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Kyoto University
ON CLASSIFICATION OF Q-FANO 3-FOLDS OF GORENSTEIN INDEX 2 AND FANO INDEX $\frac{1}{2}$

HIROMICHI TAKAGI

Notation and Conventions.

\begin{itemize}
  \item $\sim$ linear equivalence
  \item $\equiv$ numerical equivalence
  \item ODP ordinary double point, i.e., singularity analytically isomorphic to \(\{xy + z^2 + u^2 = 0 \subset \mathbb{C}^4\}\)
  \item QODP singularity analytically isomorphic to \(\{xy+z^2+u^2 = 0 \subset \mathbb{C}^4/Z_2(1,1,1,0)\}\)
  \item \(F_n\) Hirzebruch surface of degree \(n\)
  \item \(F_{n,0}\) surface which is obtained by the contraction of the negative section of \(F_n\)
  \item \(Q_3\) smooth 3-dimensional quadric.
  \item \(B_i\) (1 \(\leq\) \(i\) \(\leq\) 5) \(Q\)-factorial Gorenstein terminal Fano 3-fold of Fano index 2, and with Picard number 1 and \((-K)^3 = 8i\), where \(K\) is the canonical divisor
  \item \(A_{2i}\) (1 \(\leq\) \(i\) \(\leq\) 11 and \(i\) \(\neq\) 10) \(Q\)-factorial Gorenstein terminal Fano 3-fold of Fano index 1, and with Picard number 1 and \((-K)^3 = 2i\)
  \item contraction of \((m,n)\)-type extremal contraction whose exceptional locus has dimension \(m\) and the image of the exceptional locus has dimension \(n\)
\end{itemize}

0. INTRODUCTION

In this article, we will work over \(\mathbb{C}\), the complex number field.

**Definition 0.0 (Q-Fano variety).** Let \(X\) be a normal projective variety. We say that \(X\) is a \(Q\)-Fano variety (resp. weak \(Q\)-Fano variety) if \(X\) has only terminal singularities and \(-K_X\) is ample (resp. nef and big).

Let \(I(X) := \min\{I|IK_X\text{ is a Cartier divisor}\}\) and we call \(I(X)\) the Gorenstein index of \(X\).

Write \(I(X)(-K_X) \equiv r(X)H(X)\), where \(H(X)\) is a primitive Cartier divisor and \(r(X) \in \mathbb{N}\). (Note that \(H(X)\) is unique since Pic\(X\) is torsion free.) Then we call \(\frac{r(X)}{I(X)}\) the Fano index of \(X\) and denote it by \(F(X)\).

**Remark 0.1.**

(1) We can allow that a \(Q\)-Fano variety or a weak \(Q\)-Fano variety has worse singularities than terminal. When we have to treat such a variety in this paper, we indicate singularities which we allow, e.g., 'a \(Q\)-Fano 3-fold with only canonical singularities';

(2) if \(X\) is Gorenstein in Definition 0.0, we say that \(X\) is a Fano variety (resp. a weak Fano variety).

\textit{Key words and phrases.} Q-Fano 3-fold, Extremal contraction.


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HIROMICHI TAKAGI

For the classification theory of varieties, a Q-factorial Q-Fano variety with Picard number 1 is important because it is an output of the minimal model program. Here we mention the known result about the classification of Q-Fano 3-folds:

(1) G. Fano started the classification of smooth Fano 3-folds and it was completed by V. A. Iskovskih [I1] ~ [I4], V. V. Shokurov [Sh1], [Sh2], T. Fujita [Fu1] ~ [Fu3], S. Mori and S. Mukai [MM1] ~ [MM3];

(2) S. Mukai [Mu] classified indecomposable Gorenstein Fano 3-folds with canonical singularities by using vector bundles;

(3) T. Sano [San1] and independently F. Campana and H. Flenner [CF] classified non Gorenstein Fano 3-folds of Fano indices > 1;

(4) T. Sano [San2] classified non Gorenstein Fano 3-folds of Fano indices 1 and with only cyclic quotient terminal singularities. Recently T. Minagawa [Mi1] proved that non Gorenstein Q-Fano 3-folds with Fano indices 1 can be deformed to one with only cyclic quotient terminal singularities;

(5) A. R. Fletcher [Fl] gave the classification of Q-Fano 3-folds which are weighted complete intersections of codimension 1 or 2. Recently S. Altinok [Al] (see also [RM2]) obtained a list of Q-Fano 3-folds which are subvarieties in a weighted projective space of codimension 3 or 4.

On the other hand K. Takeuchi [T1] simplified and amplified V. A. Iskovskih's method of classification by using the theory of the extremal ray. In particular he reproved the Shokurov's theorem [Sh2], the existence of lines on a smooth Fano 3-fold of Fano index 1 and with Picard number 1 by simple numerical calculations.

