<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>On Classification of Q-Fano 3-Folds of Gorenstein Index 2 and Fano Index 1/2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Takagi, Hiromichi</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>代数幾何学シンポジューム記録 1999: 8-20</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1999</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="http://hdl.handle.net/2433/214710">http://hdl.handle.net/2433/214710</a></td>
</tr>
<tr>
<td><strong>Type</strong></td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td><strong>Textversion</strong></td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
ON CLASSIFICATION OF $\mathbb{Q}$-FANO 3-FOLDS OF GORENSTEIN INDEX 2 AND FANO INDEX $\frac{1}{2}$

HIROMICHI TAKAGI

Notation and Conventions.

\begin{itemize}
  \item $\sim$ linear equivalence
  \item $\equiv$ numerical equivalence
  \item ODP ordinary double point, i.e., singularity analytically isomorphic to \{xy + z^2 + u^2 = 0 $\subset$ $\mathbb{C}^4$\}
  \item QODP singularity analytically isomorphic to \{xy+z^2+u^2 = 0 $\subset$ $\mathbb{C}^4/\mathbb{Z}_2(1,1,1,0)$\}
  \item $F_n$ Hirzebruch surface of degree $n$
  \item $F_{n,0}$ surface which is obtained by the contraction of the negative section of $F_n$
  \item $Q_3$ smooth 3-dimensional quadric.
  \item $B_i$ (1 $\leq$ $i$ $\leq$ 5) $\mathbb{Q}$-factorial Gorenstein terminal Fano 3-fold of Fano index 2, and with Picard number 1 and $(-K)^3 = 8i$, where $K$ is the canonical divisor
  \item $A_{2i}$ (1 $\leq$ $i$ $\leq$ 11 and $i \neq 10$) $\mathbb{Q}$-factorial Gorenstein terminal Fano 3-fold of Fano index 1, and with Picard number 1 and $(-K)^3 = 2i$
  \item contraction of $(m,n)$-type extremal contraction whose exceptional locus has dimension $m$ and the image of the exceptional locus has dimension $n$
\end{itemize}

0. INTRODUCTION

In this article, we will work over $\mathbb{C}$, the complex number field.

Definition 0.0 (Q-Fano variety). Let $X$ be a normal projective variety. We say that $X$ is a $\mathbb{Q}$-Fano variety (resp. weak $\mathbb{Q}$-Fano variety) if $X$ has only terminal singularities and $-K_X$ is ample (resp. nef and big).

Let $I(X) := \min\{I|IK_X$ is a Cartier divisor$\}$ and we call $I(X)$ the Gorenstein index of $X$.

Write $I(X)(-K_X) \equiv r(X)H(X)$, where $H(X)$ is a primitive Cartier divisor and $r(X) \in \mathbb{N}$. (Note that $H(X)$ is unique since Pic$X$ is torsion free.) Then we call $r(X)$ the Fano index of $X$ and denote it by $F(X)$.

Remark 0.1.

(1) We can allow that a $\mathbb{Q}$-Fano variety or a weak $\mathbb{Q}$-Fano variety has worse singularities than terminal. When we have to treat such a variety in this paper, we indicate singularities which we allow, e.g., 'a $\mathbb{Q}$-Fano 3-fold with only canonical singularities';

(2) if $X$ is Gorenstein in Definition 0.0, we say that $X$ is a Fano variety (resp. a weak Fano variety).
HIROMICHI TAKAGI

For the classification theory of varieties, a $Q$-factorial $Q$-Fano variety with Picard number 1 is important because it is an output of the minimal model program. Here we mention the known result about the classification of $Q$-Fano 3-folds:

1. G. Fano started the classification of smooth Fano 3-folds and it was completed by V. A. Iskovskih [I1] $\sim$ [I4], V. V. Shokurov [Sh1], [Sh2], T. Fujita [Fu1] $\sim$ [Fu3], S. Mori and S. Mukai [MM1] $\sim$ [MM3];
2. S. Mukai [Mu] classified indecomposable Gorenstein Fano 3-folds with canonical singularities by using vector bundles;
3. T. Sano [San1] and independently F. Campana and H. Flenner [CF] classified non Gorenstein Fano 3-folds of Fano indices $> 1$;
4. T. Sano [San2] classified non Gorenstein Fano 3-folds of Fano indices 1 and with only cyclic quotient terminal singularities. Recently T. Minagawa [Mi1] proved that non Gorenstein $Q$-Fano 3-folds with Fano indices 1 can be deformed to one with only cyclic quotient terminal singularities;
5. A. R. Fletcher [Fl] gave the classification of $Q$-Fano 3-folds which are weighted complete intersections of codimension 1 or 2. Recently S. Altinok [Al] (see also [RM2]) obtained a list of $Q$-Fano 3-folds which are subvarieties in a weighted projective space of codimension 3 or 4.

On the other hand K. Takeuchi [T1] simplified and amplified V. A. Iskovskih’s method of classification by using the theory of the extremal ray. In particular he reproved the Shokurov’s theorem [Sh2], the existence of lines on a smooth Fano 3-fold of Fano index 1 and with Picard number 1 by simple numerical calculations.

