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<th>Title</th>
<th>On Classification of Q-Fano 3-Folds of Gorenstein Index 2 and Fano Index 1/2</th>
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<td>Author(s)</td>
<td>Takagi, Hiromichi</td>
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Kyoto University
ON CLASSIFICATION OF Q-FANO 3-FOLDS OF
GORENSTEIN INDEX 2 AND FANO INDEX $\frac{1}{2}$

HIROMICHI TAKAGI

Notation and Conventions.

$\sim$ linear equivalence
$\cong$ numerical equivalence
ODP ordinary double point, i.e., singularity analytically isomorphic to \{xy + $z^2 + u^2 = 0 \subset \mathbb{C}^4$\}
QODP singularity analytically isomorphic to \{xy+$z^2+u^2 = 0 \subset \mathbb{C}^4/\mathbb{Z}_2(1,1,1,0)$\}
$F_n$ Hirzebruch surface of degree $n$
$F_{n,0}$ surface which is obtained by the contraction of the negative section of $F_n$
$Q_3$ smooth 3-dimensional quadric.

$B_i$ (1 \leq i \leq 5) Q-factorial Gorenstein terminal Fano 3-fold of Fano index 2, and with Picard number 1 and $(-K)^3 = 8i$, where $K$ is the canonical divisor

$A_{2i}$ (1 \leq i \leq 11 and $i \neq 10$) Q-factorial Gorenstein terminal Fano 3-fold of Fano index 1, and with Picard number 1 and $(-K)^3 = 2i$

contraction of $(m,n)$-type extremal contraction whose exceptional locus has dimension $m$ and the image of the exceptional locus has dimension $n$

0. INTRODUCTION

In this article, we will work over $\mathbb{C}$, the complex number field.

Definition 0.0 (Q-Fano variety). Let $X$ be a normal projective variety. We say that $X$ is a Q-Fano variety (resp. weak Q-Fano variety) if $X$ has only terminal singularities and $-K_X$ is ample (resp. nef and big).

Let $I(X) := \min\{I|IK_X$ is a Cartier divisor} and we call $I(X)$ the Gorenstein index of $X$.

Write $I(X)(-K_X) \equiv r(X)H(X)$, where $H(X)$ is a primitive Cartier divisor and $r(X) \in \mathbb{N}$. (Note that $H(X)$ is unique since Pic$X$ is torsion free.) Then we call $r(X)$ the Fano index of $X$ and denote it by $F(X)$.

Remark 0.1.

(1) We can allow that a Q-Fano variety or a weak Q-Fano variety has worse singularities than terminal. When we have to treat such a variety in this paper, we indicate singularities which we allow, e.g., 'a Q-Fano 3-fold with only canonical singularities';

(2) if $X$ is Gorenstein in Definition 0.0, we say that $X$ is a Fano variety (resp. a weak Fano variety).

Key words and phrases. Q-Fano 3-fold, Extremal contraction.

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For the classification theory of varieties, a $Q$-factorial $Q$-Fano variety with Picard number 1 is important because it is an output of the minimal model program. Here we mention the known result about the classification of $Q$-Fano 3-folds:

1. G. Fano started the classification of smooth Fano 3-folds and it was completed by V. A. Iskovskikh [I1] $\sim$ [I4], V. V. Shokurov [Sh1], [Sh2], T. Fujita [Fu1] $\sim$ [Fu3], S. Mori and S. Mukai [MM1] $\sim$ [MM3];
2. S. Mukai [Mu] classified indecomposable Gorenstein Fano 3-folds with canonical singularities by using vector bundles;
3. T. Sano [San1] and independently F. Campana and H. Flenner [CF] classified non Gorenstein Fano 3-folds of Fano indices $>1$;
4. T. Sano [San2] classified non Gorenstein Fano 3-folds of Fano indices 1 and with only cyclic quotient terminal singularities. Recently T. Minagawa [Mi1] proved that non Gorenstein $Q$-Fano 3-folds with Fano indices 1 can be deformed to one with only cyclic quotient terminal singularities;
5. A. R. Fletcher [Fl] gave the classification of $Q$-Fano 3-folds which are weighted complete intersections of codimension 1 or 2. Recently S. Altinok [Al] (see also [RM2]) obtained a list of $Q$-Fano 3-folds which are subvarieties in a weighted projective space of codimension 3 or 4.

On the other hand K. Takeuchi [T1] simplified and amplified V. A. Iskovskih's method of classification by using the theory of the extremal ray. In particular he reproved the Shokurov's theorem [Sh2], the existence of lines on a smooth Fano 3-fold of Fano index 1 and with Picard number 1 by simple numerical calculations.

We formulate a slight generalization of Takeuchi's construction for a $Q$-factorial $Q$-Fano 3-fold $X$ with $\rho(X) = 1$ and give a classification of a $Q$-factorial $Q$-Fano 3-fold with the following properties:

Main Assumption 0.2.

1. $\rho(X) = 1$;
2. $I(X) = 2$;
3. $F(X) = \frac{1}{2}$;
4. $h^0(-K_X) \geq 4$;
5. there exists an index 2 point $P$ such that

$$(X, P) \simeq (\{xy + z^2 + u^a = 0\}/\mathbb{Z}_2(1,1,1,0), o)$$

for some $a \in \mathbb{N}$.

