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Determinantal varieties associated to rank two vector bundles on projective spaces and splitting theorems

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(joint work with Hideyasu Sumihiro)

0. Introduction

In this paper, we work over an algebraically closed field $k$. Let $E$ be a rank 2 vector bundle on $n$-dimensional projective spaces $\mathbb{P}_k^n$.

In [3], H. Sumihiro showed the following theorem in the case $\text{char } k = 0$.

**Theorem 0.1.** Let $P$ be a 4- or 5-dimensional projective linear subspace of $\mathbb{P}_k^n$ and $E = E|_P$ be the restriction of $E$ to $P$. Then $E$ splits into line bundles if and only if $H^1(P, \text{End}(E)) = 0$.

The aim of this article is to prove that this theorem holds affirmatively true in char $k = p > 0$. The proof is almost same with the one of char $k = 0$, namely, is obtained by studying some geometric structures of the Hilbert scheme of $\mathbb{P}_k^n$ at determinantal subvarieties. In char $k = p > 0$, however, since we can not use Kodaira vanishing theorem and Le-Potier vanishing theorem, we have to observe some vanishings of cohomologies appeared in [3] carefully.

1. Determinantal Varieties

We first recall the definition and some properties of determinantal varieties associated to 2-bundles (cf. [3]).

1.1. Definition of determinantal varieties. Let $E$ be a rank 2 vector bundle on an $\mathbb{P}_k^n$, $\pi : P(E) \to \mathbb{P}_k^n$ the projective bundle associated to $E$ over $\mathbb{P}_k^n$, $L_E$ the tautological line bundle on $P(E)$ and let $G = \text{Grass}(H^0(E), m + 1)$ be the Grassmann variety which parametrizes $(m+1)$-dimensional linear subspaces of $H^0(\mathbb{P}_k^n, E)$. We assume that $E$ is very ample, i.e., $L_E$ is a very ample line bundle. Then we can take $s = (s_1, s_2, \ldots, s_{m+1}) \in G$ ($(s_i \in H^0(\mathbb{P}_k^n, E))$) with $n = 2m$ (resp. $n = 2m + 1$) satisfying the following condition

1) $Y = Y_s = D_1 \cap D_2 \cap \cdots \cap D_{m+1}$ is a smooth closed subscheme of $P(E)$

(*)

2) $W(s_1) \cap W(s_2) \cap \cdots \cap W(s_{m+1}) = \emptyset$,

where $D_i$ is the tautological divisor on $P(E)$ defined by $s_i$ and $W(s_i)$ is the zero locus on $\mathbb{P}_k^n$ of $s_i (1 \leq i \leq m + 1)$.

Let $X_s = \pi(Y_s)$. Then we can show that $X_s$ is a closed subscheme of $\mathbb{P}_k^n$ which is isomorphic to $Y_s$ through $\pi$ with the following defining equations:

$s_i \wedge s_j = 0 \quad (1 \leq i \leq j \leq m+1)$.

**Definition 1.1.** We call the closed subscheme $X = X_s$ of $\mathbb{P}_k^n$ the (smooth) determinantal variety associated to $E$ defined by $s \in G$. 


Remark 1.1. $X$ depends on the choice of $s \in G$ subject to the condition $(\ast)$. 

As for determinantal varieties, we obtain the following.

Theorem 1.1. Let the notation be as above.

1) $U = \{s \in G|s \text{ satisfies the condition } (\ast)\}$ is a Zariski open subset of $G$.

2) There exists a closed subscheme $\Xi$ of $\mathbb{P}_k^n \times U$ such that the second projection $q : \Xi \subset \mathbb{P}_k^n \times U \to U$ is faithfully flat and $X_s = q^{-1}(s)$ for any $s \in U$. Thus smooth determinantal varieties associated to $E$ form a smooth family over an open subset of $G$ and hence they are diffeomorphic to each other.

When $n = 4$ or $5$, let $I_X$ be the defining ideal of a determinantal subvariety $X$ in $\mathbb{P}^n$. Then $I_X$ has the following resolution of vector bundles.

Lemma 1.2. In above notation, there exists an exact sequence

$$0 \to E^*(-c_1) \to \bigoplus \mathcal{O}_{\mathbb{P}^n}(-c_1) \to I_X \to 0,$$

where $c_1$ is the first Chern number of $E$.