We formulate a slight generalization of Takeuchi’s construction for a $Q$-factorial $Q$-Fano 3-fold $X$ with $\rho(X) = 1$ and give a classification of a $Q$-factorial $Q$-Fano 3-fold with the following properties:

**Main Assumption 0.2.**

1. $\rho(X) = 1$;
2. $I(X) = 2$;
3. $F(X) = \frac{1}{2}$;
4. $h^0(-K_X) \geq 4$;
5. there exists an index 2 point $P$ such that

$$(X, P) \simeq (\{xy + z^2 + u^a = 0\}/\mathbb{Z}_2(1, 1, 1, 0), o)$$

for some $a \in \mathbb{N}$.

**Takeuchi’s construction 0.3.** Here we explain a slight generalization of Takeuchi’s construction. Let $X$ be a $Q$-factorial $Q$-Fano 3-fold with $\rho(X) = 1$. Suppose that we are given a birational morphism $f : Y \to X$ with the following properties:

1. $Y$ is a weak $Q$-Fano 3-fold;
2. $f$ is an extremal divisorial contraction such that $f$-exceptional locus $E$ is a prime $Q$-Cartier divisor.

Then we obtain the following diagram:

$$Y_0 := Y \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \cdots \xrightarrow{g_k} Y_k \xrightarrow{f'} X'$$

$$X$$
Q-FANO 3-FOLDS

where

1. \( Y_0 \rightarrow Y_1 \) is a flop or a flip and \( Y_i \rightarrow Y_{i+1} \) is a flip for \( i \geq 1 \);
2. \( f' \) is a crepant divisorial contraction (in this case, \( i = 0 \)) or an extremal contraction which is not isomorphic in codimension 1.

We use the following notation:

\( Y' := Y_k \);

\( E_i := \) the strict transform of \( E \) on \( Y_i \);

\( \hat{E} := \) the strict transform of \( E \) on \( Y' \);

\( e := E^3 - E_1^3 \) if \( Y_0 \rightarrow Y_1 \) is a flop or \( := 0 \) otherwise;

\( d_i := (-K_{Y_i})^3 - (-K_{Y_{i+1}})^3 \) (resp. \( a_i := \frac{E \cdot l_i}{(-K_{Y_i})^3} \)) if \( Y_i \rightarrow Y_{i+1} \) is a flip, where \( l_i \) is a flipping curve, or \( := 0 \) (resp. \( := 0 \)) otherwise;

\( z \) and \( u \) is defined as follows:

If \( f' \) is birational, then let \( E' \) be the exceptional divisor of \( f' \) and set \( E' = z(-K_{Y'}) - u\hat{E} \) or if \( f' \) is not birational, then let \( L \) be the pull back of an ample generator of \( \text{Pic} X' \) and set \( L = z(-K_{Y'}) - u\hat{E} \).

We note the following:

\[ (-K_{Y'})^2 \hat{E} = (-K_Y)^2 E - \sum a_i d_i; \]

\[ (-K_{Y'})\hat{E}^2 = (-K_Y)E^2 - \sum a_i^2 d_i; \]

\[ \hat{E}^3 = E^3 - e - \sum a_i^3 d_i; \]

(2) On the other hand the value or the relation of the value (expressed with \( z \) and \( u \)) of \( (-K_{Y'})^3 \), \( (-K_{Y'})^2 \hat{E} \), \( (-K_{Y'})\hat{E}^2 \) and \( \hat{E}^3 \) are restricted by the properties of \( f' \).

By these (1) and (2), we obtain equations of Diophantine type.

Under Main Assumption 0.2, Construction 0.3 works for a suitable choice of \( f \) and we can solve the equations as noted above.

**Main Theorem.** Let \( X \) be as in Main Assumption 0.2. Let \( f : Y \rightarrow X \) be the weighted blow up at \( P \) with weight \( \frac{1}{2}(1, 1, 1, 2) \). Then \( Y \) is a weak Q-Fano 3-fold.

Consider the diagram as in 0.3. Let \( h := h^\iota(-K_X) \), \( N := \text{aw}(X) \) and \( n := \sum \text{aw}(Y_i, P_{ij}) \) (the summation is taken over the index 2 points on flipping curves), where \( \text{aw}(X) \) is the number of \( \frac{1}{2}(1, 1, 1) \)-singularities which we obtain by deforming non Gorenstein points of \( X \) locally and \( \text{aw}(Y_i, P_{ij}) \) is defined similarly. Then we can solve the equations above and obtain a geographic classification of \( X \) as below (? in the table means that we don't know the existence of an example) :

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Main Assumption} & \text{Construction} & \text{Main Theorem} & \text{Geographic Classification} \\
\hline
0.2 & 0.3 & & \\
\hline
\end{array}
\]
HIROMICHI TAKAGI

\[ \begin{array}{|c|c|c|c|c|c|c|} \hline
(-K_X)^3 & N & e & n & z & (-K_Y \cdot C) & f', X' \\
\hline
\frac{3}{2} & 1 & 15 & 0 & 1 & / & (2,0)_{14}, (-K_X)^3 = \frac{3}{2}, I(X') = 2 \\
\frac{3}{2} & 1 & / & / & 1 & / & \text{crep. div.}, (-K_X)^3 = 2, I(X') = 1 \\
3 & 2 & 12 & 0 & 1 & / & (2,0)_{8}, A_4 \\
\frac{7}{2} & 3 & 10 & 0 & 1 & 1 & (2,1), A_6 \\
4 & 4 & 8 & 0 & 1 & 2 & (2,1), A_8 \\
4 & 4 & 9 & 3 & 1 & / & (2,0)_{11}, A_{10} \\
\frac{9}{2} & 5 & 6 & 0 & 1 & 3 & (2,1), A_{10} \\
\frac{9}{2} & 5 & 8 & 3 & 1 & / & (2,0)_{15}, A_{16} \\
\frac{9}{2} & 5 & 9 & 0 & 2 & / & (3,1), \deg F = 6 \\
\frac{7}{2} & 6 & 4 & 0 & 1 & 4 & (2,1), A_{12} \\
\hline
\end{array} \]

