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ON CLASSIFICATION OF $\mathbb{Q}$-FANO 3-FOLDS OF GORENSTEIN INDEX 2 AND FANO INDEX $\frac{1}{2}$

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Notation and Conventions.

$\sim$ linear equivalence

$\equiv$ numerical equivalence

ODP ordinary double point, i.e., singularity analytically isomorphic to $\{xy+z^2+u^2=0\subset\mathbb{C}^4\}$

QODP singularity analytically isomorphic to $\{xy+z^2+u^2=0\subset\mathbb{C}^4/\mathbb{Z}_2(1,1,1,0)\}$

$\mathbb{F}_n$ Hirzebruch surface of degree n

$\mathbb{F}_{n,0}$ surface which is obtained by the contraction of the negative section of $\mathbb{F}_n$

Q_3 smooth 3-dimensional quadric.

B_i $(1\le i\le 5)$ $\mathbb{Q}$-factorial Gorenstein terminal Fano 3-fold of Fano index 2, and with Picard number 1 and $(-K)^3=8i$, where K is the canonical divisor

A_{2i} $(1\le i\le 11$ and $i\ne 10)$ $\mathbb{Q}$-factorial Gorenstein terminal Fano 3-fold of Fano index 1, and with Picard number 1 and $(-K)^3=2i$

contraction of (m,n)-type extremal contraction whose exceptional locus has dimension m and the image of the exceptional locus has domension n

0. Introduction

In this article, we will work over $\mathbb{C}$, the complex number field.

Definition 0.0 ($\mathbb{Q}$-Fano variety). Let X be a normal projective variety. We say that X is a $\mathbb{Q}$-Fano variety (resp. weak $\mathbb{Q}$-Fano variety) if X has only terminal singularities and $-K_X$ is ample (resp. nef and big).

Let $I(X):=\min\{I|IK_X$ is a Cartier divisor$\}$ and we call $I(X)$ the Gorenstein index of X.

Write $I(X)(-K_X)\equiv r(X)H(X)$, where $H(X)$ is a primitive Cartier divisor and $r(X)\in\mathbb{N}$. (Note that $H(X)$ is unique since Pic$X$ is torsion free.) Then we call $\frac{r(X)}{I(X)}$ the Fano index of X and denote it by $F(X)$.

Remark 0.1.

(1) We can allow that a $\mathbb{Q}$-Fano variety or a weak $\mathbb{Q}$-Fano variety has worse singularities than terminal. When we have to treat such a variety in this paper, we indicate singularities which we allow, e.g., 'a $\mathbb{Q}$-Fano 3-fold with only canonical singularities';

(2) if X is Gorenstein in Definition 0.0, we say that X is a Fano variety (resp. a weak Fano variety).

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For the classification theory of varieties, a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano variety with Picard number 1 is important because it is an output of the minimal model program. Here we mention the known result about the classification of $\mathbb{Q}$-Fano 3-folds:

(1) G. Fano started the classification of smooth Fano 3-folds and it was completed by V. A. Iskovskih [I1] $\sim$ [I4], V. V. Shokurov [Sh1], [Sh2], T. Fujita [Fu1] $\sim$ [Fu3], S. Mori and S. Mukai [MM1] $\sim$ [MM3];
(2) S. Mukai [Mu] classified indecomposable Gorenstein Fano 3-folds with canonical singularities by using vector bundles;
(3) T. Sano [San1] and independently F. Campana and H. Flenner [CF] classified non Gorenstein Fano 3-folds of Fano indices > 1;
(4) T. Sano [San2] classified non Gorenstein Fano 3-folds of Fano indices 1 and with only cyclic quotient terminal singularities. Recently T. Minagawa [Mi1] proved that non Gorenstein $\mathbb{Q}$-Fano 3-folds with Fano indices 1 can be deformed to one with only cyclic quotient terminal singularities;
(5) A. R. Fletcher [Fl] gave the classification of $\mathbb{Q}$-Fano 3-folds which are weighted complete intersections of codimension 1 or 2. Recently S. Altinok [Al] (see also [RM2]) obtained a list of $\mathbb{Q}$-Fano 3-folds which are subvarieties in a weighted projective space of codimension 3 or 4.

On the other hand K. Takeuchi [T1] simplified and amplified V. A. Iskovskih 's method of classification by using the theory of the extremal ray. In particular he reproved the Shokurov's theorem [Sh2], the existence of lines on a smooth Fano 3-fold of Fano index 1 and with Picard number 1 by simple numerical calculations.

We formulate a slight generalization of Takeuchi's construction for a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold X with $\rho(X) = 1$ and give a classification of a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold with the following properties:

Main Assumption 0.2.