We formulate a slight generalization of Takeuchi's construction for a Q-factorial Q-Fano 3-fold X with ρ(X) = 1 and give a classification of a Q-factorial Q-Fano 3-fold with the following properties:

**Main Assumption 0.2.**

(1) ρ(X) = 1;
(2) I(X) = 2;
(3) F(X) = 1/2;
(4) h^0(-K_X) ≥ 4;
(5) there exists an index 2 point P such that

\[(X, P) \simeq (\{xy + z^2 + ua = 0\}/\mathbb{Z}_2(1,1,1,0), o)\]

for some a ∈ N.

**Takeuchi's construction 0.3.** Here we explain a slight generalization of Takeuchi's construction. Let X be a Q-factorial Q-Fano 3-fold with ρ(X) = 1. Suppose that we are given a birational morphism f : Y → X with the following properties:

(1) Y is a weak Q-Fano 3-fold;
(2) f is an extremal divisorial contraction such that f-exceptional locus E is a prime Q-Cartier divisor.

Then we obtain the following diagram:

\[
Y_0 := Y \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \ldots \xrightarrow{g_{k-1}} Y_k \xrightarrow{g_k} Y \xrightarrow{f} X',
\]

where f is a prime Q-Cartier divisor.
where

1. $Y_0 \rightarrow Y_1$ is a flop or a flip and $Y_i \rightarrow Y_{i+1}$ is a flip for $i \geq 1$;
2. $f'$ is a crepant divisorial contraction (in this case, $i = 0$) or an extremal contraction which is not isomorphic in codimension 1.

We use the following notation:

$Y' := Y_k$;

$E_i :=$ the strict transform of $E$ on $Y_i$;

$E :=$ the strict transform of $E$ on $Y'$;

$e := E^3 - E_1^3$ if $Y_0 \rightarrow Y_1$ is a flop or $:= 0$ otherwise;

$d_i := (-K_{Y_i})^3 - (-K_{Y_{i+1}})^3$ (resp. $a_i := \frac{E_i l_i}{(-K_{Y_i}) l_i}$) if $Y_i \rightarrow Y_{i+1}$ is a flip, where

$l_i$ is a flipping curve, or $:= 0$ (resp. $:= 0$) otherwise;

$z$ and $u$ is defined as follows:

If $f'$ is birational, then let $E'$ be the exceptional divisor of $f'$ and set $E' := z(-K_{Y'}) - uE$ or if $f'$ is not birational, then let $L$ be the pull back of an ample generator of $\text{Pic} X'$ and set $L := z(-K_{Y'}) - uE$.

We note the following:

1. $(-K_{Y'})^2 E = (-K_Y)^2 E - \sum a_i d_i$;

2. $(-K_{Y'})^2 E = (-K_Y)^2 E^2 - \sum a_i^2 d_i$;

$E^3 = E^3 - e - \sum a_i^3 d_i$;

On the other hand the value or the relation of the value (expressed with $z$ and $u$) of $(-K_{Y'})^3$, $(-K_{Y'})^2 E$, $(-K_{Y'}) E^2$ and $E^3$ are restricted by the properties of $f'$.

By these (1) and (2), we obtain equations of Diophantine type.

Under Main Assumption 0.2, Construction 0.3 works for a suitable choice of $f$ and we can solve the equations as noted above.

**Main Theorem.** Let $X$ be as in Main Assumption 0.2. Let $f : Y \rightarrow X$ be the weighted blow up at $P$ with weight $\frac{1}{2}(1, 1, 1, 2)$. Then $Y$ is a weak $\mathbb{Q}$-Fano 3-fold.

Consider the diagram as in 0.3. Let $h := h^i(-K_X)$, $N := \text{aw}(X)$ and $n := \sum \text{aw}(Y_i, P_{ij})$ (the summation is taken over the index 2 points on flipping curves), where $\text{aw}(X)$ is the number of $\frac{1}{2}(1, 1, 1)$-singularities which we obtain by deforming non Gorenstein points of $X$ locally and $\text{aw}(Y_i, P_{ij})$ is defined similarly. Then we can solve the equations above and obtain a geographic classification of $X$ as below (?? in the table means that we don't know the existence of an example):
HIROMICHI TAKAGI

\[ (-K_X)^3 \]

<table>
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<tr>
<th>( N )</th>
<th>( e )</th>
<th>( n )</th>
<th>( z )</th>
<th>(-K_Y, C)</th>
<th>(-K_X^3, X')</th>
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<tr>
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<td>0</td>
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</table>

\( z = u \) if \( f' \) is not a crepant divisorial contraction.

\( u = 2 \) if \( f' \) is a crepant divisorial contraction.