\[ z = u \text{ if } f' \text{ is not a crepant divisorial contraction.} \]

\[ u = 2 \text{ if } f' \text{ is a crepant divisorial contraction.} \]

\[ F := \text{a general fiber of } f' \text{ if } f' \text{ is (3,1)-type.} \]

See Appendix for \((2,0)_{14}\.\]

\[ g(C) = 0 \text{ in case } f' \text{ is of type } E_1 \text{ and every singularity of } Y \text{ is a } 1/2(1,1,1)-\text{singularity.} \]

\[ \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline
(-K_X)^3 & h = 5 & N & e & n & z & \deg \Delta & \deg F & f', X' \\
\hline
\frac{3}{2} & 1 & 9 & 0 & 1 & / & 3 & (3,1) \\
5 & 2 & 8 & 1 & 1 & / & 4 & (3,1) \\
?\frac{11}{2} & 3 & 7 & 2 & 1 & / & 5 & (3,1) \\
?\frac{11}{2} & 3 & 8 & 0 & 2 & 8 & / & (3,2), F_2,0 \\
?6 & 4 & 7 & 1 & 2 & 6 & / & (3,2), F_2,0 \\
?6 & 4 & 6 & 3 & 1 & / & 6 & (3,1) \\
?\frac{13}{2} & 5 & 6 & 2 & 2 & 4 & / & (3,2), F_2,0 \\
\hline
\end{array} \]

\[ z = u. \]

\[ \Delta := \text{the discriminant divisor of } f' \text{ if } f' \text{ is (3,2)-type.} \]

\[ F := \text{a general fiber of } f' \text{ if } f' \text{ is (3,1)-type.} \]

\[ \begin{array}{|c|c|c|c|c|c|c|c|} \hline
(-K_X)^3 & h = 6 & N & e & n & z & \deg \Delta & (-K_Y \cdot C) & f', X' \\
\hline
\frac{13}{2} & 1 & 7 & 0 & 1 & 7 & / & (3,2), F^2_2 \\
\frac{7}{2} & 2 & 7 & 0 & 4 & / & 35 & (2,1), [5] \\
\frac{7}{2} & 2 & 6 & 1 & 1 & 6 & / & (3,2), F^2_2 \\
\frac{13}{2} & 3 & 7 & 0 & 2 & / & 9 & (2,1), [2], I(X') = 2 \\
\frac{13}{2} & 3 & 6 & 1 & 4 & / & 30 & (2,1), [5] \\
?\frac{13}{2} & 3 & 5 & 2 & 1 & 5 & / & (3,2), F^2_2 \\
?8 & 4 & 4 & 3 & 1 & 4 & / & (3,2), F^2_2 \\
?\frac{13}{2} & 5 & 3 & 4 & 1 & 3 & / & (3,2), F^2_2 \\
\hline
\end{array} \]
Q-FANO 3-FOLDS

Type [i] means the Q-Fano 3-fold of type [i] which was classified by T. Sano in [San2].

<table>
<thead>
<tr>
<th>$(-K_X)^3$</th>
<th>$N$</th>
<th>$e$</th>
<th>$n$</th>
<th>$z$</th>
<th>$(-K_{Y'.C})$</th>
<th>$f', X'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{h}{2}$</td>
<td>1</td>
<td>6</td>
<td>0</td>
<td>3</td>
<td>36</td>
<td>$(2, 1), P^3$</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>6</td>
<td>0</td>
<td>2</td>
<td>18</td>
<td>$(2, 1), [3]$</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>32</td>
<td>$(2, 1), P^3$</td>
</tr>
<tr>
<td>$\frac{19}{2}$</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td>15</td>
<td>$(2, 1), [3]$</td>
</tr>
<tr>
<td>$\frac{19}{2}$</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>28</td>
<td>$(2, 1), P^3$</td>
</tr>
</tbody>
</table>

Type [i] means the Q-Fano 3-fold of type [i] which was classified by T. Sano in [San2].

$u = z + 1.$

<table>
<thead>
<tr>
<th>$(-K_X)^3$</th>
<th>$N$</th>
<th>$e$</th>
<th>$n$</th>
<th>$z$</th>
<th>$(-K_{Y'.C})$</th>
<th>$f',X'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{21}{2}$</td>
<td>1</td>
<td>6</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>$(2, 1), B_3$</td>
</tr>
<tr>
<td>$\frac{21}{2}$</td>
<td>1</td>
<td>5</td>
<td>0</td>
<td>2</td>
<td>27</td>
<td>$(2, 1), Q_3$</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>24</td>
<td>$(2, 1), Q_3$</td>
</tr>
</tbody>
</table>

$u = z + 1.$

<table>
<thead>
<tr>
<th>$(-K_X)^3$</th>
<th>$N$</th>
<th>$e$</th>
<th>$n$</th>
<th>$z$</th>
<th>$u$</th>
<th>$(-K_{Y'.C})$</th>
<th>$f',X'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{25}{2}$</td>
<td>1</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>10</td>
<td>$(2, 1), B_4$</td>
</tr>
</tbody>
</table>

$z = 1$ and $u = 2.$

In particular we have $(-K_X)^3 \leq 15$ and $h^0(-K_X) \leq 10.$

Based on this result, we can derive the following properties for $X$ as in the main theorem:

**Theorem A.** If any index 2 point satisfies the assumption (5) of 0.2, then $| -K_X |$ has a member with only canonical singularities.