(1) $\rho(X) = 1$;
(2) $I(X) = 2$;
(3) $F(X) = \frac{1}{2}$;
(4) $h^0(-K_X) \geq 4$;
(5) there exists an index 2 point P such that

$$(X, P) \simeq (\{xy + z^2 + u^a = 0\}/\mathbb{Z}_2(1,1,1,0), o)$$

for some $a \in \mathbb{N}$.

Takeuchi's construction 0.3. Here we explain a slight generalization of Takeuchi's construction. Let X be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold with $\rho(X) = 1$. Suppose that we are given a birational morphism $f : Y \to X$ with the following properties:

(1) Y is a weak $\mathbb{Q}$-Fano 3-fold;
(2) f is an extremal divisorial contraction such that f-exceptional locus E is a prime $\mathbb{Q}$-Cartier divisor.

Then we obtain the following diagram:

$$\begin{array}{ccccccc} & & Y_0 := Y & \overset{g_0}{\dashrightarrow} & Y_1 \ldots & \overset{g_{k-1}}{\dashrightarrow} & Y_k & & \\ & & f \swarrow & & & & \searrow f' & & \\ X & & & & & & & & X' \end{array} ,$$

where

(1) $Y_0 \dashrightarrow Y_1$ is a flop or a flip and $Y_i \dashrightarrow Y_{i+1}$ is a flip for $i \geq 1$;
(2) f' is a crepant divisorial contraction (in this case, $i = 0$) or an extremal contraction which is not isomorphic in codimension 1.

We use the following notation:

$Y' := Y_k$;

$E_i :=$ the strict transform of E on Y_i;

$\tilde{E} :=$ the strict transform of E on Y';

$e := E^3 - {E_1}^3$ if $Y_0 \dashrightarrow Y_1$ is a flop or $:= 0$ otherwise;

$d_i := (-K_{Y_i})^3 - (-K_{Y_{i+1}})^3$ (resp. $a_i := \frac{E_i.l_i}{(-K_{Y_i}).l_i}$) if $Y_i \dashrightarrow Y_{i+1}$ is a flip, where l_i is a flipping curve, or $:= 0$ (resp. $:= 0$) otherwise;

z and u is defined as follows:

If f' is birational, then let E' be the exceptional divisor of f' and set $E' \equiv z(-K_{Y'}) - u\tilde{E}$ or if f' is not birational, then let L be the pull back of an ample generator of PicX' and set $L \equiv z(-K_{Y'}) - u\tilde{E}$.

We note the following:

(1)

$$(-K_{Y'})^2\tilde{E} = (-K_Y)^2 E - \sum a_i d_i;$$

$$(-K_{Y'})\tilde{E}^2 = (-K_Y)E^2 - \sum {a_i}^2 d_i;$$

$$\tilde{E}^3 = E^3 - e - \sum {a_i}^3 d_i;$$

(2) On the other hand the value or the relation of the value (expressed with z and u) of $(-K_{Y'})^3$, $(-K_{Y'})^2\tilde{E}$, $(-K_{Y'})\tilde{E}^2$ and $\tilde{E}^3$ are restricted by the properties of f'.

By these (1) and (2), we obtain equations of Diophantine type.

Under Main Assumption 0.2, Construction 0.3 works for a suitable choice of f and we can solve the equations as noted above.

Main Theorem. *Let X be as in Main Assumption 0.2. Let $f : Y \to X$ be the weighted blow up at P with weight $\frac{1}{2}(1,1,1,2)$. Then Y is a weak $\mathbb{Q}$-Fano 3-fold.*

Consider the diagram as in 0.3. Let $h := h^0(-K_X)$, $N := aw(X)$ and $n := \sum aw(Y_i, P_{ij})$ (the summation is taken over the index 2 points on flipping curves), where $aw(X)$ is the number of $\frac{1}{2}(1,1,1)$-singularities which we obtain by deforming non Gorenstein points of X locally and $aw(Y_i, P_{ij})$ is defined similarly. Then we can solve the equations above and obtain a geographic classification of X as below (? in the table means that we don't know the existence of an example) :

$h=4$

$(-K_X)^3$	N	e	n	z	$(-K_{Y'}.C)$	f', X'
$\frac{5}{2}$	1	15	0	1	/	$(2,0)_4, (-K_{X'})^3 = \frac{5}{2}, I(X') = 2$
$\frac{5}{2}$	1	/	/	1	/	*crep. div.*, $(-K_{X'})^3 = 2, I(X') = 1$
3	2	12	0	1	/	$(2,0)_8,\ A_4$
$\frac{7}{2}$	3	10	0	1	1	$(2,1),\ A_6$
4	4	8	0	1	2	$(2,1),\ A_8$
4	4	9	3	1	/	$(2,0)_1,\ A_{10}$
$?\frac{9}{2}$	5	6	0	1	3	$(2,1),\ A_{10}$
$?\frac{9}{2}$	5	8	3	1	/	$(2,0)_5,\ A_{16}$
$?\frac{9}{2}$	5	9	0	2	/	$(3,1), \deg F = 6$
?5	6	4	0	1	4	$(2,1),\ A_{12}$

$z = u$ if f' is not a crepant divisorial contraction.