\( F := \text{a general fiber of } f' \) if \( f' \) is \( (3,1) \)-type.

See Appendix for \( (2,0)_4 \).

\( g(C) = 0 \) in case \( f' \) is of type \( E_1 \) and every singularity of \( Y \) is a \( \frac{1}{2}(1,1,1) \)-singularity.

\[ (-K_X)^3 \]

<table>
<thead>
<tr>
<th>( N )</th>
<th>( e )</th>
<th>( n )</th>
<th>( z )</th>
<th>( \deg \Delta )</th>
<th>( \deg F )</th>
<th>(-K_Y, C)</th>
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<td>9</td>
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<td>3</td>
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</tr>
</tbody>
</table>

\( z = u \).

\( \Delta := \text{the discriminant divisor of } f' \) if \( f' \) is \( (3,2) \)-type.

\( F := \text{a general fiber of } f' \) if \( f' \) is \( (3,1) \)-type.
Q-FANO 3-FOLDS

Type [i] means the Q-Fano 3-fold of type [i] which was classified by T. Sano in [San2].

\[
(-K_X)^3 h = 7
\]

| \(\frac{13}{2}\) | 1 | 6 | 0 | 3 | 36 | \((2,1), P^3\) |
| 9 | 2 | 6 | 0 | 2 | 18 | \((2,1), P^3\) |
| 9 | 2 | 5 | 1 | 3 | 32 | \((2,1), P^3\) |
| \(\frac{19}{2}\) | 3 | 5 | 1 | 2 | 15 | \((2,1), P^3\) |
| \(\frac{19}{2}\) | 3 | 4 | 2 | 3 | 28 | \((2,1), P^3\) |

Type [i] means the Q-Fano 3-fold of type [i] which was classified by T. Sano in [San2].

\[ u = z + 1. \]

\[
(-K_X)^3 h = 8
\]

| \(\frac{21}{2}\) | 1 | 6 | 0 | 1 | 6 | \((2,1), B_3\) |
| \(\frac{13}{2}\) | 1 | 5 | 0 | 2 | 27 | \((2,1), Q_3\) |
| 11 | 2 | 4 | 1 | 2 | 24 | \((2,1), Q_3\) |

\[ u = z + 1. \]

\[
(-K_X)^3 h = 9
\]

| \(\frac{25}{2}\) | 1 | 5 | 0 | 1 | 10 | \((2,1), B_4\) |

\[ z = 1 \text{ and } u = 2. \]

In particular we have \((-K_X)^3 \leq 15\) and \(h^0(-K_X) \leq 10\).

Based on this result, we can derive the following properties for \(X\) as in the main theorem:

**Theorem A.** If any index 2 point satisfies the assumption (5) of 0.2, then \(|-K_X|\) has a member with only canonical singularities.

So the general elephant conjecture by M. Reid is affirmative for such an \(X\).
Theorem B. Let $X$ be a $Q$-factorial $Q$-Fano 3-fold with $(1)\sim(4)$ of 0.2. Let $N := \text{aw}(X)$. Then if $N > 1$ (resp. $N = 1$), $X$ can be transformed to a $Q$-factorial $Q$-Fano 3-fold $\tilde{Z}'$ with $(1)\sim(4)$ of 0.2 and with only $QODP$'s or $\frac{1}{2}(1,1,1)$-singularities as its singularities and $h^0(-K_{\tilde{Z}'}) = h$ and $\text{aw}(\tilde{Z}') = N - 1$ (resp. a smooth Fano 3-fold $\tilde{Z}'$ with $\rho(\tilde{Z}') = 1$, $F(\tilde{Z}') = 1$ and $h^0(-K_{\tilde{Z}'}) = h$) as follows:

$$\tilde{Y} \xrightarrow{\tilde{f}} \tilde{X} \xrightarrow{g} \tilde{Z} \xrightarrow{\tilde{g}} \tilde{Z}',$$

where $* \xrightarrow{\text{def}} **$ means that $**$ is a small deformation of $*$;

$\tilde{X}$ is a $Q$-Fano 3-fold as in 0.2 and with only $ODP$'s, $QODP$'s or $\frac{1}{2}(1,1,1)$-singularities as its singularities;

$\tilde{f}: \tilde{Y} \rightarrow \tilde{X}$ is chosen as $f$ in the main theorem;

$\tilde{g}: \tilde{Y} \rightarrow \tilde{Z}$ be the anti-canonical model.

This is an analogue to the Reid's fantasy about Calabi-Yau 3-folds [RM1].

Theorem C. If any index 2 point is a $\frac{1}{2}(1,1,1)$-singularity, $X$ can be embedded into a weighted projective space $\mathbb{P}(1^h, 2^N)$, where $h := h^0(-K_X)$ and $N$ is the number of $\frac{1}{2}(1,1,1)$-singularities on $X$.