So the general elephant conjecture by M. Reid is affirmative for such an $X.$
Theorem B. Let $X$ be a Q-factorial Q-Fano 3-fold with (1) of 0.2. Let $N := \text{aw}(X)$. Then if $N > 1$ (resp. $N = 1$), $X$ can be transformed to a Q-factorial Q-Fano 3-fold $\tilde{Z}'$ with (1) of 0.2 and with only QODP's or $\frac{1}{2}(1,1,1)$-singularities as its singularities and $h^0(\mathcal{K}_{\tilde{Z}'}) = h$ and $\text{aw}(\tilde{Z}') = N - 1$ (resp. a smooth Fano 3-fold $\tilde{Z}'$ with $\rho(\tilde{Z}') = 1$, $F(\tilde{Z}') = 1$ and $h^0(\mathcal{K}_{\tilde{Z}'}) = h$) as follows:

$$\tilde{Y} \xrightarrow{\tilde{f}} \tilde{X} \xrightarrow{\tilde{g}} \tilde{Z} \xrightarrow{\text{def}} \tilde{Z'},$$

where $\star \xrightarrow{\text{def}} \star' \star$ means that $\star'$ is a small deformation of $\star$;

$\tilde{X}$ is a Q-Fano 3-fold as in 0.2 and with only QODP's or $\frac{1}{2}(1,1,1)$-singularities as its singularities;

$\tilde{f} : \tilde{Y} \to \tilde{X}$ is chosen as $f$ in the main theorem;

$\tilde{g} : \tilde{Y} \to \tilde{Z}$ be the anti-canonical model.

This is an analogue to the Reid's fantasy about Calabi-Yau 3-folds [RM1].

Theorem C. If any index 2 point is a $\frac{1}{2}(1,1,1)$-singularity, $X$ can be embedded into a weighted projective space $\mathbb{P}(h, 2N)$, where $h := h^0(\mathcal{K}_X)$ and $N$ is the number of $\frac{1}{2}(1,1,1)$-singularities on $X$.

We hope that this fact can be used for the classification of Mukai's type (see [Mu]).

1. Examples

We consider the case that $h^0(\mathcal{K}_X) = 4$ and $N = 4$. By the table of the main theorem, there are two possibilities of $X$ in this case. We assume that every singularity of $Y$ is a $\frac{1}{2}(1,1,1)$-singularity. Then one of the following holds:

[1]. $f'$ is an extremal divisorial contraction which contracts a divisor $E'$ to a curve $C$ and $| -K_Y - E'| \neq \emptyset$. $X'$ is a $(2,2,2)$-complete intersection in $\mathbb{P}^6$ and satisfies the following properties:

1. $X'$ is factorial;
2. $C$ is a smooth conic;
3. $X'$ has 3 singularities $P_0 \sim P_2$ on $C$ and $P_i$ is an ODP or the singularity analytically isomorphic to the origin of $\{xy + z^2 + w^3 = 0\} \subset \mathbb{C}^4$. Outside $P_i$'s, $X'$ is smooth.

[2]. $f'$ is blowing up at a smooth point $Q := f'(E')$ and $| -K_Y - E'| \neq \emptyset$. $X'$ is smooth, isomorphic to $A_{10}$ and there exist exactly three lines through the point $Q$.

We will construct examples for these cases by the following three steps:

Step 1. We construct $X'$ satisfying the properties as stated as in [1] or [2];

Step 2. We construct $f'$ satisfying the properties as stated as in [1] or [2];

Step 3. We construct $f : Y \to X$ as in the main theorem from $Y'$.

[1].

Step 1 for [1]. We construct $X'$ with only ODP's.
Claim 1. Let $V$ (resp. $X'$) be a $(2,2)$-complete intersection in $\mathbb{P}^6$ (resp. a quadric section of $V$) with the following properties:

1. $V$ (resp. $X'$) contains a smooth conic $C$;
2. $V$ (resp. $X'$) has three ODP's $P_0 \sim P_2$ on $C$ and outside $P_i$'s, $V$ (resp. $X'$) is smooth.

Then $X'$ is factorial.

Proof. We claim that $V$ contains the plane $P$ spanned by $C$. Let $\sigma$ be the pencil which consists of quadrics in $\mathbb{P}^6$ containing $V$. Since $P_i$ is an ODP on $V$, there is a quadric in $\sigma$ which is singular at $P_i$. If there is a quadric in $\sigma$ which is singular at all $P_i$'s, then it is singular on $P$ and hence $V$ is singular along $C$, a contradiction. So $\sigma$ is generated by two quadrics which are singular at some $P_i$. But such quadrics contains $P$ and hence $V$ contains $P$.