$u = 2$ if f' is a crepant divisorial contraction.

$F :=$ a general fiber of f' if f' is $(3,1)$-type.

See Appendix for $(2,0)_i$.

$g(C) = 0$ in case f' is of type E_1 and every singularity of Y is a $\frac{1}{2}(1,1,1)$−singularity.

$h=5$

$(-K_X)^3$	N	e	n	z	$\deg\Delta$	$\deg F$	f', X'
$\frac{9}{2}$	1	9	0	1	/	3	$(3,1)$
5	2	8	1	1	/	4	$(3,1)$
$?\frac{11}{2}$	3	7	2	1	/	5	$(3,1)$
$?\frac{11}{2}$	3	8	0	2	8	/	$(3,2), \mathbb{F}_{2,0}$
?6	4	7	1	2	6	/	$(3,2), \mathbb{F}_{2,0}$
?6	4	6	3	1	/	6	$(3,1)$
$?\frac{13}{2}$	5	6	2	2	4	/	$(3,2), \mathbb{F}_{2,0}$

$z = u$.

$\Delta :=$ the discriminant divisor of f' if f' is $(3,2)$-type.

$F :=$ a general fiber of f' if f' is $(3,1)$-type.

$h=6$

$(-K_X)^3$	N	e	n	z	$\deg\Delta$	$(-K_{Y'}.C)$	f', X'
$\frac{13}{2}$	1	7	0	1	7	/	$(3,2), \mathbb{P}^2$
7	2	7	0	4	/	35	$(2,1)$,[5]
?7	2	6	1	1	6	/	$(3,2), \mathbb{P}^2$
$\frac{15}{2}$	3	7	0	2	/	9	$(2,1), [2], I(X') = 2$
$\frac{15}{2}$	3	6	1	4	/	30	$(2,1)$,[5]
$?\frac{15}{2}$	3	5	2	1	5	/	$(3,2), \mathbb{P}^2$
?8	4	4	3	1	4	/	$(3,2), \mathbb{P}^2$
$?\frac{17}{2}$	5	3	4	1	3	/	$(3,2), \mathbb{P}^2$

Type [i] means the $\mathbb{Q}$-Fano 3-fold of type [i] which was classified by T.Sano in [San2].

$h = 7$

$(-K_X)^3$	N	e	n	z	$(-K_{Y'}.C)$	f', X'
$\frac{17}{2}$	1	6	0	3	36	$(2,1), \mathbb{P}^3$
9	2	6	0	2	18	$(2,1)$,[3]
9	2	5	1	3	32	$(2,1), \mathbb{P}^3$
$\frac{19}{2}$	3	5	1	2	15	$(2,1)$,[3]
$\frac{19}{2}$	3	4	2	3	28	$(2,1), \mathbb{P}^3$

Type [i] means the $\mathbb{Q}$-Fano 3-fold of type [i] which was classified by T.Sano in [San2].

$$u = z + 1.$$

$h = 8$

$(-K_X)^3$	N	e	n	z	$(-K_{Y'}.C)$	f, X'
$\frac{21}{2}$	1	6	0	1	6	$(2,1), B_3$
$\frac{21}{2}$	1	5	0	2	27	$(2,1), Q_3$
11	2	4	1	2	24	$(2,1), Q_3$

$$u = z + 1.$$

$h = 9$

$(-K_X)^3$	N	e	n	z	u	$(-K_{Y'}.C)$	f', X'
$\frac{25}{2}$	1	5	0	1	2	10	$(2,1), B_4$

$h = 10$

$(-K_X)^3$	N	e	n	$\deg\Delta$	$(-K_{Y'}.C)$	f', X'
$\frac{29}{2}$	1	4	0	/	14	$(2,1), B_5$
$\frac{29}{2}$	1	6	0	0	/	$(3,2), \mathbb{P}^2$
15	2	3	1	/	12	$(2,1), B_5$

$$z = 1 \textit{ and } u = 2.$$

In particular we have $(-K_X)^3 \leq 15$ and $h^0(-K_X) \leq 10$.

Based on this result, we can derive the following properties for X as in the main theorem:

Theorem A. *if any index 2 point satisfies the assumption (5) of 0.2, then $|-K_X|$ has a member with only canonical singularities.*

So the general elephant conjecture by M. Reid is affirmative for such an X.