1.2. Tangent bundle and normal bundle of determinantal varieties. In this subsection, we consider when $n = 4$ or $5$, i.e., $m = 2$.

1.2.1. Let $E$ be a very ample rank two bundle on $\mathbb{P}_k^n$ and $X$ a determinantal variety associated to $E$. Let $H$ be the restriction of a hyperplane of $\mathbb{P}^n$ to $X$ and $D$ the restriction of a tautological divisor of $P(E)$ to $X$ through the isomorphism $\pi$.

Then we can obtain the following diagram of exact sequences:

$$
\begin{array}{cccc}
0 & \to & T_{P(E)/\mathbb{P}^n}|Y & \to \mathcal{O}_X(2D - c_1H) \\
\downarrow & & \downarrow & \alpha \\
0 & \to & T_Y & \to T_{P(E)}|Y & \to N_{Y/P(E)} & \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \to & T_X & \to T_{\mathbb{P}^n}|X & \to N_{X/\mathbb{P}^n} & \to 0, \\
\downarrow & & \downarrow & & & \downarrow \\
0 & & 0 & & 0 & \\
\end{array}
$$

where $\alpha$ is an injection induced by the snake lemma. Since $N_{Y/P(E)} \simeq \bigoplus \mathcal{O}_X(D)$ by the condition of $Y$, we obtain the following.

Proposition 1.3. There exists an exact sequence

$$0 \to \mathcal{O}_X(2D - c_1H) \to \bigoplus \mathcal{O}_X(D) \to N_{X/\mathbb{P}^n} \to 0.$$
1.2.2. From the exact sequence of the above proposition, we have the following exact sequence

$$0 \to H^0(\mathcal{O}_X(2D - c_1H)) \to \bigoplus_3 H^0(\mathcal{O}_X(D)) \to H^0(N_{X/P^n})$$

$$\to H^1(\mathcal{O}_X(2D - c_1H)) \to \bigoplus_3 H^1(\mathcal{O}_X(D)).$$

Now we recall $Y = D_1 \cap D_2 \cap D_3$. Consider the canonical exact sequence

$$(*)_1 \quad 0 \to \mathcal{O}_{P(E)}(D - c_1H) \to \mathcal{O}_{P(E)}(2D - c_1H) \to \mathcal{O}_{D_1}(2D - c_1H) \to 0,$$

from which we obtain the following exact sequence:

$$0 \to H^0(\mathcal{O}_{P(E)}(D - c_1H)) \to H^0(\mathcal{O}_{P(E)}(2D - c_1H)) \to H^0(\mathcal{O}_{D_1}(2D - c_1H))$$

$$\to H^1(\mathcal{O}_{P(E)}(D - c_1H)) \to H^1(\mathcal{O}_{P(E)}(2D - c_1H)) \to H^1(\mathcal{O}_{D_1}(2D - c_1H))$$

$$\to H^2(\mathcal{O}_{P(E)}(D - c_1H)).$$

Since $H^i(D - c_1H) = H^i(E^*)$ $(0 \leq i \leq 4)$ and we can show that $H^0(E^*) = 0$ and $H^1(E^*) = H^{n-1}(E \otimes K_{P^n}) = 0$, where $K_{P^n}$ is the canonical divisor of $P^n$, it turns out that $H^i(\mathcal{O}_{P(E)}(2D - c_1H)) \simeq H^i(\mathcal{O}_{D_1}(2D - c_1H)) (i = 0, 1)$ if $H^2(E^*) = H^{n-2}(E \otimes K_{P^n}) = 0$.

Considering the exact sequences similarly

$$(*)_2 \quad 0 \to \mathcal{O}_{D_1}(D - c_1H) \to \mathcal{O}_{D_1}(2D - c_1H) \to \mathcal{O}_{D_1 \cap D_2}(2D - c_1H) \to 0,$$

$$0 \to \mathcal{O}_{D_1 \cap D_2}(D - c_1H) \to \mathcal{O}_{D_1 \cap D_2}(2D - c_1H) \to \mathcal{O}_Y(2D - c_1H) \to 0,$$

$$(*)_3 \quad 0 \to \mathcal{O}_{D_1}(-c_1H) \to \mathcal{O}_{D_1}(D - c_1H) \to \mathcal{O}_{D_1 \cap D_2}(D - c_1H) \to 0,$$