Let $\nu : \tilde{V} \rightarrow V$ be the composition of the blowing ups at $P_0 \sim P_2$ and $F_i$ the exceptional divisor over $P_i$. Let $\tilde{X}'$ be the strict transform of $X'$ on $V$ and $H$ the total transform of a hyperplane section of $V$. Then $\tilde{X}' \sim 2H - F_0 - F_1 - F_2$. Note that $|H - F_1 - F_2|$ is free outside the strict transform $l_{ij}$ of the line through $P_i$ and $P_j$ and $|H - F_k|$ is free (note that $l_{ij}$ is contained in $V$ since $l_{ij} \subset P$). By this, we can easily see that $|\tilde{X}'|$ is free and $\tilde{X}'$ is numerically trivial only for $l_{ij}$'s $((i,j) = (0,1), (1,2), (2,0))$.

Let $\phi$ be the morphism defined by $|\tilde{X}'|$. Then $\phi$-exceptional curves are $l_{ij}$'s. We will prove that $\text{Leff}(\tilde{V}, \tilde{X}')$ holds and $\tilde{X}'$ meets every effective divisor on $\tilde{V}$. By [H, p.165, Proposition 1.1] and the argument of [H, p.172, the proof of Theorem 1.5], it suffices to prove that $\text{cd}(\tilde{V} - \tilde{X}') < 3$, i.e., for any coherent sheaf $F$ on $\tilde{V} - \tilde{X}'$, $H^i(\tilde{V} - \tilde{X}', F) = 0$ for all $i \geq 3$. Let $\tilde{V} := \phi(\tilde{V})$ and $\tilde{X}' := \phi(\tilde{X}')$. Consider the Leray spectral sequence

$$E^{p,q}_2 = H^p(\tilde{V} - \tilde{X}', R^q\phi'_*F) \Rightarrow E^{p+q} = H^{p+q}(\tilde{V} - \tilde{X}', F),$$

where $\phi' := \phi|_{\tilde{V} - \tilde{X}'}$. Since $\tilde{V} - \tilde{X}'$ is affine and the dimension of every fiber of $\phi$ $\leq 1$, we have $E^{p,q}_2 = 0$ for $p \geq 1$ or $q \geq 2$ whence $E^{p+q} = 0$ for $p + q \geq 2$. So the assertion follows.

Furthermore since $\tilde{X}'$ is nef and big, $H^i(\tilde{V}, O(-n\tilde{X}')) = 0$ for $n \geq 1$ and $i = 1, 2$ by KKV vanishing theorem. Hence by the Grothendieck-Lefschetz theorem [G, p.135, 3.18] (or [H, p.178, Theorem 3.1]), we have $\text{Pic}\tilde{X}' \simeq \text{Pic}\tilde{V} \simeq \mathbb{Z}^4$. So $\rho(\tilde{X}'/X') = 3$ which imply that $X'$ is factorial.

We will give a pair $(V, X')$ satisfying the condition of Claim 1. Let $C$ be a smooth conic in $\mathbb{P}^6$ and $P_0 \sim P_2$ three points on $C$. We can choose a coordinate of $\mathbb{P}^6$ such that $C = \{x_0x_1 + x_1x_2 + x_2x_0 = x_3 = x_4 = x_5 = x_6 = 0\}$ and $P_i = \{x_j = 0\}$ for $j \neq i$.

Claim 2. Let $X'$ be a $(2,2,2)$-complete intersection in $\mathbb{P}^6$ satisfying the following conditions:

1. $X'$ is factorial;
2. $X'$ contains a smooth conic $C$;
3. $X'$ has three ODP's $P_0 \sim P_2$ on $C$ and outside $P_i$'s, $X'$ is smooth.

Then $X'$ is the intersection of three quadrics $Q_1 \sim Q_3$ of the following forms by permuting $P_i$'s if necessary:
HIROMICHI TAKAGI

\[
Q_1 := \{m_0x_0 + m_1x_1 + q_1 = 0\};
\]
\[
Q_2 := \{pm_1x_1 + m_2x_2 + q_2 = 0\};
\]
\[
Q_3 := \{x_0x_1 + x_1x_2 + \sum_{i=3}^{6} l_ix_i = 0\},
\]

where \( p \in \mathbb{C}, \) \( m_i \) (resp. \( q_i \)) is a linear form (resp. a quadratic form) of \( x_3 \sim x_6 \)
and \( l_i \) is a linear form of \( x_0 \sim x_6 \).

Conversely if \( X' = Q_1 \cap Q_2 \cap Q_3 \), where \( Q_i \) is of the form as above and \( m_i, q_i \)
and \( l_i \) are suitably general, then \( X' \) satisfies (1) \sim (3).

**Proof.** Let \( \gamma \) be the net which consists of quadrics containing \( X' \). \( \gamma \) contains a
member \( Q_1 \) which is singular at \( P_2 \). Then \( Q_1 \) is of the form as above. If \( m_1 = m_2 = 0 \), then \( Q_1 \) is singular on the plane \( P \) spanned by \( C \) and hence \( X' \) is singular along \( C \), a contradiction. Hence \( m_1 \neq 0 \) or \( m_2 \neq 0 \). By permuting \( P_1 \) and \( P_2 \) if necessary, we may assume that \( m_1 \neq 0 \). \( \gamma \) contains a member \( Q_2 \) which is singular
at \( P_0 \). \( Q_2 \) is of the form as

\[
\{m_1'x_1 + m_2x_2 + q_2 = 0\},
\]

where \( m_1' \) and \( m_2 \) (resp. \( q_2 \)) are linear forms (resp. a quadratic form) of \( x_3 \sim x_6 \).
\( \gamma \) also contains a member \( Q' \) which is singular at \( P_1 \). If \( Q_1, Q_2 \) and \( Q' \) generate \( \gamma \), then \( X' \) contains the plane \( P \), a contradiction to the factoriality and \( F(X') = 1 \).