Theorem B. *Let X be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold with (1)~(4) of 0.2. Let $N := aw(X)$. Then if $N > 1$ (resp. $N = 1$), X can be transformed to a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold $\tilde{Z}'$ with (1)~(4) of 0.2 and with only QODP's or $\frac{1}{2}(1,1,1)$-singularities as its singularities and $h^0(-K_{\tilde{Z}'}) = h$ and $aw(\tilde{Z}') = N - 1$ (resp. a smooth Fano 3-fold $\tilde{Z}'$ with $\rho(\tilde{Z}') = 1$, $F(\tilde{Z}') = 1$ and $h^0(-K_{\tilde{Z}'}) = h$) as follows:*

$$\begin{array}{ccccccc} & & & \tilde{Y} & & & \\ & & \tilde{f} \swarrow & & \searrow \tilde{g} & & \\ X & \overset{\text{def}}{\longrightarrow} & \tilde{X} & & \tilde{Z} & \overset{\text{def}}{\longrightarrow} & \tilde{Z}', \end{array}$$

where $* \overset{\text{def}}{\longrightarrow} **$ means that $**$ is a small deformation of $*$;

$\tilde{X}$ is a $\mathbb{Q}$-Fano 3-fold as in 0.2 and with only ODP's, QODP's or $\frac{1}{2}(1,1,1)$-singularities as its singularities;

$\tilde{f} : \tilde{Y} \to \tilde{X}$ is chosen as f in the main theorem;

$\tilde{g} : \tilde{Y} \to \tilde{Z}$ be the anti-canonical model.

This is an analogue to the Reid's fantasy about Calabi-Yau 3-folds [RM1].

Theorem C. *If any index 2 point is a $\frac{1}{2}(1,1,1)$-singularity, X can be embedded into a weighted projective space $\mathbb{P}(1^h, 2^N)$, where $h := h^0(-K_X)$ and N is the number of $\frac{1}{2}(1,1,1)$-singularities on X.*

We hope that this fact can be used for the classification of Mukai's type (see [Mu]).

1. Examples

We consider the case that $h^0(-K_X) = 4$ and $N = 4$. By the table of the main theorem, there are two possibilities of X in this case. We assume that every singularity of Y is a $\frac{1}{2}(1,1,1)$-singularity. Then one of the following holds:

[1]. f' is an extremal divisorial contraction which contracts a divisor E' to a curve C and $|-K_{Y'} - E'| \neq \phi$. X' is a $(2,2,2)$-complete intersection in $\mathbb{P}^6$ and satisfies the following properties:

(1) X' is factorial;
(2) C is a smooth conic;
(3) X' has 3 singularities $P_0 \sim P_2$ on C and P_i is an ODP or the singularity analytically isomorphic to the origin of $\{xy + z^2 + w^3 = 0\} \subset \mathbb{C}^4$. Outside P_i's, X' is smooth.

[2]. f' is blowing up at a smooth point $Q := f'(E')$ and $|-K_{Y'} - E'| \neq \phi$. X' is smooth, isomorphic to A_{10} and there exist exactly three lines through the point Q.

We will construct examples for these cases by the following three steps:

Step 1. We construct X' satisfying the properties as stated as in [1] or [2];

Step 2. We construct f' satisfying the properties as stated as in [1] or [2];

Step 3. We construct $f : Y \to X$ as in the main theorem from Y'.

[1].

Step 1 for [1]. We construct X' with only ODP's.

Claim 1. *Let V (resp. X') be a $(2,2)$-complete intersection in $\mathbb{P}^6$ (resp. a quadric section of V) with the following properties:*

(1) *V (resp. X') contains a smooth conic C;*
(2) *V (resp. X') has three ODP's $P_0 \sim P_2$ on C and outside P_i's, V (resp. X') is smooth.*

Then X' is factorial.

Proof. We claim that V contains the plane P spanned by C. Let σ be the pencil which consists of quadrics in $\mathbb{P}^6$ containing V. Since P_i is an ODP on V, there is a quadric in σ which is singular at P_i. If there is a quadric in σ which is singular at all P_i's, then it is singular on P and hence V is singular along C, a contradiction. So σ is generated by two quadrics which are singular at some P_i. But such quadrics contains P and hence V contains P.

Let $\nu : \tilde{V} \to V$ be the composition of the blowing ups at $P_0 \sim P_2$ and F_i the exceptional divisor over P_i. Let $\tilde{X}'$ be the strict transform of X' on $\tilde{V}$ and H the total transform of a hyperplane section of V. Then $\tilde{X}' \sim 2H - F_0 - F_1 - F_2$. Note that $|H - F_i - F_j|$ is free outside the strict transform l_{ij} of the line through P_i and P_j and $|H - F_k|$ is free (note that l_{ij} is contained in V since $l_{ij} \subset P$). By this, we can easily see that $|\tilde{X}'|$ is free and $\tilde{X}'$ is numerically trivial only for l_{ij}'s ($(i,j) = (0,1), (1,2), (2,0)$).