Takeuchi's construction 0.3. Here we explain a slight generalization of Takeuchi's construction. Let $X$ be a $Q$-factorial $Q$-Fano 3-fold with $\rho(X) = 1$. Suppose that we are given a birational morphism $f : Y \to X$ with the following properties:

1. $Y$ is a weak $Q$-Fano 3-fold;
2. $f$ is an extremal divisorial contraction such that $f$-exceptional locus $E$ is a prime $Q$-Cartier divisor.

Then we obtain the following diagram:

$$
\begin{array}{ccc}
Y_0 := Y & \xrightarrow{g_0} & Y_1 \ldots \xrightarrow{g_{k-1}} Y_k \\
\downarrow f & & \downarrow f' \\
X & & X'
\end{array}
$$
where

1. $Y_0 \to Y_1$ is a flop or a flip and $Y_i \to Y_{i+1}$ is a flip for $i \geq 1$;
2. $f'$ is a crepant divisorial contraction (in this case, $i = 0$) or an extremal contraction which is not isomorphic in codimension 1.

We use the following notation:

- $Y' := Y_k$;
- $E_i :=$ the strict transform of $E$ on $Y_i$;
- $\hat{E} :=$ the strict transform of $E$ on $Y'$;
- $e := E^3 - E_1^3$ if $Y_0 \to Y_1$ is a flop or $:= 0$ otherwise;
- $d_i := (-K_{Y_i})^3 - (-K_{Y_{i+1}})^3$ (resp. $a_i := \frac{E_{i+1}}{-(-K_{Y_{i+1}})}$) if $Y_i \to Y_{i+1}$ is a flip, where $l_i$ is a flipping curve, or $:= 0$ (resp. $:= 0$) otherwise;
- $z$ and $u$ is defined as follows:
  - If $f'$ is birational, then let $E'$ be the exceptional divisor of $f'$ and set $E' \equiv z(-K_{Y'}) - uE$ or if $f'$ is not birational, then let $L$ be the pull back of an ample generator of $\text{Pic}X'$ and set $L \equiv z(-K_{Y'}) - uE$.

We note the following:

1. 
   
   $$(-K_{Y'})^2 \hat{E} = (-K_Y)^2 E - \sum a_i d_i;$$

2. 
   
   $$(-K_{Y'})\hat{E}^2 = (-K_Y)E^2 - \sum a_i^2 d_i;$$

   $$\hat{E}^3 = E^3 - e - \sum a_i^3 d_i;$$

(2) On the other hand the value or the relation of the value (expressed with $z$ and $u$) of $(-K_{Y'})^3$, $(-K_{Y'})^2 \hat{E}$, $(-K_{Y'})\hat{E}^2$ and $\hat{E}^3$ are restricted by the properties of $f'$.

By these (1) and (2), we obtain equations of Diophantine type.

Under Main Assumption 0.2, Construction 0.3 works for a suitable choice of $f$ and we can solve the equations as noted above.

**Main Theorem.** Let $X$ be as in Main Assumption 0.2. Let $f : Y \to X$ be the weighted blow up at $P$ with weight $1/2(1, 1, 1, 2)$. Then $Y$ is a weak $Q$-Fano 3-fold.

Consider the diagram as in 0.3. Let $h := h^0(-K_X)$, $N := \text{aw}(X)$ and $n := \sum \text{aw}(Y_i, P_{i,j})$ (the summation is taken over the index 2 points on flipping curves), where $\text{aw}(X)$ is the number of $1/2(1, 1, 1)$-singularities which we obtain by deforming non Gorenstein points of $X$ locally and $\text{aw}(Y_i, P_{i,j})$ is defined similarly. Then we can solve the equations above and obtain a geographic classification of $X$ as below (in the table means that we don't know the existence of an example).
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<table>
<thead>
<tr>
<th>$(K_X)^3$</th>
<th>$N$</th>
<th>$e$</th>
<th>$n$</th>
<th>$z$</th>
<th>$(-K_Y.C)$</th>
<th>$f', X'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{2}{9}$</td>
<td>1</td>
<td>15</td>
<td>0</td>
<td>1</td>
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<td>/</td>
<td>/</td>
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<td>/</td>
<td>crep. div., $(-K_X)^3 = 2, I(X') = 1$</td>
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<td>1</td>
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<td>8</td>
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<td>2</td>
<td>$(2,1), A_6$</td>
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<tr>
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<td>4</td>
<td>9</td>
<td>3</td>
<td>1</td>
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<td>3</td>
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<td>1</td>
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<td>$(2,0)<em>{15}, A</em>{16}$</td>
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<td>9</td>
<td>0</td>
<td>2</td>
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<td>$(3,1), deg F = 6$</td>
</tr>
<tr>
<td>?5</td>
<td>6</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>$(2,1), A_{12}$</td>
</tr>
</tbody>
</table>

$z = u$ if $f'$ is not a crepant divisorial contraction.
$u = 2$ if $f'$ is a crepant divisorial contraction.