Hence \( Q' \) is contained in the pencil generated by \( Q_1 \) and \( Q_2 \). So \( m_1' = pm_1 \) for
some \( p \in \mathbb{C} \) and

\[
Q = \{-pm_0x_0 + m_2x_2 + (q_2 - pq_1) = 0\}.
\]

Since \( X' \) does not contain \( P \) as noted above, \( \gamma \) contains a member \( Q_3 \) of the form
as in the statement. \( Q_3 \) is not contained in the pencil generated by \( Q_1 \) and \( Q_2 \) and hence \( Q_i \)'s generate \( \gamma \).

Conversely let \( X' := Q_1 \cap Q_2 \cap Q_3 \), where \( Q_i \) is of the form as above and \( m_i, q_i \)
and \( l_i \) are suitably general. We can easily check that \( X' \) satisfies (2) and (3). Set \( V := Q_1 \cap Q_2 \). We may assume that \( V \) satisfies the condition of Claim 1. Hence by
Claim 1, \( X' \) is factorial. \( \square \)

**Step 2 for [1].** Let \( \nu' : \tilde{X}' \to X' \) be the composition of the blowing ups at
\( P_0 \sim P_{N-2} \) and \( F_i' \) the exceptional divisor over \( P_i \). Let \( \mu' : \tilde{X}' \to \tilde{X} \) be the
blowing up along the strict transform \( \tilde{C} \) of \( C \) and \( F' \) the \( \mu' \)-exceptional divisor. We will
denote the strict transforms of the two fibers of \( F_i \simeq \mathbb{P}^1 \times \mathbb{P}^1 \) through \( F_i \cap \tilde{C} \)
by \( l_{ij} \) (\( i = 1, 2 \)). Note that \(-K_{\tilde{X}}, l_{ij} = 0 \). We can easily see that \(|-K_{\tilde{X}}| \) is free by
\( \mathcal{P} \cap X' = C \), where \( \mathcal{P} \) is the plane spanned by \( C \) and \(-K_{\tilde{X}} \) is big. Hence \( l_{ij} \)'s are
flopping curves on \( \tilde{X}' \) and we can see that the classes of \( l_{11} \) and \( l_{22} \) belong to the
same ray. Let \( \tilde{X}' \to \tilde{X}'^+ \) be the flop. Then the strict transforms of \( F_i \)'s on \( \tilde{X}'^+ \)
are \( \mathbb{P}^2 \)'s and we can contract them to \( \frac{1}{2}(1,1,1) \)-singularities. Let \( g' : \tilde{X}'^+ \to Y' \) be
the contraction morphism, \( f' : Y' \to X' \) the natural morphism and \( E' \) the strict transform of \( F' \).

We will see that \(|-K_{Y'}, -E'| \neq \phi \). Let \( F'^+ \) be the strict transform of \( F' \) on
\( \tilde{X}'^+ \). Then \(-K_{\tilde{X}', +} - F'^+ = g'^*(-K_{Y'}, -E') \). Furthermore \( h^0(-K_{\tilde{X}', +} - F'^+) = \)
h^0(-K_{X'}, -F')$. Hence it suffices to prove that $h^0(-K_{X'}, |F'|) \leq 3$ since $h^0(-K_{X'}) = 4$. Since there is a smooth member of $| -K_{X'}|$, we have $N_{C'/X'} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-2)$. Hence $F' \simeq \mathbb{F}_1$ and $-K_{X'}|_{F'} \simeq C_0 + l$, where $C_0$ is the minimal section of $F'$ and $l$ is a fiber of $F'$. So we are done.

**Step 3 for [1].** Since $Y'$ has only $\frac{1}{2}(1,1,1)$-singularities and $-K_{Y'}$ is nef and big, we can construct a similar diagram $Y_0' := Y' \rightarrow Y_1' \rightarrow Y_2' \rightarrow \cdots \rightarrow Y_i' \rightarrow Y := Y' \rightarrow X$ to 0.3 by considering extremal rays, where $Y_i' \rightarrow Y_{i+1}'$ is a flop or a flip for $i = 0$ and a flip for $i \geq 1$. Let $E_i$ (resp. $E_i$) be the strict transform of $E$ on $Y_i'$ (resp. $Y$). Let $R_i$ be the extremal ray which is other than the ray associated to $f'$ for $i = 0$ or the $K_Y$-negative extremal ray for $i \geq 1$. By similar calculations to 0.3, we have

\begin{align*}
(1) \quad (-K_{Y'})^2 E &= 1 + \sum a_i'd_i'; \\
(2) \quad (-K_Y)^2 E &= -2 - \sum a_i'^2 d_i'; \\
(3) \quad E_3 &= -6 + \sum a_i'^3 d_i' + e',
\end{align*}

where $e'$, $a_i'$ and $d_i'$ are similarly defined to 0.3 with respect to $-K_{Y'}$ and $E_i$ and furthermore we can see that $a_i'$ is a non negative integer.