We hope that this fact can be used for the classification of Mukai's type (see [Mu]).

1. Examples

We consider the case that $h^0(-K_X) = 4$ and $N = 4$. By the table of the main theorem, there are two possibilities of $X$ in this case. We assume that every singularity of $Y$ is a $\frac{1}{2}(1,1,1)$-singularity. Then one of the following holds:

1. $f'$ is an extremal divisorial contraction which contracts a divisor $E'$ to a curve $C$ and $| - K_Y - E'| \neq \phi$. $X'$ is a $(2, 2, 2)$-complete intersection in $\mathbb{P}^6$ and satisfies the following properties:
   (1) $X'$ is factorial;
   (2) $C$ is a smooth conic;
   (3) $X'$ has 3 singularities $P_0 \sim P_2$ on $C$ and $P_i$ is an ODP or the singularity analytically isomorphic to the origin of $\{xy + z^2 + w^3 = 0\} \subset \mathbb{C}^4$. Outside $P_i$'s, $X'$ is smooth.

2. $f'$ is blowing up at a smooth point $Q := f'(E')$ and $| - K_{Y'} - E'| \neq \phi$. $X'$ is smooth, isomorphic to $A_{10}$ and there exist exactly three lines through the point $Q$.

We will construct examples for these cases by the following three steps:

Step 1. We construct $X'$ satisfying the properties as stated as in [1] or [2];

Step 2. We construct $f'$ satisfying the properties as stated as in [1] or [2];

Step 3. We construct $f: Y \rightarrow X$ as in the main theorem from $Y'$.

[1].

Step 1 for [1]. We construct $X'$ with only ODP's.
Claim 1. Let $V$ (resp. $X'$) be a $(2,2)$-complete intersection in $\mathbb{P}^6$ (resp. a quadric section of $V$) with the following properties:

1. $V$ (resp. $X'$) contains a smooth conic $C$;
2. $V$ (resp. $X'$) has three ODP's $P_0 \sim P_2$ on $C$ and outside $P_i$'s, $V$ (resp. $X'$) is smooth.

Then $X'$ is factorial.

Proof. We claim that $V$ contains the plane $P$ spanned by $C$. Let $\sigma$ be the pencil which consists of quadrics in $\mathbb{P}^6$ containing $V$. Since $P_i$ is an ODP on $V$, there is a quadric in $\sigma$ which is singular at $P_i$. If there is a quadric in $\sigma$ which is singular at all $P_i$'s, then it is singular on $P$ and hence $V$ is singular along $C$, a contradiction. So $\sigma$ is generated by two quadrics which are singular at some $P_i$. But such quadrics contains $P$ and hence $V$ contains $P$.

Let $\nu : \bar{V} \rightarrow V$ be the composition of the blowing ups at $P_0 \sim P_2$ and $F_i$ the exceptional divisor over $P_i$. Let $\bar{X}'$ be the strict transform of $X'$ on $V$ and $H$ the total transform of a hyperplane section of $V$. Then $\bar{X}' \sim 2H - F_0 - F_1 - F_2$. Note that $[H - F_i - F_j]$ is free outside the strict transform $l_{ij}$ of the line through $P_i$ and $P_j$ and $[H - F_k]$ is free (note that $l_{ij}$ is contained in $V$ since $l_{ij} \subset P$). By this, we can easily see that $|\bar{X}'|$ is free and $\bar{X}'$ is numerically trivial only for $l_{ij}$'s $((i,j) = (0,1), (1,2), (2,0))$.

Let $\phi$ be the morphism defined by $|\bar{X}'|$. Then $\phi$-exceptional curves are $l_{ij}$'s. We will prove that $\text{Eff}(\bar{V}, \bar{X}')$ holds and $\bar{X}'$ meets every effective divisor on $\bar{V}$. By [H, p.165, Proposition 1.1] and the argument of [H, p.172, the proof of Theorem 1.5], it suffices to prove that $\text{cd}(\bar{V} - \bar{X}') < 3$, i.e., for any coherent sheaf $F$ on $\bar{V} - \bar{X}'$, $H^i(\bar{V} - \bar{X}', F) = 0$ for all $i \geq 3$. Let $\bar{V} := \phi(\bar{V})$ and $\bar{X}' := \phi(\bar{X}')$. Consider the Leray spectral sequence

$$E^{pq}_2 = H^p(V - X', R^q\phi_* F) \Rightarrow E^{p+q} = H^{p+q}(\bar{V} - \bar{X}', F),$$

where $\phi' := \phi|_{\bar{V} - \bar{X}'}$. Since $\bar{V} - \bar{X}'$ is affine and the dimension of every fiber of $\phi$ is 1, we have $E^{pq}_2 = 0$ for $p \geq 1$ or $q \geq 2$ whence $E^{p+q} = 0$ for $p + q \geq 2$. So the assertion follows.