$$0 \to \mathcal{O}_{D_1 \cap D_2}(-D - c_1H) \to \mathcal{O}_{P(E)}(-c_1H) \to \mathcal{O}_{D_1}(-c_1H) \to 0,$$

we obtain isomorphisms $H^i(\mathcal{O}_{D_1}(2D - c_1H)) \simeq H^i(\mathcal{O}_{D_1 \cap D_2}(2D - c_1H))$ and $H^i(\mathcal{O}_{D_1 \cap D_2}(2D - c_1H)) \simeq H^i(\mathcal{O}_Y(2D - c_1H)) (i = 0, 1)$. Summing up the above, we conclude that $H^i(\mathcal{O}_X(2D - c_1H)) \simeq H^i(\mathcal{O}_{P^n}, S^2(E)(-c_1)).$

On the other hand, since there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_{P^n} \longrightarrow \mathcal{E}nd(E) \longrightarrow S^2(E)(-c_1) \longrightarrow 0,$$

we have a canonical isomorphism $H^1(S^2(E)(-c_1)) \simeq H^1(\mathcal{E}nd(E))$ and $\dim H^0(S^2(E)(-c_1)) = \dim H^0(\mathcal{E}nd(E)) - 1$.

In addition we easily see that $\dim H^0(\mathcal{O}_X(D)) = \dim H^0(E) - 3$.

Summarizing the above, we get the following proposition.

**Proposition 1.4.** Assume that $n = 4$ or $5$ and $H^{n-2}(E \otimes K_{P^n}) = 0$. If $H^1(\mathcal{E}nd(E)) = 0$, then

$$\dim H^0(N_{X/P^n}) = 3(\dim H^0(E) - 3) - \dim H^0(\mathcal{E}nd(E)) + 1$$

**Remark 1.2.** When $\text{char } k = 0$, we get $H^i(E^*) \simeq H^{n-i}(E \otimes K_{P^n}) = 0$ for $0 \leq i \leq n - 2$ by Le-Potier vanishing theorem. So we do not need the assumption $H^{n-2}(E \otimes K_{P^n}) = 0$ in the above proposition. Also the proof itself becomes slightly simpler because we can use the vanishing theorems.
2. Hilbert Schemes

In this section, we assume that \( n = 4 \) or \( 5 \). Let \( \mathcal{Hilb} \) be the Hilbert scheme of \( \mathbb{P}^n \).

2.1. Let \( \varphi : U \ni s \mapsto X_s \in \mathcal{Hilb} \) be the morphism induced by Theorem 1.1. Let \( \text{Aut}(E) \) be the automorphism group of \( E \). Then \( \text{Aut}(E) \) is a reduced connected linear algebraic group of dimension \( \dim H^0(\text{End}(E)) \).

For every element \( g \in \text{Aut}(E) \) and \( s = (s_1, s_2, s_3) \in G \), we define
\[
g \cdot s = (g(s_1), g(s_2), g(s_3)),
\]
where \( g(s_i) \) is the composite of \( s_i \) with \( g \). Then it defines an action of \( \text{Aut}(E) \) on \( G \) and we have
\[
g \cdot s_i \wedge g \cdot s_j = \det g \cdot s_i \wedge s_j (1 \leq i \leq j \leq 3),
\]
where \( \det : \text{Aut}(E) \ni g \mapsto \det(g) \in k^* = k \setminus \{0\} \) is the determinant character. Hence \( X_{g,s} = X_s \). Therefore \( \text{Aut}(E) \) acts on \( U \) and \( \varphi \) is an orbit morphism, i.e., \( \varphi \) is constant on any orbit \( O(s) = \{g \cdot s | g \in \text{Aut}(E)\} \).

Then we have the following.

Lemma 2.1. The stabilizer \( \text{Stab}(s) \) of \( s \in U \) is equal to the multiplicative group \( k^* \).