**Claim 3.** $E_i . R_i < 0$.

**Proof.** We can prove the assertion by induction. For $i = 0$, $E_0 . R_0 < 0$ can be directly checked. Assume that the assertion holds for the numbers less than $i$. So the other extremal ray than $R_i$ is positive for $E_i$. Since $-K_{Y'}$ is free outside a finite number of curves, $-K_{Y'}|_{E_i}$ is numerically equivalent to an effective 1-cycle. Hence by $-K_{Y'} E_i^2 = -K_{Y'} E_2 = -2$, we have $E_i . R_i < 0$. □

By this claim, we know that $f$ is an divisorial contraction whose exceptional divisor is $E$. If $f$ is a crepant divisorial contraction, then $l = 0$. But $(-K_{Y'})^2 E = 1$, a contradiction. Hence $f$ is a $K_Y$-negative contraction. Assume that $f$ is (2,1)-type which contracts $E$ to a curve $C'$. Then $(-K_{X'} C') = (-K_Y + E) (-K_{Y'}) E = -1 - \sum d_i' a_i'(a_i' - 1) < 0$, a contradiction since $X$ is a Q-Fano 3-fold.

By the classification of a (2,0)-type contraction from a 3-fold with only index 2 terminal singularities (see Appendix), if $f$ is such an contraction, then we have $-K_Y E_2 \geq -2$. On the other hand $-K_Y E_2 \leq -K_{Y'} E_2 = -2$. Hence there is no flip. So $(-K_Y)^2 E = (-K_{Y'})^2 E = 1$ and hence again by the classification of a contraction as above, $f$ is the blow up at a $\frac{1}{2}(1,1,1)$-singularity or the weighted blow up at a QODP with weight $(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, 1)$ (we use the coordinate as stated in the definition of QODP). In any case $X$ is a Q-Fano 3-fold with $I(X) = 2$. We can easily check that $(-K_X)^3 = 4$ and $aw(X) = 4$. Furthermore by this, $F(X)$ must be $\frac{1}{2}$. So $X$ is what we want.

[2].
HIROMICHI TAKAGI

Step 1 for [2]. The Grassmannian $G(2, 5)$ (parameterizing 2-dimensional subspaces of 5-dimensional vector space) can be embedded into $\mathbb{P}^9$ by the Plücker embedding. Its defining equations are $x_{ij}x_{kl} - x_{ik}x_{jl} + x_{jk}x_{il} = 0$ for all $1 \leq i < j < k < l \leq 5$, where $x_{pq}$ $(1 \leq p < q \leq 5)$ is a Plücker coordinate. Let $Q$ be the point defined by $x_{pq} = 0$ for any $(p, q) \neq (1, 2)$. Let $l_1$ (resp. $l_2$) be the line $\in G(2, 5)$ defined by $x_{pq} = 0$ for any $(p, q) \neq (1, 2), (1, 3)$ (resp. $(p, q) \neq (1, 2), (2, 4)$). Let $l_3$ be the line $\in G(2, 5)$ defined by the equations $x_{pq} = r_{pq}x_{12}$ for $(p, q) \neq (1, 2)$ such that $r_{34} = r_{35} = r_{45} = 0$, $r_{13}r_{24} - r_{23}r_{14} = 0$, $r_{13}r_{25} - r_{23}r_{15} = 0$, $r_{14}r_{25} - r_{24}r_{15} = 0$ and $r_{15}r_{25} \neq 0$. Let $H$ be the 3-plane spanned by $l_1$, $l_2$ and $l_3$. Then $G(2, 5) \cap H = l_1 \cup l_2 \cup l_3$. Hence by [MM3, Proposition 6.8], there are two hyperplane $H_1$, $H_2$ and a quadric $Q$ such that $X' := G(2, 5) \cap H_1 \cap H_2 \cap Q$ is smooth and $X'$ contains $l_1$, $l_2$ and $l_3$. Since the tangent space of $X'$ at $Q$ also contains all the lines on $X'$ through $Q$, it is equal to $H$. Hence there are only three lines on $X'$ through $Q$.

Step 2 for [2]. Let $f' : Y' \rightarrow X'$ be the blow up at $Q$ and $E'$ the exceptional divisor. Let $l'_1$, $l'_2$ and $l'_3$ be the transforms of $l_1$, $l_2$ and $l_3$ on $Y'$. Since $Bs(-K_{Y'}) = l'_1 \cup l'_2 \cup l'_3$, the rank of the natural map $H^0(-K_{Y'}) \rightarrow H^0(O(-K_{Y'}|_{E'}))$ is 3. Hence there is a unique member $E$ of $-K_{Y'} - E'$ such that $h^0(-K_{Y'}) = 4$.