Furthermore since $\bar{X}'$ is nef and big, $H^i(\bar{V}, \mathcal{O}(-n\bar{X}')) = 0$ for $n \geq 1$ and $i = 1,2$ by KKV vanishing theorem. Hence by the Grothendieck-Lefschetz theorem [G, p.135, 3.18] (or [H, p.178, Theorem 3.1]), we have $\text{Pic} \bar{X}' \sim \text{Pic} \bar{V} \sim \mathbb{Z}^4$. So $\rho(\bar{X}'/X') = 3$ which imply that $X'$ is factorial. □

We will give a pair $(V,X')$ satisfying the condition of Claim 1. Let $C$ be a smooth conic in $\mathbb{P}^6$ and $P_0 \sim P_2$ three points on $C$. We can choose a coordinate of $\mathbb{P}^6$ such that $C = \{x_0x_1 + x_1x_2 + x_2x_0 = x_3 = x_4 = x_5 = x_6 = 0\}$ and $P_i = \{x_j = 0 \text{ for } j \neq i\}$.

Claim 2. Let $X'$ be a $(2,2,2)$-complete intersection in $\mathbb{P}^6$ satisfying the following conditions:

1. $X'$ is factorial;
2. $X'$ contains a smooth conic $C$;
3. $X'$ has three ODP's $P_0 \sim P_2$ on $C$ and outside $P_i$'s, $X'$ is smooth.

Then $X'$ is the intersection of three quadrics $Q_1 \sim Q_3$ of the following forms by permuting $P_i$'s if necessary:
HIROMICHI TAKAGI

\[ Q_1 := \{m_0 x_0 + m_1 x_1 + q_1 = 0\}; \]
\[ Q_2 := \{pm_1 x_1 + m_2 x_2 + q_2 = 0\}; \]
\[ Q_3 := \{x_0 x_1 + x_1 x_2 + x_2 x_0 + \sum_{i=3}^{6} l_i x_i = 0\}, \]

where \( p \in \mathbb{C} \), \( m_i \) (resp. \( q_i \)) is a linear form (resp. a quadratic form) of \( x_3 \sim x_6 \) and \( l_i \) is a linear form of \( x_0 \sim x_6 \).

Conversely if \( X' = Q_1 \cap Q_2 \cap Q_3 \), where \( Q_i \) is of the form as above and \( m_i, q_i \) and \( l_i \) are suitably general, then \( X' \) satisfies (1) \( \sim \) (3).

**Proof.** Let \( \gamma \) be the net which consists of quadrics containing \( X' \). \( \gamma \) contains a member \( Q_1 \) which is singular at \( P_2 \). Then \( Q_1 \) is of the form as above. If \( m_1 = m_2 = 0 \), then \( Q_1 \) is singular on the plane \( P \) spanned by \( C \) and hence \( X' \) is singular along \( C \), a contradiction. Hence \( m_1 \neq 0 \) or \( m_2 \neq 0 \). By permuting \( P_1 \) and \( P_2 \) if necessary, we may assume that \( m_1 \neq 0 \). \( \gamma \) contains a member \( Q_2 \) which is singular at \( P_0 \). \( Q_2 \) is of the form as

\[ \{m_1' x_1 + m_2 x_2 + q_2 = 0\}, \]

where \( m_1' \) and \( m_2 \) (resp. \( q_2 \)) are linear forms (resp. a quadratic form) of \( x_3 \sim x_6 \). \( \gamma \) also contains a member \( Q' \) which is singular at \( P_1 \). If \( Q_1, Q_2 \) and \( Q' \) generate \( \gamma \), then \( X' \) contains the plane \( P \), a contradiction to the factoriality and \( F(X') = 1 \). Hence \( Q' \) is contained in the pencil generated by \( Q_1 \) and \( Q_2 \). So \( m_1' = p m_1 \) for some \( p \in \mathbb{C} \) and

\[ Q = \{-p m_0 x_0 + m_2 x_2 + (q_2 - pq_1) = 0\}. \]

Since \( X' \) does not contain \( P \) as noted above, \( \gamma \) contains a member \( Q_3 \) of the form as in the statement. \( Q_3 \) is not contained in the pencil generated by \( Q_1 \) and \( Q_2 \) and hence \( Q_i \)'s generate \( \gamma \).