As a trivial corollary of the above lemma and Proposition 1.3, we observe the following.

a) Every orbit has the same dimension \( \dim \text{Aut}(E)/k^* \). Hence the action of \( \text{Aut}(E) \) on \( U \) is closed, i.e., every orbit is closed in \( U \).

b) \( \dim O(s) = \dim H^0(\text{End}(E)) - 1 \)

Proposition 2.2. Under the same assumptions in Proposition 1.4, if \( H^1(\text{End}(E)) = 0 \) then
\[
\dim \overline{\varphi(U)} = \dim H^0(N_{X_s/\mathbb{P}^n}).
\]

Proof. Using the exact sequence in Proposition 1.3, we see that \( \varphi^{-1}(\varphi(s)) (s \in U) \) consists of finitely many orbits. Hence
\[
\dim \overline{\varphi(U)} = \dim U - \dim O(s) = \dim \text{Grass}(H^0(E), 3) - \dim H^0(\text{End}(E)) + 1
\]
\[
= 3(\dim H^0(E) - 3) - \dim H^0(\text{End}(E)) + 1.
\]

So the result follows by Proposition 1.4.

2.2. Let \( \mathcal{Hilb}^0 \) be an irreducible component of \( \mathcal{Hilb} \) containing \( \overline{\varphi(U)} \) and \( T_{X_s,\mathcal{Hilb}} \) the Zariski tangent space of \( \mathcal{Hilb} \) at \( X_s \). Then it is known that \( T_{X_s,\mathcal{Hilb}} \simeq H^0(N_{X_s/\mathbb{P}^n}) \). So we have the following proposition from Proposition 2.2.

Proposition 2.3. Under the same assumptions in Proposition 1.4, if \( H^1(\text{End}(E)) = 0 \) then

1) \( \mathcal{Hilb}^0 \) coincides with \( \overline{\varphi(U)} \).

2) \( \mathcal{Hilb}^0 \) is smooth at the determinantal subvarieties associated to \( E \).
3. Proof of Theorem

Let $\text{PGL}(n + 1, k)$ be the automorphism group of $\text{P}^n$ and let $T_\sigma : \text{P}^n \ni x \mapsto \sigma x \in \text{P}^n$ be the transformation of $\text{P}^n$ defined by $\sigma \in \text{PGL}(n + 1, k)$.

Since it is well-known that $E$ splits if and only if $\overline{E}$ splits, we may assume that $E$ is a rank two vector bundle on $\text{P}^n$ ($n$ being either 4 or 5). In addition after multiplying $E$ by a suitable line bundle, we may assume that $E$ is a very ample vector bundle enjoying the assumption in Proposition 1.4.

Suppose that $H^1(\text{End}(E)) = 0$. Hence it follows from Proposition 2.3 that $\sigma \varphi(U) = \varphi(U)$ for every element $\sigma \in \text{PGL}(n + 1, k)$. Since $\varphi(U)$ is a constructible set, there exist two elements $s, t \in U$ satisfying $X_{\sigma^*(s)} = X_t$, where $X_{\sigma^*(s)}$ is the determinantal subvariety associated to $T_{\sigma}^*(E)$ defined by $\sigma^*(s) = (T_{\sigma}^*(s_1), T_{\sigma}^*(s_2), T_{\sigma}^*(s_3))$. Consider the resolutions of the defining ideal sheaves $I_X$ of $X_t$ and $I_{X_{\sigma^*(s)}}$ of $X_{\sigma^*(s)}$ respectively (cf. Lemma 1.2):

$$
\begin{array}{cccccc}
0 & \longrightarrow & E^* & \longrightarrow & \bigoplus_3 \mathcal{O}_{\text{P}^n} & \longrightarrow & I_{X_t} \otimes \mathcal{O}(c_1) & \longrightarrow & 0 \\
\psi & & \downarrow & & & \cong & & \downarrow & \\
0 & \longrightarrow & T_{\sigma}^*(E^*) & \longrightarrow & \bigoplus_3 \mathcal{O}_{\text{P}^n} & \longrightarrow & I_{X_{\sigma^*(s)}} \otimes \mathcal{O}(c_1) & \longrightarrow & 0.
\end{array}
$$

Then it is observed that there exists an isomorphism $\psi : \bigoplus_3 \mathcal{O}_{\text{P}^n} \to \bigoplus_3 \mathcal{O}_{\text{P}^n}$ such that $\psi$ makes the diagram in (**) commutative and so we see that $T_{\sigma}^*(E)$ is isomorphic to $E$, i.e., $E$ is a homogeneous vector bundle. Since every homogeneous bundle on $\text{P}^n$ of rank $r < n$ is a direct sum of line bundles even if $\text{char } k = p > 0$ (cf. [2]), we can complete the proof of Theorem 0.1.

References