Step 3 for [2]. Since $-K_{Y'} + E'$ is free and $-K_{Y'} + E'$ is numerically trivial only for $l'_1$, $l'_2$ and $l'_3$ and positive for a curve in $E'$, they are numerically equivalent and span an extremal ray $R$ of $NE(Y')$. Since $Bs(-K_{Y'}) = l'_1 \cup l'_2 \cup l'_3$ and $-K_{Y'}|_{E'} < 0$, Supp $R = l'_1 \cup l'_2 \cup l'_3$. Furthermore by $Bs(-K_{Y'}) = l'_1 \cup l'_2 \cup l'_3$ again, there is a smooth anti-canonical divisor $D$ ([MM3, Proposition 6.8]). Hence the contraction of $l'_1$, $l'_2$ and $l'_3$ is a log flopping contraction for the pair $(Y', D)$ and the log flop exists. Let $Y' \dashrightarrow Y'_0$ be the log flop. Since $D.l'_i = -1$, the normal bundle of $l'_i$ is of type $(-1, -2)$. Hence $Y'_0$ has three $\frac{1}{2}(1, 1, 1)$-singularities. Since $-K_{Y'_0}$ is nef and big, we can construct a similar diagram $Y'_0 \dashrightarrow Y'_1 \dashrightarrow \cdots Y'_i \dashrightarrow Y'_{i+1} \cdots Y := Y'_i \dashrightarrow X$ to Lemma 3.2 by considering extremal rays, where $Y'_i \dashrightarrow Y'_{i+1}$ is a flop or a flip for $i = 0$ and a flip if $i \geq 1$. Let $E_i$ be the strict transform of $E$ on $Y'_i$.

Similarly to Step 3 for [1], we can see that $f$ is the blow up at a $\frac{1}{2}(1, 1, 1)$-singularity or the weighted blow up at a QODP with weight $(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1)$. In any case $X$ is a Q-Fano 3-fold with $I(X) = 2$. Since $(-K_X)^3 = 4$ and $N = 4$, $F(X)$ must be $\frac{3}{2}$. So $X$ is what we want.

APPENDIX

In this appendix, we give the table of a (2, 0)-type contraction from a 3-fold with only index 2 terminal singularities.

Proposition. Let $X$ be a 3-fold with only index 2 terminal singularities and $f : X \rightarrow (Y, Q)$ a contraction of (2, 0)-type to a germ $(Y, Q)$ which contracts a prime divisor $E$ to $Q$. Then the following holds:

1. Assume that $E$ contains no index 2 point. Then one of the following holds:

   \[(2, 0)_1 : (E, -E|_E) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \text{ and } Q \text{ is a smooth point} ;\]

   \[(2, 0)_2 : (E, -E|_E) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1)|_{\mathbb{P}^1 \times \mathbb{P}^1}) \text{ and } (Y, Q) \cong (xy + zw = 0) \subset \mathbb{C}^4, o);\]
Q-FANO 3-FOLDS

(2, 0)_3 : (E, -E|_E) \simeq (\mathbb{F}_{2,0}, \mathcal{O}_{\mathbb{P}^2}(1)|_{\mathbb{F}_{2,0}}) \text{ and } (Y, Q) \simeq (((xy + z^2 + w^k = 0) \subset \mathbb{C}^4), o)(k \geq 3);

(2, 0)_4 : (E, -E|_E) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) \text{ and } Q \text{ is a } \frac{1}{2}(1,1,1)-\text{singularity.}

Furthermore for all cases, $f$ is the blow up of $Q$.

(2) Assume that $E$ contains an index 2 point. Then one of the following holds:

(2, 0)_5 : (E, -E|_E) \simeq (\mathbb{F}_{2,0}, l), \text{ where } l \text{ is a ruling of } \mathbb{F}_{2,0}.

$Q$ is a smooth point and $f$ is a weighted blow up with weight $(2, 1, 1)$.

In particular we have $K_X = f^*K_Y + 3E$;

(2, 0)_6 : $K_X = f^*K_Y + E$ and $Q$ is a Gorenstein singular point. $E^3 = \frac{1}{2}$;

(2, 0)_7 : $K_X = f^*K_Y + E$ and $Q$ is a Gorenstein singular point. $E^3 = \frac{3}{2}$;

(2, 0)_8 : $K_X = f^*K_Y + E$ and $Q$ is a Gorenstein singular point. $E^3 = 1$;

(2, 0)_9 : $K_X = f^*K_Y + E$ and $Q$ is a Gorenstein singular point. $E^3 = 2$;

(2, 0)_{10} : (E, -E|_E) \simeq (((xy + w^2 = 0) \subset \mathbb{F}(1,1,2,1)), \mathcal{O}(2)).

(Y, Q) \simeq (((xy + z^k + w^3 = 0) \subset C^4/\mathbb{Z}_2(1,1,0,1)), o).

$f$ is a weighted blow up with a weight $\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}\right)$.

In particular we have $K_X = f^*K_Y + \frac{1}{2}E$;

(2, 0)_{11} : (E, -E|_E) \simeq (\mathbb{F}_{2,0}, 3l).

$Q$ is a $\frac{1}{3}(2,1,1)$-singularity and $f$ is a weighted blow up with a weight $\frac{1}{3}(2,1,1)$.

In particular we have $K_X = f^*K_Y + \frac{1}{3}E$;
HIROMICHI TAKAGI

References


[MM3] ———, *Classification of Fano 3-folds with $B_2 \geq 2$, I*, to the memory of Dr. Takehiko MIYATA, Algebraic and Topological Theories, 1985, pp. 496–545.


Q-FANO 3-FOLDS

[T3] _____, a private letter to the author.

RIMS, KYOTO UNIVERSITY, KITASHIRAKAWA, SAKYO-KU, 606-8502 KYOTO, JAPAN
E-mail address: takagi@kurims.kyoto-u.ac.jp