Let ϕ be the morphism defined by $|\tilde{X}'|$. Then ϕ-exceptional curves are l_{ij}'s. We will prove that $\mathrm{Leff}(\tilde{V}, \tilde{X}')$ holds and $\tilde{X}'$ meets every effective divisor on $\tilde{V}$. By [H, p.165, Proposition 1.1] and the argument of [H, p.172, the proof of Theorem 1.5], it suffices to prove that $\mathrm{cd}(\tilde{V} - \tilde{X}') < 3$, i.e., for any coherent sheaf F on $\tilde{V} - \tilde{X}'$, $H^i(\tilde{V} - \tilde{X}', F) = 0$ for all $i \geq 3$. Let $\overline{V} := \phi(\tilde{V})$ and $\overline{X'} := \phi(\tilde{X}')$. Consider the Leray spectral sequence

$$E_2^{pq} = H^p(\overline{V} - \overline{X'}, R^q\phi'_*F) \Rightarrow E^{p+q} = H^{p+q}(\tilde{V} - \tilde{X}', F),$$

where $\phi' := \phi|_{\tilde{V} - \tilde{X}'}$. Since $\overline{V} - \overline{X'}$ is affine and the dimension of every fiber of ϕ ≤ 1, we have $E_2^{pq} = 0$ for $p \geq 1$ or $q \geq 2$ whence $E^{p+q} = 0$ for $p + q \geq 2$. So the assertion follows.

Furthermore since $\tilde{X}'$ is nef and big, $H^i(\tilde{V}, \mathcal{O}(-n\tilde{X}')) = 0$ for $n \geq 1$ and $i = 1, 2$ by KKV vanishing theorem. Hence by the Grothandieck-Lefschetz theorem [G, p.135, 3.18] (or [H, p.178, Theorem 3.1]), we have $\mathrm{Pic}\tilde{X}' \simeq \mathrm{Pic}\tilde{V} \simeq \mathbb{Z}^4$. So $\rho(\tilde{X}'/X') = 3$ which imply that X' is factorial. $\square$

We will give a pair (V, X') satisfying the condition of Claim 1. Let C be a smooth conic in $\mathbb{P}^6$ and $P_0 \sim P_2$ three points on C. We can choose a coordinate of $\mathbb{P}^6$ such that $C = \{x_0x_1 + x_1x_2 + x_2x_0 = x_3 = x_4 = x_5 = x_6 = 0\}$ and $P_i = \{x_j = 0 \text{ for } j \neq i\}$.

Claim 2. *Let X' be a $(2,2,2)$-complete intersection in $\mathbb{P}^6$ satisfying the following conditions:*

(1) *X' is factorial;*
(2) *X' contains a smooth conic C;*
(3) *X' has three ODP's $P_0 \sim P_2$ on C and outside P_i's, X' is smooth.*

Then X' is the intersection of three quadrics $Q_1 \sim Q_3$ of the following forms by permuting P_i's if necessary:

$$Q_1 := \{m_0x_0 + m_1x_1 + q_1 = 0\};$$

$$Q_2 := \{pm_1x_1 + m_2x_2 + q_2 = 0\};$$

$$Q_3 := \{x_0x_1 + x_1x_2 + x_2x_0 + \sum_{i=3}^{6} l_ix_i = 0\},$$

where $p \in \mathbb{C}$, m_i (resp. q_i) is a linear form (resp. a quadratic form) of $x_3 \sim x_6$ and l_i is a linear form of $x_0 \sim x_6$.

Conversely if $X' = Q_1 \cap Q_2 \cap Q_3$, where Q_i is of the form as above and m_i, q_i and l_i are suitably general, then X' satisfies (1) $\sim$ (3).

Proof. Let γ be the net which consists of quadrics containing X'. γ contains a member Q_1 which is singular at P_2. Then Q_1 is of the form as above. If $m_1 = m_2 = 0$, then Q_1 is singular on the plane P spanned by C and hence X' is singular along C, a contradiction. Hence $m_1 \neq 0$ or $m_2 \neq 0$. By permuting P_1 and P_2 if necessary, we may assume that $m_1 \neq 0$. γ contains a member Q_2 which is singular at P_0. Q_2 is of the form as

$$\{m_1'x_1 + m_2x_2 + q_2 = 0\},$$

where m_1' and m_2 (resp. q_2) are linear forms (resp. is a quadratic form) of $x_3 \sim x_6$. γ also contains a member Q' which is singular at P_1. If Q_1, Q_2 and Q' generate γ, then X' contains the plane P, a contradiction to the factoriality and $F(X') = 1$. Hence Q' is contained in the pencil generated by Q_1 and Q_2. So $m_1' = pm_1$ for some $p \in \mathbb{C}$ and

$$Q = \{-pm_0x_0 + m_2x_2 + (q_2 - pq_1) = 0\}.$$

Since X' does not contain P as noted above, γ contains a member Q_3 of the form as in the statement. Q_3 is not contained in the pencil generated by Q_1 and Q_2 and hence Q_i's generate γ.

