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Determinantal varieties associated to rank two vector bundles on projective spaces and splitting theorems

Shigehiro Tagami
(joint work with Hideyasu Sumihiro)

0. Introduction

In this paper, we work over an algebraically closed field k. Let E be a rank 2 vector bundle on n-dimensional projective spaces $\mathbf{P}_k^n$.

In [3], H. Sumihiro showed the following theorem in the case char $k = 0$.

Theorem 0.1. *Let P be a 4- or 5-dimensional projective linear subspace of $\mathbf{P}_k^n$ and $\bar{E} = E|P$ be the restriction of E to P. Then E splits into line bundles if and only if $H^1(P, \mathcal{E}nd(\bar{E})) = 0$.*

The aim of this article is to prove that this theorem holds affirmatively true in char $k = p > 0$. The proof is almost same with the one of char $k = 0$, namely, is obtained by studying some geometric structures of the Hilbert scheme of $\mathbf{P}_k^n$ at determinantal subvarieties. In char $k = p > 0$, however, since we can not use Kodaira vanishing theorem and Le-Potier vanishing theorem, we have to observe some vanishings of cohomologies appeared in [3] carefully.

1. Determinantal Varieties

We first recall the definition and some properties of determinantal varieties associated to 2-bundles (cf. [3]).

1.1. **Definition of determinantal varieties.** Let E be a rank 2 vector bundle on an $\mathbf{P}_k^n$, $\pi : P(E) \to \mathbf{P}_k^n$ the projective bundle associated to E over $\mathbf{P}_k^n$, L_E the tautological line bundle on $P(E)$ and let $G = \mathrm{Grass}(H^0(E), m+1)$ be the Grassmann variety which parametrizes $(m+1)$-dimensional linear subspaces of $H^0(\mathbf{P}_k^n, E)$. We assume that E is very ample, i.e., L_E is a very ample line bundle. Then we can take $s = \langle s_1, s_2, \dots, s_{m+1} \rangle \in G$ ($s_i \in H^0(\mathbf{P}_k^n, E)$) with $n = 2m$ (resp. $n = 2m+1$) satisfying the following condition

(*)
1) $Y = Y_s = D_1 \cap D_2 \cap \cdots \cap D_{m+1}$ is a smooth closed subscheme of $P(E)$ of pure codimension $m+1$,
2) $W(s_1) \cap W(s_2) \cap \cdots \cap W(s_{m+1}) = \emptyset$,

where D_i is the tautological divisor on $P(E)$ defined by s_i and $W(s_i)$ is the zero locus on $\mathbf{P}_k^n$ of s_i $(1 \le i \le m+1)$.

Let $X_s = \pi(Y_s)$. Then we can show that X_s is a closed subscheme of $\mathbf{P}_k^n$ which is isomorphic to Y_s through π with the following defining equations:

$$s_i \wedge s_j = 0 \quad (1 \le i \le j \le m+1).$$

Definition 1.1. We call the closed subscheme $X = X_s$ of $\mathbf{P}_k^n$ the (smooth) determinantal variety associated to E defined by $s \in G$.

Remark 1.1. X depends on the choice of $s \in G$ subject to the condition $(*)$.

As for determinantal varieties, we obtain the following.

Theorem 1.1. *Let the notaion be as above.*

1) *$U = \{s \in G | s$ satisfies the condition $(*)\}$ is a Zariski open subset of G.*
2) *There exists a closed subscheme Ξ of $\mathbf{P}^n_k \times U$ such that the second projection $q : \Xi \subset \mathbf{P}^n_k \times U \to U$ is faithfully flat and $X_s = q^{-1}(s)$ for any $s \in U$. Thus smooth determinantal varieties associated to E form a smooth family over an open subset of G and hence they are diffeomorphic to each other.*

When $n = 4$ or 5, let I_X be the defining ideal of a determinantal subvariety X in $\mathbf{P}^n$. Then I_X has the following resolution of vector bundles.

Lemma 1.2. *In above notaion, there exists on an exact sequence*

$$0 \longrightarrow E^*(-c_1) \longrightarrow \bigoplus^3 \mathcal{O}_{\mathbf{P}^n}(-c_1) \longrightarrow I_X \longrightarrow 0,$$

where c_1 is the first Chern number of E.

1.2. **Tangent bundle and normal bundle of determinantal varieties.** In this subsection, we consider when $n = 4$ or 5, i.e., $m = 2$.