$F :=$ a general fiber of $f'$ if $f'$ is $(3,1)$-type.

See Appendix for $(2,0)_{4a}$.

$g(C) = 0$ in case $f'$ is of type $E_1$ and every singularity of $Y$ is a $\frac{1}{2}(1,1,1)$—singularity.

<table>
<thead>
<tr>
<th>$(K_X)^3$</th>
<th>$N$</th>
<th>$e$</th>
<th>$n$</th>
<th>$z$</th>
<th>$deg \Delta$</th>
<th>$deg F$</th>
<th>$f', X'$</th>
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<td>1</td>
<td>9</td>
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<td>8</td>
<td>1</td>
<td>1</td>
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<td>4</td>
<td>$(3,1)$</td>
</tr>
<tr>
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<td>3</td>
<td>7</td>
<td>2</td>
<td>1</td>
<td>/</td>
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<tr>
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<td>8</td>
<td>0</td>
<td>2</td>
<td>8</td>
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</tr>
<tr>
<td>$\frac{7}{6}$</td>
<td>4</td>
<td>7</td>
<td>1</td>
<td>2</td>
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</tr>
<tr>
<td>$\frac{13}{6}$</td>
<td>4</td>
<td>6</td>
<td>3</td>
<td>1</td>
<td>/</td>
<td>6</td>
<td>$(3,1)$</td>
</tr>
<tr>
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<td>5</td>
<td>6</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>/</td>
<td>$(3,2), P_{2,0}$</td>
</tr>
</tbody>
</table>

$z = u$.

$\Delta :=$ the discriminant divisor of $f'$ if $f'$ is $(3,2)$-type.

$F :=$ a general fiber of $f'$ if $f'$ is $(3,1)$-type.

<table>
<thead>
<tr>
<th>$(K_X)^3$</th>
<th>$N$</th>
<th>$e$</th>
<th>$n$</th>
<th>$z$</th>
<th>$deg \Delta$</th>
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<td>0</td>
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<td>7</td>
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<tr>
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<td>2</td>
<td>7</td>
<td>0</td>
<td>4</td>
<td>/</td>
<td>35</td>
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<tr>
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<td>1</td>
<td>1</td>
<td>6</td>
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<td>$(3,2), P_{2}$</td>
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<td>7</td>
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<td>4</td>
<td>1</td>
<td>3</td>
<td>/</td>
<td>$(3,2), P_{2}$</td>
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</table>
Q-FANO 3-FOLDS

Type [i] means the Q-Fano 3-fold of type [i] which was classified by T. Sano in [San2].

<table>
<thead>
<tr>
<th>$(-K_X)^3$</th>
<th>$h = 7$</th>
<th>$N$</th>
<th>$e$</th>
<th>$n$</th>
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<tr>
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<td>3</td>
<td>32</td>
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<td>1</td>
<td>2</td>
<td>15</td>
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</table>

Type [i] means the Q-Fano 3-fold of type [i] which was classified by T. Sano in [San2].

$u = z + 1.$

<table>
<thead>
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<th>$(-K_X)^3$</th>
<th>$h = 8$</th>
<th>$N$</th>
<th>$e$</th>
<th>$n$</th>
<th>$z$</th>
<th>$(-K_{Y'}.C)$</th>
<th>$f', X'$</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>6</td>
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<td></td>
</tr>
<tr>
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<td>2</td>
<td>24</td>
<td>$(2,1), Q_3$</td>
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$u = z + 1.$

<table>
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<tr>
<th>$(-K_X)^3$</th>
<th>$h = 9$</th>
<th>$N$</th>
<th>$e$</th>
<th>$n$</th>
<th>$z$</th>
<th>$u$</th>
<th>$(-K_{Y'}.C)$</th>
<th>$f', X'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$25/2$</td>
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<td>1</td>
<td>2</td>
<td>10</td>
<td>$(2,1), B_4$</td>
<td></td>
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</table>

$z = 1$ and $u = 2.$

In particular we have $(-K_X)^3 \leq 15$ and $h^0(-K_X) \leq 10.$

Based on this result, we can derive the following properties for $X$ as in the main theorem:

**Theorem A.** if any index 2 point satisfies the assumption (5) of 0.2, then $|−K_X|$ has a member with only canonical singularities.

So the general elephant conjecture by M. Reid is affirmative for such an $X.$
Theorem B. Let $X$ be a $Q$-factorial $Q$-Fano 3-fold with (1)~(4) of 0.2. Let $N := aw(X)$. Then if $N > 1$ (resp. $N = 1$), $X$ can be transformed to a $Q$-factorial $Q$-Fano 3-fold $\tilde{Z}'$ with (1)~(4) of 0.2 and with only $QODP's$ or $\frac{1}{2}(1,1,1)$-singularities as its singularities and $h^0(-K_{\tilde{Z}'}) = h$ and $aw(\tilde{Z}') = N - 1$ (resp. a smooth Fano 3-fold $\tilde{Z}'$ with $\rho(\tilde{Z}') = 1$, $F(\tilde{Z}') = 1$ and $h^0(-K_{\tilde{Z}'}) = h$) as follows:

\[ \begin{array}{ccc} \tilde{Y} & \overset{\text{def}}{\longrightarrow} & \tilde{X} \\ f & \searrow & \searrow \tilde{g} \\ & \tilde{Z} & \overset{\text{def}}{\longrightarrow} \tilde{Z}', \end{array} \]

where $* \overset{\text{def}}{\rightarrow} **$ means that ** is a small deformation of *

