). *The abelian group $A(K)/\tau B(k)$ is finitely generated.*

We apply the above theorem to the case $k = \bar{\mathbf{Q}}$, $K = \bar{\mathbf{Q}}(S)$ and $A = J(C_K)$ the Jacobian variety of the curve C_K. Note that there is the canonical isomorphism $J(C_K)(K) \simeq \mathrm{CH}^1(C_K)_{\mathrm{hom}}$. To complete the proof of (2), we have to show $B = 0$. By definition, $\mathrm{Image}(H_{1,\text{ét}}(B, \mathbf{Q}_\ell))$ is contained in $H_{1,\text{ét}}(C_{\overline{K}}, \mathbf{Q}_\ell)^{\mathrm{Gal}(\overline{K}/K)}$. By the Picard-Lefschetz theory, we have $H_{1,\text{ét}}(C_{\overline{K}}, \mathbf{Q}_\ell)^{\mathrm{Gal}(\overline{K}/K)} = 0$, which means $B = 0$.

3.4. **Proof of** (3.5). Since the map (3.5) factors as follows

$\mathrm{Ext}^1_{\mathrm{LS}(S^{an})}(\mathbf{Z}_S(0), R^1 f_* \mathbf{Z}_{U_S}(2))_{\mathrm{tor}} \to \mathrm{Ext}^1_{\mathrm{LS}(S^{an})}(\mathbf{Z}_S(0), \mathbf{Z}_S(1))_{\mathrm{tor}} \to \mathrm{Ext}^1_{\mathrm{VMHS}(S)}(\mathbf{Z}_S(0), \mathbf{Z}_S(1))$

it suffices to show

$$\mathrm{Ext}^1_{\mathrm{LS}(S^{an})}(\mathbf{Z}_S(0), \mathbf{Z}_S(1))_{\mathrm{tor}} = 0. \tag{3.10}$$

However, there are the isomorphisms

$$\mathrm{Ext}^1_{\mathrm{LS}(S^{an})}(\mathbf{Z}_S(0), \mathbf{Z}_S(1)) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{S}(S^{an})}(\mathbf{Z}_S(0), \mathbf{Z}_S(1)) \simeq H^1_{\mathrm{sing}}(S^{an}, \mathbf{Z}(1))$$

where $\mathcal{S}(S^{an})$ denotes the category of abelian sheaves on the analytic site S^{an}. By the universal coefficient theorem ([Mac] Chapter III, Theorem 4.1.), there is the isomorphism $H^1_{\mathrm{sing}}(S^{an}, \mathbf{Z}(1)) \simeq \mathrm{Hom}(H_1(S^{an}, \mathbf{Z}), \mathbf{Z}(1))$, which is clearly torsion-free. This completes the proof of (3.10).

References

[A1] M.Asakura: Motives and algebraic de Rham cohomology. (to appear).

[A2] ———: Arithmetic Hodge structure and higher Abel-Jacobi maps. (preprint)

[A3] ———: On the K_1-groups of algebraic curves. (preprint)

[AS] M.Asakura and S.Saito: Filtration on Chow groups and higher Abel-Jacobi maps. (preprint).

[B1] S.Bloch: Lectures on algebraic cycles. Duke Univ. Math. Ser. Vol IV, Duke Univ. Durham, NC, 1980.

[B2] ———: Algebraic cycles and higher K-theory. Adv. in Math. **61** (1986), 267-304.

[Ca] J.Carlson, *Extensions of mixed Hodge structures,* Journées de geométrie algébrique d'Angers 1979, Sijthoff and Noordhoff, pp.107-127.

[CH] J.Carlson and R.Hain, *Extensions of variations of mixed Hodge structures,* Astérisque, 179-180 (1989) 39-65.

[DG] R.Donagi and M.Green: Anew proof of the symmetrizer lemma and a stronger weak Torelli theorem for projective hypersurfaces. J. Diff. Geom. 20 (1984), 459-461.

[EV] H.Esnault and E.Viehweg, *Deligne-Beilinson cohomology,* In Beilinson's conjectures on special values of L-functions, 43–91, Perspect. Math., 4, Academic Press, Boston, MA, 1988.

[EZ] F.El Zein and S.Zucker: Extendability of normal functions associated to algebraic cycles. in Topics in transcendental algebraic geometry, edited by P.Griffiths, Princeton, 1984 pp.269-288.

[G] M.Green: Koszul cohomology and Geometry, in Cornalba, Gomez-Mond, and Verjovsky, Lectures on Riemann surfaces, ICTP, Trieste, Italy, 177-200.

[H] R.Hain, *Algebraic cycles and extensions of variations of mixed Hodge structures,* In Complex geometry and Lie theory, Proc. Sympos. Pure Math. vol.53, AMS, 1991, pp.175-221.

[vdK] van der Kallen: Generators and relations in algebraic K-theory. Proceedings of the International Congress of Mathematicians (Helsinki, 1978), pp. 305–310, Acad. Sci. Fennica, Helsinki, 1980.

[K] M.Kashiwara, *A study of variation of mixed Hodge structure,* Publ. RIMS. Kyoto Univ. **22** (1986), 991-1024.

[La1] S.Lang: Abelian Varieties. Interscience, New York, 1959.

[La2] ———: Fundamentals of Diophantine Geometry. Springer-Verlag, New York, 1983.

[LN] S.Lang and A.Néron: Rational points of abelian varieties over function fields. Amer. J. Math 81 No.1 (1959) 95-118.

[Mac] S.MacLane: *Homology.* Springer-Verlag, 1963.
[Et] J.Milne: Étale cohomology. Princeton, 1980.
[Mi] J.Milnor: Introduction to Algebraic K-theory. Ann. of Math Studies 72, Princeton 1970.
[Q] D.Quillen: Higher algebraic K-theory. I. pp. 85–147. Lecture Notes in Math. Vol. 341, Springer, Berlin 1973.
[SaM1] M.Saito: Modules de Hodge polarisables. Publ. RIMS. Kyoto Univ. 24 (1988), 849-995.
[SaM2] ———: Mixed Hodge modules. Publ. RIMS. Kyoto Univ. 26 (1990), 221-333.
[SaM3] ———, *On the formalism of mixed sheaves,* (RIMS preprint).
[SaM4] ———, *Arithmetic Mixed Hodge structure,* (RIMS preprint).
[SaS] S.Saito: Motives and filtration on Chow groups. Invent. Math. 125 (1996), 149-196.
[Sr] V.Srinivas: Algebraic K-theory. Progress in Math. Vol 90 (1991), Birkhäuser.
[SZ] J.Steenbrink and S.Zucker, *Variation of mixed Hodge structure,* Invent. math. **80** (1985) no.3, 489-542.

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