1.2.1. Let E be a very ample rank two bundle on $\mathbf{P}^n_k$ and X a determinantal variety associated to E. Let H be the restriction of a hyperplane of $\mathbf{P}^n$ to X and D the restriction of a tautological divisor of $P(E)$ to X through the isomorphism π.

Then we can obtain the following diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T_{P(E)/\mathbf{P}^n}|Y & \xrightarrow{\sim} & \mathcal{O}_X(2D - c_1 H) & & \\
 & & & & \downarrow & & \downarrow \alpha & & \\
0 & \longrightarrow & T_Y & \longrightarrow & T_{P(E)}|Y & \longrightarrow & N_{Y/P(E)} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_X & \longrightarrow & T_{\mathbf{P}^n}|X & \longrightarrow & N_{X/\mathbf{P}^n} & \longrightarrow & 0, \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where α is an injection induced by the snake lemma. Since $N_{Y/P(E)} \simeq \bigoplus^3 \mathcal{O}_X(D)$ by the condition of Y, we obtain the following.

Proposition 1.3. *There exists an exact sequence*

$$0 \longrightarrow \mathcal{O}_X(2D - c_1 H) \longrightarrow \bigoplus^3 \mathcal{O}_X(D) \longrightarrow N_{X/\mathbf{P}^n} \longrightarrow 0.$$

1.2.2. From the exact sequence of the above proposition, we have the following exact sequence

$$0 \to H^0(\mathcal{O}_X(2D - c_1H)) \to \bigoplus^3 H^0(\mathcal{O}_X(D)) \to H^0(N_{X/\mathbf{P}^n})$$
$$\to H^1(\mathcal{O}_X(2D - c_1H)) \to \bigoplus^3 H^1(\mathcal{O}_X(D)).$$

Now we recall $Y = D_1 \cap D_2 \cap D_3$. Consider the canonical exact sequence

$$(*)_1 \qquad 0 \to \mathcal{O}_{P(E)}(D - c_1H) \to \mathcal{O}_{P(E)}(2D - c_1H) \to \mathcal{O}_{D_1}(2D - c_1H) \to 0,$$

from which we obtain the following exact sequence:

$$0 \to H^0(\mathcal{O}_{P(E)}(D - c_1H)) \to H^0(\mathcal{O}_{P(E)}(2D - c_1H)) \to H^0(\mathcal{O}_{D_1}(2D - c_1H))$$
$$\to H^1(\mathcal{O}_{P(E)}(D - c_1H)) \to H^1(\mathcal{O}_{P(E)}(2D - c_1H)) \to H^1(\mathcal{O}_{D_1}(2D - c_1H))$$
$$\to H^2(\mathcal{O}_{P(E)}(D - c_1H)).$$

Since $H^i(D - c_1H) = H^i(E^*)$ $(0 \leq i \leq 4)$ and we can show that $H^0(E^*) = 0$ and $H^1(E^*) = H^{n-1}(E \otimes K_{\mathbf{P}^n}) = 0$, where $K_{\mathbf{P}^n}$ is the canonical divisor of $\mathbf{P}^n$, it turns out that $H^i(\mathcal{O}_{P(E)}(2D - c_1H)) \simeq H^i(\mathcal{O}_{D_1}(2D - c_1H))$ $(i = 0, 1)$ if $H^2(E^*) = H^{n-2}(E \otimes K_{\mathbf{P}^n}) = 0$.

Considering the exact sequences similarly

$$(*)_2 \qquad \begin{aligned} &0 \to \mathcal{O}_{D_1}(D - c_1H) \to \mathcal{O}_{D_1}(2D - c_1H) \to \mathcal{O}_{D_1 \cap D_2}(2D - c_1H) \to 0, \\ &0 \to \mathcal{O}_{P(E)}(-c_1H) \to \mathcal{O}_{P(E)}(D - c_1H) \to \mathcal{O}_{D_1}(D - c_1H) \to 0, \end{aligned}$$

$$(*)_3 \qquad \begin{aligned} &0 \to \mathcal{O}_{D_1 \cap D_2}(D - c_1H) \to \mathcal{O}_{D_1 \cap D_2}(2D - c_1H) \to \mathcal{O}_Y(2D - c_1H) \to 0, \\ &0 \to \mathcal{O}_{D_1}(-c_1H) \to \mathcal{O}_{D_1}(D - c_1H) \to \mathcal{O}_{D_1 \cap D_2}(D - c_1H) \to 0, \\ &0 \to \mathcal{O}_{P(E)}(-D - c_1H) \to \mathcal{O}_{P(E)}(-c_1H) \to \mathcal{O}_{D_1}(-c_1H) \to 0, \end{aligned}$$