$\tilde{X}$ is a $Q$-Fano 3-fold as in 0.2 and with only $ODP's$, $QODP's$ or $\frac{1}{2}(1,1,1)$-singularities as its singularities;

$f : Y \rightarrow \tilde{X}$ is chosen as $f$ in the main theorem;

$\tilde{g} : \tilde{Y} \rightarrow \tilde{Z}$ be the anti-canonical model.

This is an analogue to the Reid's fantasy about Calabi-Yau 3-folds [RM1].

Theorem C. If any index 2 point is a $\frac{1}{2}(1,1,1)$-singularity, $X$ can be embedded into a weighted projective space $\mathbb{P}(h,2N)$, where $h := h^0(-K_X)$ and $N$ is the number of $\frac{1}{2}(1,1,1)$-singularities on $X$.

We hope that this fact can be used for the classification of Mukai's type (see [Mu]).

1. Examples

We consider the case that $h^0(-K_X) = 4$ and $N = 4$. By the table of the main theorem, there are two possibilities of $X$ in this case. We assume that every singularity of $Y$ is a $\frac{1}{2}(1,1,1)$-singularity. Then one of the following holds:

[1]. $f'$ is an extremal divisorial contraction which contracts a divisor $E'$ to a curve $C$ and $| - K_{Y'} - E' | \neq \phi$. $X'$ is a $(2,2,2)$-complete intersection in $\mathbb{P}^6$ and satisfies the following properties:

1. $X'$ is factorial;
2. $C$ is a smooth conic;
3. $X'$ has 3 singularities $P_0 \sim P_2$ on $C$ and $P_i$ is an ODP or the singularity analytically isomorphic to the origin of \{xy + z^2 + w^3 = 0\} $\subset \mathbb{C}^4$. Outside $P_i$'s, $X'$ is smooth.

[2]. $f'$ is blowing up at a smooth point $Q := f'(E')$ and $| - K_{Y'} - E' | \neq \phi$. $X'$ is smooth, isomorphic to $A_{10}$ and there exist exactly three lines through the point $Q$.

We will construct examples for these cases by the following three steps:

Step 1. We construct $X'$ satisfying the properties as stated as in [1] or [2];
Step 2. We construct $f'$ satisfying the properties as stated as in [1] or [2];
Step 3. We construct $f : Y \rightarrow X$ as in the main theorem from $Y'$.

[1].

Step 1 for [1]. We construct $X'$ with only ODP's.
Claim 1. Let $V$ (resp. $X'$) be a $(2,2)$-complete intersection in $\mathbb{P}^6$ (resp. a quadric section of $V$) with the following properties:

1. $V$ (resp. $X'$) contains a smooth conic $C$;
2. $V$ (resp. $X'$) has three ODP's $P_0 \sim P_2$ on $C$ and outside $P_i$'s, $V$ (resp. $X'$) is smooth.

Then $X'$ is factorial.

Proof. We claim that $V$ contains the plane $P$ spanned by $C$. Let $\sigma$ be the pencil which consists of quadrics in $\mathbb{P}^6$ containing $V$. Since $P_i$ is an ODP on $V$, there is a quadric in $\sigma$ which is singular at $P_i$. If there is a quadric in $\sigma$ which is singular at all $P_i$'s, then it is singular on $P$ and hence $V$ is singular along $C$, a contradiction. So $\sigma$ is generated by two quadrics which are singular at some $P_i$. But such quadrics contains $P$ and hence $V$ contains $P$.

Let $\nu : \tilde{V} \to V$ be the composition of the blowing ups at $P_0 \sim P_2$ and $F_i$ the exceptional divisor over $P_i$. Let $\tilde{X}'$ be the strict transform of $X'$ on $\tilde{V}$ and $H$ the total transform of a hyperplane section of $V$. Then $\tilde{X}' \sim 2H - F_0 - F_1 - F_2$.

Let $u : V \to \mathbb{P}^6$ be the composition of the blowing ups at $P_0 \sim P_2$ and $F_i$ the exceptional divisor over $P_i$. Let $X'$ be the strict transform of $X'$ on $V$ and $H$ the total transform of a hyperplane section of $V$. Then $X' \sim 2H - F_0 - F_1 - F_2$. Note that $|H - F_i - F_j|$ is free outside the strict transform $l_{ij}$ of the line through $P_i$ and $P_j$ and $|H - F_k|$ is free (note that $l_{ij}$ is contained in $V$ since $l_{ij} \subset P$). By this, we can easily see that $|\tilde{X}'|$ is free and $\tilde{X}'$ is numerically trivial only for $l_{ij}$'s $((i,j) = (0,1), (1,2), (2,0)).$