Conversely let \( X' := Q_1 \cap Q_2 \cap Q_3 \), where \( Q_i \) is of the form as above and \( m_i, q_i \) and \( l_i \) are suitably general. We can easily check that \( X' \) satisfies (2) \( \sim \) (3). Set \( V := Q_1 \cap Q_2 \). We may assume that \( V \) satisfies the condition of Claim 1. Hence by Claim 1, \( X' \) is factorial. \( \square \)

**Step 2 for [1].** Let \( \nu' : \tilde{X}' \to X' \) be the composition of the blowing ups at \( P_0 \sim P_{N-2} \) and \( F_i' \) the exceptional divisor over \( P_i \). Let \( \mu' : \tilde{X}' \to \tilde{X}' \) be the blowing up along the strict transform \( \tilde{C} \) of \( C \) and \( F' \) the \( \mu' \)-exceptional divisor. We will denote the strict transforms of the two fibers of \( F_i \simeq \mathbb{P}^1 \times \mathbb{P}^1 \) through \( F_i \cap \tilde{C} \) by \( l_{ij} \) (\( j = 1, 2 \)). Note that \( -K_{\tilde{X}}.l_{ij} = 0 \). We can easily see that \( | -K_{\tilde{X}} | \) is free by \( P \cap X' = C \), where \( P \) is the plane spanned by \( C \) and \( -K_{\tilde{X}} \), is big. Hence \( l_{ij} \)'s are flopping curves on \( \tilde{X}' \) and we can see that the classes of \( l_{i1} \) and \( l_{i2} \) belong to the same ray. Let \( \tilde{X}' \to \tilde{X}'^+ \) be the flop. Then the strict transforms of \( F_i \)'s on \( \tilde{X}'^+ \) are \( \mathbb{P}^2 \)'s and we can contract them to \( \frac{1}{2}(1,1,1) \)-singularities. Let \( g' : \tilde{X}'^+ \to Y' \) be the contraction morphism, \( f' : Y' \to X' \) the natural morphism and \( E' \) the strict transform of \( F' \).

We will see that \( | -K_{Y'} - E' | \neq \phi \). Let \( F'^+ \) be the strict transform of \( F' \) on \( \tilde{X}'^+ \). Then \( -K_{\tilde{X}}^+ - F'^+ = g'^*( -K_{Y'} - E') \). Furthermore \( h^0( -K_{\tilde{X}}^+ - F'^+) = \)
Q-FANO 3-FOLDS

\( h^0(-K_{X'}, -F') \). Hence it suffices to prove that \( h^0(-K_{X'}, |F'|) \leq 3 \) since \( h^0(-K_{X'}) = 4 \). Since there is a smooth member of \( |-K_{X'}| \), we have \( N_{F'/X'} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-2) \).

Hence \( F' \simeq \mathbb{P}^1 \) and \( -K_{X'}|_{F'} \sim C_0 + l \), where \( C_0 \) is the minimal section of \( F' \) and \( l \) is a fiber of \( F' \). So we are done.

**Step 3 for [1].** Since \( Y' \) has only \( \frac{1}{3}(1,1,1) \)-singularities and \( -K_{Y'} \) is nef and big, we can construct a similar diagram \( Y_0' := Y' \dashrightarrow Y_1' \ldots Y_i' \dashrightarrow Y_{i+1}' \ldots Y := Y_1' \xrightarrow{f} X \) to 0.3 by considering extremal rays, where \( Y_i' \dashrightarrow Y_{i+1}' \) is a flop or a flip for \( i = 0 \) and a flip for \( i \geq 1 \). Let \( \tilde{E}_i \) (resp. \( E \)) be the strict transform of \( E \) on \( Y_i' \) (resp. \( Y \)). Let \( R_i \) be the extremal ray which is other than the ray associated to \( f' \) for \( i = 0 \) or the \( K_{Y_i} \)-negative extremal ray for \( i \geq 1 \). By similar calculations to 0.3, we have

\[
\begin{align*}
(1) & \quad (-K_{Y'})^2 E = 1 + \sum a_i d_i; \\
(2) & \quad (-K_Y E^2 = -2 - \sum a_i d_i; \\
(3) & \quad E^3 = -6 + \sum a_i d_i + e',
\end{align*}
\]

where \( e', a_i \) and \( d_i \) are similarly defined to 0.3 with respect to \( -K_{Y'} \) and \( \tilde{E}_i \) and furthermore we can see that \( a_i \) is a non negative integer.

**Claim 3.** \( \tilde{E}_i.R_i < 0 \).

**Proof.** We can prove the assertion by induction. For \( i = 0 \), \( \tilde{E}_0.R_0 < 0 \) can be directly checked. Assume that the assertion holds for the numbers less than \( i \). So the other extremal ray than \( R_i \) is positive for \( E_i \). Since \( -K_{Y_i} \) is free outside a finite number of curves, \( -K_{Y_i}|_{E_i} \) is numerically equivalent to an effective 1-cycle. Hence by \( -K_{Y_i} E_i^2 \leq -K_{Y_j} E^2 = -2 \), we have \( E_i.R_i < 0 \).