Conversely let $X' := Q_1 \cap Q_2 \cap Q_3$, where Q_i is of the form as above and m_i, q_i and l_i are suitably general. We can easily check that X' satisfies (2) and (3). Set $V := Q_1 \cap Q_2$. We may assume that V satisfies the condition of Claim 1. Hence by Claim 1, X' is factorial. □

Step 2 for [1]. Let $\nu' : \tilde{X}' \to X'$ be the composition of the blowing ups at $P_0 \sim P_{N-2}$ and F_i' the exceptional divisor over P_i. Let $\mu' : \hat{X}' \to \tilde{X}'$ be the blowing up along the strict transform $\tilde{C}$ of C and F' the μ'-exceptional divisor. We will denote the strict transforms of the two fibers of $F_i \simeq \mathbb{P}^1 \times \mathbb{P}^1$ through $F_i \cap \tilde{C}$ by l_{ij} $(j = 1, 2)$. Note that $-K_{\hat{X}'}.l_{ij} = 0$. We can easily see that $|-K_{\hat{X}'}|$ is free by $P \cap X' = C$, where P is the plane spanned by C and $-K_{\hat{X}'}$ is big. Hence l_{ij}'s are flopping curves on $\hat{X}'$ and we can see that the classes of l_{i1} and l_{i2} belong to the same ray. Let $\hat{X}' \dashrightarrow \hat{X}'^+$ be the flop. Then the strict transforms of F_i''s on $\hat{X}'^+$ are $\mathbb{P}^2$'s and we can contract them to $\frac{1}{2}(1,1,1)$-singularities. Let $g' : \hat{X}'^+ \to Y'$ be the contraction morphism, $f' : Y' \to X'$ the natural morphism and E' the strict transform of F'.

We will see that $|-K_{Y'} - E'| \neq \phi$. Let F'^+ be the strict transform of F' on $\hat{X}'^+$. Then $-K_{\hat{X}'^+} - F'^+ = g'^*(-K_{Y'} - E')$. Furthermore $h^0(-K_{\hat{X}'^+} - F'^+) =$

$h^0(-K_{\hat{X}'}-F')$. Hence it suffices to prove that $h^0(-K_{\hat{X}'}|_{F'}) \leq 3$ since $h^0(-K_{\hat{X}'}) = 4$. Since there is a smooth member of $|-K_{\hat{X}'}|$, we have $\mathcal{N}_{\tilde{C}/\hat{X}'} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-2)$. Hence $F' \simeq \mathbb{F}_1$ and $-K_{\hat{X}'}|_{F'} \sim C_0 + l$, where C_0 is the minimal section of F' and l is a fiber of F'. So we are done.

Step 3 for [1]. Since Y' has only $\frac{1}{2}(1,1,1)$-singularities and $-K_{Y'}$ is nef and big, we can construct a similar diagram $Y_0' := Y' \dashrightarrow Y_1' \dots Y_i' \dashrightarrow Y_{i+1}' \dots Y := Y_l' \xrightarrow{f} X$ to 0.3 by considering extremal rays, where $Y_i' \dashrightarrow Y_{i+1}'$ is a flop or a flip for $i = 0$ and a flip for $i \geq 1$. Let $\tilde{E}_i$ (resp. E) be the strict transform of $\tilde{E}$ on Y_i' (resp. Y). Let R_i be the extremal ray which is other than the ray associated to f' for $i = 0$ or the K_{Y_i}-negative extremal ray for $i \geq 1$. By similar calculations to 0.3, we have

$$(1) \qquad (-K_Y)^2 E = 1 + \sum a_i' d_i';$$

$$(2) \qquad (-K_Y)E^2 = -2 - \sum {a_i'}^2 d_i';$$

$$(3) \qquad E^3 = -6 + \sum {a_i'}^3 d_i' + e',$$

where e', a_i' and d_i' are similarly defined to 0.3 with respect to $-K_{Y_i'}$ and $\tilde{E}_i$ and furthermore we can see that a_i' is a non negative integer.

Claim 3. $\tilde{E}_i.R_i < 0$.