Let $\phi$ be the morphism defined by $|\tilde{X}'|$. Then $\phi$-exceptional curves are $l_{ij}$'s. We will prove that $\text{Leff}(\tilde{V}, \tilde{X}')$ holds and $\tilde{X}'$ meets every effective divisor on $\tilde{V}$. By [H, p.165, Proposition 1.1] and the argument of [H, p.172, the proof of Theorem 1.5], it suffices to prove that $\text{cd}(\tilde{V} - \tilde{X}') < 3$, i.e., for any coherent sheaf $F$ on $\tilde{V} - \tilde{X}'$, $H^i(\tilde{V} - \tilde{X}', F) = 0$ for all $i \geq 3$. Let $\overline{V} := \phi(\tilde{V})$ and $\overline{X} := \phi(\tilde{X}')$. Consider the Leray spectral sequence

$$E_2^{pq} = H^p(\overline{V} - \overline{X}', R^q\phi_* F) \Rightarrow E_\infty^{p+q} = H^{p+q}(\overline{V} - \overline{X}', F),$$

where $\phi' := \phi|_{\overline{V} - \overline{X}'}$. Since $\overline{V} - \overline{X}'$ is affine and the dimension of every fiber of $\phi$ $\leq 1$, we have $E_2^{pq} = 0$ for $p \geq 1$ or $q \geq 2$ whence $E_\infty^{p+q} = 0$ for $p + q \geq 2$. So the assertion follows.

Furthermore since $\tilde{X}'$ is nef and big, $H^i(\tilde{V}, O(-n\tilde{X}')) = 0$ for $n \geq 1$ and $i = 1, 2$ by KKV vanishing theorem. Hence by the Grothendieck-Lefschetz theorem [G, p.135, 3.18] (or [H, p.178, Theorem 3.1]), we have $\text{Pic}\tilde{X}' \cong \text{Pic}\overline{V} \cong \mathbb{Z}$. So $\rho(\overline{X}'/X') = 3$ which imply that $X'$ is factorial. □

We will give a pair $(V, X')$ satisfying the condition of Claim 1. Let $C$ be a smooth conic in $\mathbb{P}^6$ and $P_0 \sim P_2$ three points on $C$. We can choose a coordinate of $\mathbb{P}^6$ such that $C = \{x_0x_1 + x_1x_2 + x_2x_0 = x_3 = x_4 = x_5 = x_6 = 0\}$ and $P_i = \{x_j = 0 \text{ for } j \neq i\}$.

Claim 2. Let $X'$ be a $(2,2,2)$-complete intersection in $\mathbb{P}^6$ satisfying the following conditions:

1. $X'$ is factorial;
2. $X'$ contains a smooth conic $C$;
3. $X'$ has three ODP's $P_0 \sim P_2$ on $C$ and outside $P_i$'s, $X'$ is smooth.

Then $X'$ is the intersection of three quadrics $Q_1 \sim Q_3$ of the following forms by permuting $P_i$'s if necessary:
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\[ Q_1 := \{m_0 x_0 + m_1 x_1 + q_1 = 0\}; \]
\[ Q_2 := \{p m_1 x_1 + m_2 x_2 + q_2 = 0\}; \]
\[ Q_3 := \{x_0 x_1 + x_1 x_2 + x_2 x_0 + \sum_{i=3}^{6} l_i x_i = 0\}, \]

where \(p \in \mathbb{C}\), \(m_i\) (resp. \(q_i\)) is a linear form (resp. a quadratic form) of \(x_3 \sim x_6\) and \(l_i\) is a linear form of \(x_0 \sim x_6\).

Conversely if \(X' = Q_1 \cap Q_2 \cap Q_3\), where \(Q_i\) is of the form as above and \(m_i\), \(q_i\) and \(l_i\) are suitably general, then \(X'\) satisfies (1) \(\sim\) (3).

**Proof.** Let \(\gamma\) be the net which consists of quadrics containing \(X'\). \(\gamma\) contains a member \(Q_1\) which is singular at \(P_2\). Then \(Q_1\) is of the form as above. If \(m_1 = m_2 = 0\), then \(Q_1\) is singular on the plane \(P\) spanned by \(C\) and hence \(X'\) is singular along \(C\), a contradiction. Hence \(m_1 \neq 0\) or \(m_2 \neq 0\). By permuting \(P_1\) and \(P_2\) if necessary, we may assume that \(m_1 \neq 0\). \(\gamma\) contains a member \(Q_2\) which is singular at \(P_0\). \(Q_2\) is of the form as

\[ \{m_1' x_1 + m_2 x_2 + q_2 = 0\}, \]

where \(m_1'\) and \(m_2\) (resp. \(q_2\)) are linear forms (resp. a quadratic form) of \(x_3 \sim x_6\).