we obtain isomorphisms $H^i(\mathcal{O}_{D_1}(2D - c_1H)) \simeq H^i(\mathcal{O}_{D_1 \cap D_2}(2D - c_1H))$ and $H^i(\mathcal{O}_{D_1 \cap D_2}(2D - c_1H)) \simeq H^i(\mathcal{O}_Y(2D - c_1H))$ $(i = 0, 1)$. Summing up the above, we conclude that $H^i(\mathcal{O}_X(2D - c_1H)) \simeq H^i(\mathbf{P}^n, S^2(E)(-c_1))$.

On the other hand, since there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n} \longrightarrow \mathcal{E}nd(E) \longrightarrow S^2(E)(-c_1) \longrightarrow 0,$$

we have a canonical isomorphism $H^1(S^2(E)(-c_1)) \simeq H^1(\mathcal{E}nd(E))$ and $\dim H^0(S^2(E)(-c_1)) = \dim H^0(\mathcal{E}nd(E)) - 1$.

In addition we easily see that $\dim H^0(\mathcal{O}_X(D)) = \dim H^0(E) - 3$.

Summarizing the above, we get the following proposition.

Proposition 1.4. *Assume that $n = 4$ or 5 and $H^{n-2}(E \otimes K_{\mathbf{P}^n}) = 0$. If $H^1(\mathcal{E}nd(E)) = 0$, then*

$$\dim H^0(N_{X/\mathbf{P}^n}) = 3(\dim H^0(E) - 3) - \dim H^0(\mathcal{E}nd(E)) + 1$$

Remark 1.2. When char $k = 0$, we get $H^i(E^*) \simeq H^{n-i}(E \otimes K_{\mathbf{P}^n_k}) = 0$ for $0 \leq i \leq n - 2$ by Le-Potier vanishing theorem. So we do not need the assumption $H^{n-2}(E \otimes K_{\mathbf{P}^n}) = 0$ in the above proposition. Also the proof itself becomes slightly simpler because we can use the vanishing theorems.

2. Hilbert Schemes

In this section, we assume that $n = 4$ or 5. Let $\mathcal{H}ilb$ be the Hilbert scheme of $\mathbf{P}^n$.

2.1. Let $\varphi : U \ni s \mapsto X_s \in \mathcal{H}ilb$ be the morphism induced by Theorem 1.1. Let $\mathrm{Aut}(E)$ be the automorphism group of E. Then $\mathrm{Aut}(E)$ is a reduced connected linear algebraic group of dimension $\dim H^0(\mathcal{E}nd(E))$.

For every element $g \in \mathrm{Aut}(E)$ and $s = \langle s_1, s_2, s_3 \rangle \in G$, we define

$$g \cdot s = \langle g(s_1), g(s_2), g(s_3) \rangle,$$

where $g(s_i)$ is the composite of s_i with g. Then it defines an action of $\mathrm{Aut}(E)$ on G and we have

$$g \cdot s_i \wedge g \cdot s_j = \det g \; s_i \wedge s_j (1 \leq i \leq j \leq 3),$$

where $\det : \mathrm{Aut}(E) \ni g \mapsto \det(g) \in k^* = k \setminus \{0\}$ is the determinant character. Hence $X_{g \cdot s} = X_s$. Therefore $\mathrm{Aut}(E)$ acts on U and φ is an orbit morphism, i.e., φ is constant on any orbit $O(s) = \{g \cdot s | g \in \mathrm{Aut}(E)\}$.

Then we have the following.

Lemma 2.1. *The stabilizer* $\mathrm{Stab}(s)$ *of* $s \in U$ *is equal to the multiplicative group* k^*.