By this claim, we know that \( f \) is an divisorial contraction whose exceptional divisor is \( E \). If \( f \) is a crepant divisorial contraction, then \( l = 0 \). But \( (-K_{Y'})^2 E = 1 \), a contradiction. Hence \( f \) is a \( K_Y \)-negative contraction. Assume that \( f \) is \( (2,1) \)-type which contracts \( E \) to a curve \( C' \). Then \( (-K_X.C') = (-K_Y + E)(-K_Y)E = -1 - \sum d_i a_i(a_i - 1) < 0 \), a contradiction since \( X \) is a Q-Fano 3-fold.

By the classification of a \( (2,0) \)-type contraction from a 3-fold with only index 2 terminal singularities (see Appendix), if \( f \) is such an contraction, then we have \( -K_Y E^2 \geq -2 \). On the other hand \( -K_Y E^2 \leq -K_{Y_i} E^2 = -2 \). Hence there is no flip. So \( (-K_Y)^2 E = (-K_{Y})^2 E = 1 \) and hence again by the classification of a contraction as above, \( f \) is the blow up at a \( \frac{1}{2}(1,1,1) \)-singularity or the weighted blow up at a QODP with weight \( (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 1) \) (we use the coordinate as stated in the definition of QODP). In any case \( X \) is a Q-Fano 3-fold with \( f(X) = 2 \). We can easily check that \((-K_X)^3 = 4 \) and \( aw(X) = 4 \). Furthermore by this, \( F(X) \) must be \( \frac{1}{2} \). So \( X \) is what we want.

[2].
Step 1 for [2]. The Grassmannian $G(2,5)$ (parameterizing 2-dimensional subspaces of 5-dimensional vector space) can be embedded into $\mathbb{P}^9$ by the Plücker embedding. Its defining equations are $x_{ij}x_{kl} - x_{ik}x_{jl} + x_{jk}x_{il} = 0$ for all $1 \leq i < j < k < l \leq 5$, where $x_{pq}$ $(1 \leq p < q \leq 5)$ is a Plücker coordinate. Let $Q$ be the point defined by $x_{pq} = 0$ for any $(p,q) \neq (1,2)$. Let $l_1$ (resp. $l_2$) be the line $l \subset G(2,5)$ defined by $x_{pq} = 0$ for any $(p,q) \neq (1,2),(1,3)$ (resp. $(p,q) \neq (1,2),(2,4)$). Let $l_3$ be the line $l \subset G(2,5)$ defined by the equations $x_{pq} = r_{pq}x_{12}$ for $(p,q) \neq (1,2)$ such that $r_{34} = r_{35} = r_{45} = 0$, $r_{13}r_{24} - r_{23}r_{14} = 0$, $r_{13}r_{25} - r_{23}r_{15} = 0$, $r_{14}r_{25} - r_{24}r_{15} = 0$ and $r_{15}r_{25} \neq 0$. Let $H$ be the 3-plane spanned by $l_1$, $l_2$ and $l_3$. Then $G(2,5) \cap H = l_1 \cup l_2 \cup l_3$. Hence by [MM3, Proposition 6.8], there are two hyperplane $H_1$, $H_2$ and a quadric $Q$ such that $X' := G(2,5) \cap H_1 \cap H_2 \cap Q$ is smooth and $X'$ contains $l_1$, $l_2$ and $l_3$. Since the tangent space of $X'$ at $Q$ also contains all the lines on $X'$ through $Q$, it is equal to $H$. Hence there are only three lines on $X'$ through $Q$.

Step 2 for [2]. Let $f' : Y' \to X'$ be the blow up at $Q$ and $E'$ the exceptional divisor. Let $l_1'$, $l_2'$ and $l_3'$ be the transforms of $l_1$, $l_2$ and $l_3$ on $Y'$. Since $Bs[-K_{Y'}] = l_1' \cup l_2' \cup l_3'$, the rank of the natural map $H^0(-K_{Y'}) \to H^0(O(-K_{Y'}|E'))$ is 3. Hence there is a unique member $\tilde{E}$ of $-K_{Y'} - E'$ since $h^0(-K_{Y'}) = 4$.