\(\gamma\) also contains a member \(Q'\) which is singular at \(P_1\). If \(Q_1, Q_2\) and \(Q'\) generate \(\gamma\), then \(X'\) contains the plane \(P\), a contradiction to the factoriality and \(F(X') = 1\). Hence \(Q'\) is contained in the pencil generated by \(Q_1\) and \(Q_2\). So \(m_1' = p m_1\) for some \(p \in \mathbb{C}\) and

\[ Q = \{-p m_0 x_0 + m_2 x_2 + (q_2 - pq_1) = 0\}. \]

Since \(X'\) does not contain \(P\) as noted above, \(\gamma\) contains a member \(Q_3\) of the form as in the statement. \(Q_3\) is not contained in the pencil generated by \(Q_1\) and \(Q_2\) and hence \(Q_i\)'s generate \(\gamma\).

Conversely let \(X' := Q_1 \cap Q_2 \cap Q_3\), where \(Q_i\) is of the form as above and \(m_i, q_i\) and \(l_i\) are suitably general. We can easily check that \(X'\) satisfies (2) and (3). Set \(V := Q_1 \cap Q_2\). We may assume that \(V\) satisfies the condition of Claim 1. Hence by Claim 1, \(X'\) is factorial. \(\square\)

**Step 2 for [1].** Let \(\nu' : \tilde{X}' \to X'\) be the composition of the blowing ups at \(P_0 \sim P_{N-2}\) and \(F_i\) the exceptional divisor over \(P_i\). Let \(\mu' : \tilde{X}' \to \tilde{X}'\) be the blowing up along the strict transform \(\tilde{C}\) of \(C\) and \(F'\) the \(\mu'\)-exceptional divisor. We will denote the strict transforms of the two fibers of \(F_i \simeq \mathbb{P}^1 \times \mathbb{P}^1\) through \(F_i \cap \tilde{C}\) by \(l_{ij}\) (\(i = 1, 2\)). Note that \(-K_{\tilde{X}'} l_{ij} = 0\). We can easily see that \(|-K_{\tilde{X}'}|\) is free by \(P \cap X' = C\), where \(P\) is the plane spanned by \(C\) and \(-K_{\tilde{X}'}\) is big. Hence \(l_{ij}\)'s are flopping curves on \(\tilde{X}'\) and we can see that the classes of \(l_{11}\) and \(l_{22}\) belong to the same ray. Let \(\tilde{X}' \to \tilde{X}'^+\) be the flop. Then the strict transforms of \(F_i\)'s on \(\tilde{X}'^+\) are \(\mathbb{P}^2\)'s and we can contract them to \(\frac{1}{2}(1, 1, 1)\)-singularities. Let \(g' : \tilde{X}'^+ \to Y'\) be the contraction morphism, \(f' : Y' \to X'\) the natural morphism and \(E'\) the strict transform of \(F'\).

We will see that \(|-K_{Y'} - E'| \neq \phi\). Let \(F'\) be the strict transform of \(F'\) on \(\tilde{X}'^+\). Then \(-K_{\tilde{X}'^+} - F'^+ = g'^*(-K_{Y'} - E')\). Furthermore \(h^0(-K_{\tilde{X}'^+} - F'^+) = \)}
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\( h^0(-K_{X'}, -F') \). Hence it suffices to prove that \( h^0(-K_{X'}) \leq 3 \) since \( h^0(-K_{X'}) = 4 \). Since there is a smooth member of \(-K_{X'}\), we have \( N_{C/X'} \sim \mathcal{O}(1) \oplus \mathcal{O}(-2) \).

Hence \( F' \simeq \mathbb{F}_1 \) and \(-K_{X'}|_{F'} \sim C_0 + l\), where \( C_0 \) is the minimal section of \( F' \) and \( l \) is a fiber of \( F' \). So we are done.

**Step 3 for [1].** Since \( Y' \) has only \( \frac{1}{2}(1,1,1) \)-singularities and \(-K_{Y'} \) is nef and big, we can construct a similar diagram \( Y_0' := Y' \to Y_1' \to \ldots \to Y_i' \to \ldots \to Y := Y \to X \) by considering extremal rays, where \( Y_i' \to Y_{i+1}' \) is a flop or a flip for \( i = 0 \) and a flip for \( i \geq 1 \). Let \( \tilde{E}_i \) (resp. \( E_i \)) be the strict transform of \( E \) on \( Y_i' \) (resp. \( Y \)). Let \( R_i \) be the extremal ray which is other than the ray associated to \( f' \) for \( i = 0 \) or the \( K_{Y_i} \)-negative extremal ray for \( i \geq 1 \). By similar calculations to 0.3, we have

\[
\begin{align*}
(1) & \quad (-K_{Y'})^{2} E = 1 + \sum a_i' d_i' ; \\
(2) & \quad (-K_{Y})^{2} E = -2 - \sum a_i'^2 d_i' ; \\
(3) & \quad E^3 = -6 + \sum a_i'^3 d_i' + e',
\end{align*}
\]

where \( e' \), \( a_i' \) and \( d_i' \) are similarly defined to 0.3 with respect to \(-K_{Y'} \) and \( \tilde{E}_i \) and furthermore we can see that \( a_i' \) is a non negative integer.