Proof. We can prove the assertion by induction. For $i = 0$, $\tilde{E}_0.R_0 < 0$ can be directly checked. Assume that the assertion holds for the numbers less than i. So the other extremal ray than R_i is positive for $\tilde{E}_i$. Since $-K_{Y_i'}$ is free outside a finite number of curves, $-K_{Y_i'}|_{\tilde{E}_i}$ is numerically equivalent to an effective 1-cycle. Hence by $-K_{Y_i'}\tilde{E}_i^2 \leq -K_{Y'}\tilde{E}^2 = -2$, we have $\tilde{E}_i.R_i < 0$. □

By this claim, we know that f is an divisorial contraction whose exceptional divisor is E. If f is a crepant divisorial contraction, then $l = 0$. But $(-K_{Y'})^2\tilde{E} = 1$, a contradiction. Hence f is a K_Y-negative contraction. Assume that f is $(2,1)$-type which contracts E to a curve C'. Then $(-K_X.C') = (-K_Y + E)(-K_Y)E = -1 - \sum d_i' a_i'(a_i' - 1) < 0$, a contradiction since X is a $\mathbb{Q}$-Fano 3-fold.

By the classification of a $(2,0)$-type contraction from a 3-fold with only index 2 terminal singularities (see Appendix), if f is such an contraction, then we have $-K_Y E^2 \geq -2$. On the other hand $-K_Y E^2 \leq -K_{Y'}\tilde{E}^2 = -2$. Hence there is no flip. So $(-K_Y)^2 E = (-K_{Y'})^2\tilde{E} = 1$ and hence again by the classification of a contraction as above, f is the blow up at a $\frac{1}{2}(1,1,1)$-singularity or the weighted blow up at a QODP with weight $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$ (we use the coordinate as stated in the definition of QODP) . In any case X is a $\mathbb{Q}$-Fano 3-fold with $I(X) = 2$. We can easily check that $(-K_X)^3 = 4$ and $\mathrm{aw}(X) = 4$. Furthermore by this, $F(X)$ must be $\frac{1}{2}$. So X is what we want.

[2].

Step 1 for [2]. The Grassmannian $G(2,5)$ (parameterizing 2-dimensional subspaces of 5-dimensional vector space) can be embedded into $\mathbb{P}^9$ by the Plücker embedding. Its defining equations are $x_{ij}x_{kl} - x_{ik}x_{jl} + x_{jk}x_{il} = 0$ for all $1 \leq i < j < k < l \leq 5$, where x_{pq} $(1 \leq p < q \leq 5)$ is a Plücker coordinate. Let Q be the point defined by $x_{pq} = 0$ for any $(p,q) \neq (1,2)$. Let l_1 (resp. l_2) be the line $\subset G(2,5)$ defined by $x_{pq} = 0$ for any $(p,q) \neq (1,2),(1,3)$ (resp. $(p,q) \neq (1,2),(2,4)$). Let l_3 be the line $\subset G(2,5)$ defined by the equations $x_{pq} = r_{pq}x_{12}$ for $(p,q) \neq (1,2)$ such that $r_{34} = r_{35} = r_{45} = 0$, $r_{13}r_{24} - r_{23}r_{14} = 0$, $r_{13}r_{25} - r_{23}r_{15} = 0$, $r_{14}r_{25} - r_{24}r_{15} = 0$ and $r_{15}r_{25} \neq 0$. Let H be the 3-plane spanned by l_1, l_2 and l_3. Then $G(2,5) \cap H = l_1 \cup l_2 \cup l_3$. Hence by [MM3, Proposition 6.8], there are two hyperplane H_1, H_2 and a quadric Q such that $X' := G(2,5) \cap H_1 \cap H_2 \cap Q$ is smooth and X' contains l_1, l_2 and l_3. Since the tangent space of X' at Q also contains all the lines on X' through Q, it is equal to H. Hence there are only three lines on X' through Q.

Step 2 for [2]. Let $f' : Y' \to X'$ be the blow up at Q and E' the exceptional divisor. Let ${l_1}'$, ${l_2}'$ and ${l_3}'$ be the transforms of l_1, l_2 and l_3 on Y'. Since $\mathrm{Bs}|-K_{Y'}| = {l_1}' \cup {l_2}' \cup {l_3}'$, the rank of the natural map $H^0(-K_{Y'}) \to H^0(\mathcal{O}(-K_{Y'}|_{E'}))$ is 3. Hence there is a unique member $\tilde{E}$ of $|-K_{Y'} - E'|$ since $h^0(-K_{Y'}) = 4$.