Step 3 for [2]. Since $-K_{Y'} + E'$ is free and $-K_{Y'} + E'$ is numerically trivial only for $l_1'$, $l_2'$ and $l_3'$ and positive for a curve in $E'$, they are numerically equivalent and span an extremal ray $R$ of $\overline{NE}(Y')$. Since $Bs[-K_{Y'}] = l_1' \cup l_2' \cup l_3'$ and $-K_{Y'}.l_i' < 0$, $\text{Supp } R = l_1' \cup l_2' \cup l_3'$. Furthermore by $Bs[-K_{Y'}] = l_1' \cup l_2' \cup l_3'$ again, there is a smooth anti-canonical divisor $D$ ([MM3, Proposition 6.8]). Hence the contraction of $l_1'$, $l_2'$ and $l_3'$ is a log flopping contraction for the pair $(Y',D)$ and the log flop exists. Let $Y' \to Y'_0$ be the log flop. Since $D.l_i' = -1$, the normal bundle of $l_i'$ is of type $(-1,-2)$. Hence $Y'_0$ has three $\frac{1}{2}(1,1,1)$-singularities. Since $-K_{Y'_0}$ is nef and big, we can construct a similar diagram $Y'_0 \to Y'_1 \to \cdots Y'_i \to Y := Y'_i \to X$ to Lemma 3.2 by considering extremal rays, where $Y'_i \to Y'_{i+1}$ is a flop or a flip for $i = 0$ and a flip if $i \geq 1$. Let $E_i$ be the strict transform of $E$ on $Y'_i$.

Similarly to Step 3 for [1], we can see that $f$ is the blow up at a $\frac{1}{2}(1,1,1)$-singularity or the weighted blow up at a QODP with weight $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$. In any case $X$ is a Q-Fano 3-fold with $I(X) = 2$. Since $(-K_X)^3 = 4$ and $N = 4$, $F(X)$ must be $\frac{1}{2}$. So $X$ is what we want.

APPENDIX

In this appendix, we give the table of a $(2,0)$-type contraction from a 3-fold with only index 2 terminal singularities.

Proposition. Let $X$ be a 3-fold with only index 2 terminal singularities and $f : X \to (Y,Q)$ a contraction of $(2,0)$-type to a germ $(Y,Q)$ which contracts a prime divisor $E$ to $Q$. Then the following holds:

1. Assume that $E$ contains no index 2 point. Then one of the following holds:

   $(2,0)_1 : (E, -E|_{E}) \simeq (\mathbb{P}^2, O_{\mathbb{P}^2}(1))$ and $Q$ is a smooth point ;

   $(2,0)_2 : (E, -E|_{E}) \simeq (\mathbb{P}^1 \times \mathbb{P}^1, O_{\mathbb{P}^3}(1)|_{\mathbb{P}^1 \times \mathbb{P}^1})$ and $(Y,Q) \simeq ((xy + zw = 0) \subset \mathbb{C}^4)$;
Q-PANO 3-FOLDS

$(2,0)_3 : (E,-E|_E) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)|_{\mathbb{P}^2_2,0})$ and $(Y,Q) \simeq (((xy+z^2+w^k = 0) \subset \mathbb{C}^4), o)(k \geq 3)$;

$(2,0)_4 : (E,-E|_E) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ and $Q$ is a $\frac{1}{2}(1,1,1)$-singularity.

Furthermore for all cases, $f$ is the blow up of $Q$.

(2) Assume that $E$ contains an index 2 point. Then one of the following holds:

$(2,0)_5 : (E,-E|_E) \simeq (\mathbb{P}^2, l)$, where $l$ is a ruling of $\mathbb{P}^2$.

$Q$ is a smooth point and $f$ is a weighted blow up with weight $(2,1,1)$.

In particular we have $K_X = f^*K_Y + 3E$;

$(2,0)_6 : K_X = f^*K_Y + E$ and $Q$ is a Gorenstein singular point. $E^3 = \frac{1}{2}$;

$(2,0)_7 : K_X = f^*K_Y + E$ and $Q$ is a Gorenstein singular point. $E^3 = 1$;

$(2,0)_8 : K_X = f^*K_Y + E$ and $Q$ is a Gorenstein singular point. $E^3 = \frac{3}{2}$;

$(2,0)_9 : K_X = f^*K_Y + E$ and $Q$ is a Gorenstein singular point. $E^3 = 2$;

$(2,0)_{10} : (E,-E|_E) \simeq (((xy + w^2 = 0) \subset \mathbb{P}(1,1,2,1)), \mathcal{O}(2))$.

$(Y,Q) \simeq (((xy + z^k + w^2 = 0) \subset \mathbb{C}^4/\mathbb{Z}_2(1,1,0,1)), o)$.

$f$ is a weighted blow up with a weight $\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}\right)$.

In particular we have $K_X = f^*K_Y + \frac{1}{2}E$;

$(2,0)_{11} : (E,-E|_E) \simeq (\mathbb{P}^2, 3l)$.

$Q$ is a $\frac{1}{3}(2,1,1)$-singularity and $f$ is a weighted blow up with a weight $\frac{1}{3}(2,1,1)$.

In particular we have $K_X = f^*K_Y + \frac{1}{3}E$;
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Q-FANO 3-FOLDS

[T3]  ———, a private letter to the author.

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