**Claim 3.** \( \tilde{E}_i, R_i < 0 \).

**Proof.** We can prove the assertion by induction. For \( i = 0 \), \( \tilde{E}_0, R_0 < 0 \) can be directly checked. Assume that the assertion holds for the numbers less than \( i \). So the other extremal ray than \( R_i \) is positive for \( \tilde{E}_i \). Since \(-K_{Y_i} \) is free outside a finite number of curves, \(-K_{Y_i}|_{\tilde{E}_i} \) is numerically equivalent to an effective 1-cycle. Hence by \(-K_{Y_i}, \tilde{E}_i^2 \leq -K_{Y_i}, \tilde{E}^2 = -2 \), we have \( \tilde{E}_i, R_i < 0 \). \( \square \)

By this claim, we know that \( f \) is an divisorial contraction whose exceptional divisor is \( E \). If \( f \) is a crepant divisorial contraction, then \( l = 0 \). But \((-K_{Y'})^{2} E = 1 \), a contradiction. Hence \( f \) is a \( K_Y \)-negative contraction. Assume that \( f \) is \((2,1)\)-type which contracts \( E \) to a curve \( C' \). Then \((-K_X, C') = (-K_Y + E)(-K_Y)E = -1 - \sum d_i' a_i'(a_i' - 1) < 0 \), a contradiction since \( X \) is a Q-Fano 3-fold.

By the classification of a \((2,0)\)-type contraction from a 3-fold with only index 2 terminal singularities (see Appendix), if \( f \) is such an contraction, then we have \(-K_Y, E^2 \geq -2 \). On the other hand \(-K_Y, E^2 \leq -K_Y, \tilde{E}^2 = -2 \). Hence there is no flip. So \((-K_Y)^2 E = (-K_{Y'})^2 \tilde{E} = 1 \) and hence again by the classification of a contraction as above, \( f \) is the blow up at a \( \frac{1}{2}(1,1,1) \)-singularity or the weighted blow up at a QODP with weight \((\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)\) (we use the coordinate as stated in the definition of QODP). In any case \( X \) is a Q-Fano 3-fold with \( I(X) = 2 \). We can easily check that \((-K_X)^3 = 4 \) and \( aw(X) = 4 \). Furthermore by this, \( F(X) \) must be \( \frac{1}{2} \). So \( X \) is what we want.

[2].
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Step 1 for [2]. The Grassmannian $G(2, 5)$ (parameterizing 2-dimensional subspaces of 5-dimensional vector space) can be embedded into $\mathbb{P}^9$ by the Plücker embedding. Its defining equations are $x_{ij} x_{kl} - x_{ik} x_{jl} + x_{jk} x_{il} = 0$ for all $1 \leq i < j < k < l \leq 5$, where $x_{pq}$ $(1 \leq p < q \leq 5)$ is a Plücker coordinate. Let $Q$ be the point defined by $x_{pq} = 0$ for any $(p, q) \neq (1, 2)$. Let $l_1$ (resp. $l_2$) be the line $\subset G(2, 5)$ defined by $x_{pq} = 0$ for any $(p, q) \neq (1, 2),(1, 3)$ (resp. $(p, q) \neq (1, 2),(2, 4)$). Let $l_3$ be the line $\subset G(2, 5)$ defined by the equations $x_{pq} = r_{pq} x_{12}$ for $(p, q) \neq (1, 2)$ such that $r_{34} = r_{35} = r_{45} = 0$, $r_{13} r_{24} - r_{23} r_{14} = 0$, $r_{13} r_{25} - r_{23} r_{15} = 0$, $r_{14} r_{25} - r_{24} r_{15} = 0$ and $r_{15} r_{25} \neq 0$. Let $H$ be the 3-plane spanned by $l_1$, $l_2$ and $l_3$. Then $G(2, 5) \cap H = l_1 \cup l_2 \cup l_3$. Hence by [MM3, Proposition 6.8], there are two hyperplane $H_1$, $H_2$ and a quadric $Q$ such that $X' := G(2, 5) \cap H_1 \cap H_2 \cap Q$ is smooth and $X'$ contains $l_1$, $l_2$ and $l_3$. Since the tangent space of $X'$ at $Q$ also contains all the lines on $X'$ through $Q$, it is equal to $H$. Hence there are only three lines on $X'$ through $Q$.

Step 2 for [2]. Let $f' : Y' \rightarrow X'$ be the blow up at $Q$ and $E'$ the exceptional divisor. Let $l_1'$, $l_2'$ and $l_3'$ be the transforms of $l_1$, $l_2$ and $l_3$ on $Y'$. Since $B_s(-K_{Y'}) = l_1' \cup l_2' \cup l_3'$, the rank of the natural map $H^0(-K_{Y'}) \rightarrow H^0(O(-K_{Y'}, E'))$ is $3$. Hence there is a unique member $E$ of [ $-K_{Y'} - E' |$ since $h^0(-K_{Y'}) = 4$.