As a trivial corollary of the above lemma and Proposition 1.3, we observe the following.

a) Every orbit has the same dimension $\dim \mathrm{Aut}(E)/k^*$. Hence the action of $\mathrm{Aut}(E)$ on U is closed, i.e., every orbit is closed in U.

b) $\dim O(s) = \dim H^0(\mathcal{E}nd(E)) - 1$

Proposition 2.2. *Under the same assumptions in Proposition 1.4, if* $H^1(\mathcal{E}nd(E)) = 0$ *then*

$$\dim \overline{\varphi(U)} = \dim H^0(N_{X_s/\mathbf{P}^n}).$$

Proof. Using the exact sequence in Proposition 1.3, we see that $\varphi^{-1}(\varphi(s))$ $(s \in U)$ consists of finitely many orbits. Hence

$$\begin{aligned}
\dim \overline{\varphi(U)} &= \dim U - \dim O(s) \\
&= \dim \mathrm{Grass}(H^0(E), 3) - \dim H^0(\mathcal{E}nd(E)) + 1 \\
&= 3(\dim H^0(E) - 3) - \dim H^0(\mathcal{E}nd(E)) + 1.
\end{aligned}$$

So the result follows by Proposition 1.4. □

2.2. Let $\mathcal{H}ilb^0$ be an irreducible component of $\mathcal{H}ilb$ containing $\overline{\varphi(U)}$ and $T_{X_s, \mathcal{H}ilb}$ the Zariski tangent space of $\mathcal{H}ilb$ at X_s. Then it is known that $T_{X_s, \mathcal{H}ilb} \simeq H^0(N_{X_s/\mathbf{P}^n})$. So we have the following proposition from Proposition 2.2.

Proposition 2.3. *Under the same assumptions in Proposition 1.4, if* $H^1(\mathcal{E}nd(E)) = 0$ *then*

1) $\mathcal{H}ilb^0$ *coincides with* $\overline{\varphi(U)}$.

2) $\mathcal{H}ilb^0$ *is smooth at the determinantal subvarieties associated to* E.

3. Proof of Theorem

Let $\mathrm{PGL}(n+1,k)$ be the automorphism group of $\mathbf{P}^n$ and let $T_\sigma : \mathbf{P}^n \ni x \mapsto \sigma x \in \mathbf{P}^n$ be the transformation of $\mathbf{P}^n$ defined by $\sigma \in \mathrm{PGL}(n+1,k)$.

Since it is well-known that E splits if and only if $\bar{E}$ splits, we may assume that E is a rank two vector bundle on $\mathbf{P}^n$ (n being either 4 or 5). In addition after multiplying E by a suitable line bundle, we may assume that E is a very ample vector bundle enjoying the assumption in Proposition 1.4.

Suppose that $H^1(\mathcal{E}nd(E)) = 0$. Hence it follows from Proposition 2.3 that $\overline{\sigma\varphi(U)} = \overline{\varphi(U)}$ for every element $\sigma \in \mathrm{PGL}(n+1,k)$. Since $\varphi(U)$ is a constructible set, there exist two element $s,\ t \in U$ satisfying $X_{\sigma^*(s)} = X_t$, where $X_{\sigma^*(s)}$ is the determinantal subvariey associated to $T_\sigma^*(E)$ defined by $\sigma^*(s) = \langle T_\sigma^*(s_1), T_\sigma^*(s_2), T_\sigma^*(s_3)\rangle$. Consider the resolutions of the defining ideal sheaves I_{X_t} of X_t and $I_{X_{\sigma^*(s)}}$ of $X_{\sigma^*(s)}$ respectively (cf. Lemma 1.2):

$$
(**)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & E^* & \longrightarrow & \bigoplus^3 \mathcal{O}_{\mathbf{P}^n} & \longrightarrow & I_{X_t}\otimes\mathcal{O}(c_1) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\simeq} & & \\
0 & \longrightarrow & T_\sigma^*(E^*) & \longrightarrow & \bigoplus^3 \mathcal{O}_{\mathbf{P}^n} & \longrightarrow & I_{X_{\sigma^*(s)}}\otimes\mathcal{O}(c_1) & \longrightarrow & 0.
\end{array}
$$

Then it is observed that there exists an isomorphism $\psi : \bigoplus^3 \mathcal{O}_{\mathbf{P}^n} \to \bigoplus^3 \mathcal{O}_{\mathbf{P}^n}$ such that ψ makes the diagram in $(**)$ commutative and so we see that $T_\sigma^*(E)$ is isomorphic to E, i.e., E is a homogeneous vector bundle. Since every homogeneous bundle on $\mathbf{P}^n$ of rank $r < n$ is a direct sum of line bundles even if char $k = p > 0$ (cf. [2]), we can complete the proof of Theorem 0.1.

References

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