Step 3 for [2]. Since $|-K_{Y'} + E'|$ is free and $-K_{Y'} + E'$ is numerically trivial only for ${l_1}'$, ${l_2}'$ and ${l_3}'$ and positive for a curve in E', they are numerically equivalent and span an extremal ray R of $\overline{\mathrm{NE}}(Y')$. Since $\mathrm{Bs}|-K_{Y'}| = {l_1}' \cup {l_2}' \cup {l_3}'$ and $-K_{Y'}.{l_i}' < 0$, $\mathrm{Supp}\, R = {l_1}' \cup {l_2}' \cup {l_3}'$. Furthermore by $\mathrm{Bs}|-K_{Y'}| = {l_1}' \cup {l_2}' \cup {l_3}'$ again, there is a smooth anti-canonical divisor D ([MM3, Proposition 6.8]). Hence the contraction of ${l_1}'$, ${l_2}'$ and ${l_3}'$ is a log flopping contraction for the pair (Y', D) and the log flop exists. Let $Y' \dashrightarrow Y'_0$ be the log flop. Since $D.{l_i}' = -1$, the normal bundle of ${l_i}'$ is of type $(-1,-2)$. Hence Y'_0 has three $\frac{1}{2}(1,1,1)$-singularities. Since $-K_{Y'_0}$ is nef and big, we can construct a similar diagram $Y'_0 \dashrightarrow Y'_1 \dashrightarrow \dots Y'_i \dashrightarrow Y'_{i+1} \dots Y := Y'_l \xrightarrow{f} X$to Lemma 3.2 by considering extremal rays, where $Y'_i \dashrightarrow Y'_{i+1}$ is a flop or a flip for $i = 0$ and a flip if $i \geq 1$. Let $\tilde{E}_i$ be the strict transform of $\tilde{E}$ on Y'_i.

Similarly to Step 3 for [1], we can see that f is the blow up at a $\frac{1}{2}(1,1,1)$-singularity or the weighted blow up at a QODP with weight $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$. In any case X is a $\mathbb{Q}$-Fano 3-fold with $I(X) = 2$. Since $(-K_X)^3 = 4$ and $N = 4$, $F(X)$ must be $\frac{1}{2}$. So X is what we want.

Appendix

In this appendix, we give the table of a $(2,0)$-type contraction from a 3-fold with only index 2 terminal singularities.

Proposition. *Let X be a 3-fold with only index 2 terminal singularities and $f : X \to (Y,Q)$ a contraction of $(2,0)$-type to a germ (Y,Q) which contracts a prime divisor E to Q. Then the following holds:*

(1) *Assume that E contains no index 2 point. Then one of the following holds:*

$$(2,0)_1 : (E, -E|_E) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \text{ and } Q \text{ is a smooth point };$$

$$(2,0)_2 : (E, -E|_E) \simeq (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^3}(1)|_{\mathbb{P}^1 \times \mathbb{P}^1}) \text{ and } (Y,Q) \simeq (((xy+zw=0) \subset \mathbb{C}^4), o);$$

$(2,0)_3 : (E, -E|_E) \simeq (\mathbb{F}_{2,0}, \mathcal{O}_{\mathbb{P}^3}(1)|_{\mathbb{F}_{2,0}})$ *and* $(Y, Q) \simeq (((xy+z^2+w^k = 0) \subset \mathbb{C}^4), o)(k \geq 3)$;

$(2,0)_4 : (E, -E|_E) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *and* Q *is a* $\frac{1}{2}(1,1,1)$*-singularity.*

Furthermore for all cases, f *is the blow up of* Q*.*

(2) *Assume that* E *contains an index* 2 *point. Then one of the following holds:*

$(2,0)_5 : (E, -E|_E) \simeq (\mathbb{F}_{2,0}, l)$, *where* l *is a ruling of* $\mathbb{F}_{2,0}$.
Q *is a smooth point and* f *is a weighted blow up with weight* $(2,1,1)$.
In particular we have $K_X = f^*K_Y + 3E$;

$(2,0)_6 : K_X = f^*K_Y + E$ *and* Q *is a Gorenstein singular point.* $E^3 = \frac{1}{2}$;

$(2,0)_7 : K_X = f^*K_Y + E$ *and* Q *is a Gorenstein singular point.* $E^3 = 1$;

$(2,0)_8 : K_X = f^*K_Y + E$ *and* Q *is a Gorenstein singular point.* $E^3 = \frac{3}{2}$;

$(2,0)_9 : K_X = f^*K_Y + E$ *and* Q *is a Gorenstein singular point.* $E^3 = 2$;

$(2,0)_{10} : (E, -E|_E) \simeq ((\{xy + w^2 = 0\} \subset \mathbb{P}(1,1,2,1)), \mathcal{O}(2))$.
$(Y, Q) \simeq (((xy + z^k + w^2 = 0) \subset \mathbb{C}^4/\mathbb{Z}_2(1,1,0,1)), o)$.
f *is a weighted blow up with a weight* $(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2})$.
In particular we have $K_X = f^*K_Y + \frac{1}{2}E$;

$(2,0)_{11} : (E, -E|_E) \simeq (\mathbb{F}_{2,0}, 3l)$.
Q *is a* $\frac{1}{3}(2,1,1)$*-singularity and* f *is a weighted blow up with a weight* $\frac{1}{3}(2,1,1)$.
In particular we have $K_X = f^*K_Y + \frac{1}{3}E$;

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