Step 3 for [2]. Since $-K_{Y'} + E'$ is free and $-K_{Y'} + E'$ is numerically trivial only for $l_1'$, $l_2'$ and $l_3'$ and positive for a curve in $E'$, they are numerically equivalent and span an extremal ray $R$ of $\mathcal{NE}(Y')$. Since $B_s(-K_{Y'}) = l_1' \cup l_2' \cup l_3'$ and $-K_{Y'}, l_1' < 0$, $\text{Supp } R = l_1' \cup l_2' \cup l_3'$. Furthermore by $B_s(-K_{Y'}) = l_1' \cup l_2' \cup l_3'$ again, there is a smooth anti-canonical divisor $D$ ([MM3, Proposition 6.8]). Hence the contraction of $l_1'$, $l_2'$ and $l_3'$ is a log flopping contraction for the pair $(Y', D)$ and the log flop exists. Let $Y' \dashrightarrow Y_0'$ be the log flop. Since $D.l_i' = -1$, the normal bundle of $l_i'$ is of type $(-1,-2)$. Hence $Y_0'$ has three $\frac{1}{3}(1,1,1)$-singularities. Since $-K_{Y_0'}$ is nef and big, we can construct a similar diagram $Y_0' \rightarrow Y_1' \rightarrow \cdots Y_i' \rightarrow \cdots Y_i+1' \rightarrow Y := Y_i' \xrightarrow{f} X$ to Lemma 3.2 by considering extremal rays, where $Y_i' \rightarrow Y_{i+1}'$ is a flop or a flip for $i = 0$ and a flip if $i \geq 1$. Let $E_i$ be the strict transform of $E$ on $Y_i'$.

Similarly to Step 3 for [1], we can see that $f$ is the blow up at a $\frac{1}{3}(1,1,1)$-singularity or the weighted blow up at a QODP with weight $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. In any case $X$ is a Q-Fano 3-fold with $I(X) = 2$. Since $(-K_X)^3 = 4$ and $N = 4$, $F(X)$ must be $\frac{1}{2}$. So $X$ is what we want.

APPENDIX

In this appendix, we give the table of a (2, 0)-type contraction from a 3-fold with only index 2 terminal singularities.

Proposition. Let $X$ be a 3-fold with only index 2 terminal singularities and $f : X \rightarrow (Y, Q)$ a contraction of (2, 0)-type to a germ $(Y, Q)$ which contracts a prime divisor $E$ to $Q$. Then the following holds:

(1) Assume that $E$ contains no index 2 point. Then one of the following holds:

$$(2, 0)_1 : (E, -E|_E) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \text{ and } Q \text{ is a smooth point};$$

$$(2, 0)_2 : (E, -E|_E) \simeq (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1)) \text{ and } (Y, Q) \simeq ((xy+zw = 0) \subset \mathbb{C}^4, o);$$
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\((2,0)_3 : (E, -E|_E) \simeq (\mathbb{F}_{2,0}, \mathcal{O}_{\mathbb{F}_2^2}(1)|_{\mathbb{F}_{2,0}})\) and \((Y, Q) \simeq (((xy+z^2+w^k = 0) \subset \mathbb{C}^4), o)(k \geq 3)\);

\[(2,0)_4 : (E, -E|_E) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))\) and \(Q\) is a \(\frac{1}{2}(1,1,1)\)-singularity.

Furthermore for all cases, \(f\) is the blow up of \(Q\).

(2) Assume that \(E\) contains an index 2 point. Then one of the following holds:

\[(2,0)_5 : (E, -E|_E) \simeq (\mathbb{F}_{2,0}, l)\), where \(l\) is a ruling of \(\mathbb{F}_{2,0}\).

\(Q\) is a smooth point and \(f\) is a weighted blow up with weight \((2,1,1)\).

In particular we have \(K_X = f^*K_Y + 3E]\;

\[(2,0)_6 : K_X = f^*K_Y + E\) and \(Q\) is a Gorenstein singular point. \(E^3 = \frac{1}{2}\);

\[(2,0)_7 : K_X = f^*K_Y + E\) and \(Q\) is a Gorenstein singular point. \(E^3 = 1\);

\[(2,0)_8 : K_X = f^*K_Y + E\) and \(Q\) is a Gorenstein singular point. \(E^3 = \frac{3}{2}\);

\[(2,0)_9 : K_X = f^*K_Y + E\) and \(Q\) is a Gorenstein singular point. \(E^3 = 2\);

\[(2,0)_{10} : (E, -E|_E) \simeq (((xy + z^2 = 0) \subset \mathbb{F}(1,1,2,1)), \mathcal{O}(2)).\]

\((Y, Q) \simeq (((xy + z^2 + w^k = 0) \subset \mathbb{C}^4/\mathbb{Z}_2(1,1,0,1)), o).\)

\(f\) is a weighted blow up with a weight \(\left(\frac{1}{2},\frac{1}{2},1,\frac{1}{2}\right)\).

In particular we have \(K_X = f^*K_Y + \frac{1}{2}E\);

\[(2,0)_{11} : (E, -E|_E) \simeq (\mathbb{F}_{2,0}, 3l).\)

\(Q\) is a \(\frac{1}{3}(2,1,1)\)-singularity and \(f\) is a weighted blow up with a weight \(\frac{1}{3}(2,1,1)\).

In particular we have \(K_X = f^*K_Y + \frac{1}{3}E\);
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[T3] ———, a private letter to the author.

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