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端末特異点に関係した基本群について
— LOCAL SIMPLE CONNECTEDNESS OF RESOLUTIONS OF LOG-TERMINAL SINGULARITIES —

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1. Introduction

Here I would like to explain an analytic method to study fundamental groups related with Fano varieties, log-terminal singularities (over $\mathbb{C}$). A pair (X, Δ) of a normal variety and an effective $\mathbb{Q}$-divisor is said to be *Kawamata log-terminal* (*KLT* for short), respectively *log-canonical* (*LC* for short), if the following conditions are satisfied: (i) $K_X + \Delta$ is $\mathbb{Q}$-Cartier; (ii) There exists a projective birational morphism $\mu : Y \longrightarrow X$ from a smooth variety Y with a normal crossing divisor $\sum E_i$ such that $K_Y \sim_{\mathbb{Q}} \mu^*(K_X + \Delta) + \sum e_i E_i$ holds with $e_i > -1$, respectively $e_i \geq -1$, for all i, where "$\sim_{\mathbb{Q}}$" denote the $\mathbb{Q}$-linear equivalence. Our result is as follows.

Theorem 1.1. *Let X be a normal variety and let $\mu : Y \longrightarrow X$ be a resolution of singularities. Then the induced homomorphism of fundamental groups $\mu_* : \pi_1(Y) \longrightarrow \pi_1(X)$ is an isomorphism if (X, Δ) is KLT for some Δ.*

Theorem 1.2. *Let $f : X \longrightarrow S$ be a proper surjective morphism of normal varieties with connected fibres. Assume that there exists an effective $\mathbb{Q}$-divisor Δ such that the pair (X, Δ) is KLT and that $-(K_X + \Delta)$ is f-nef and f-big. Then $f_* : \pi_1(X) \longrightarrow \pi_1(S)$ is an isomorphism.*

Corollary 1.3. *Every $\mathbb{Q}$-Fano variety is simply connected.*

As consequences of these theorems, we can see that the fundamental group is preserved by contractions of extremal rays, flips, pluricanonical morphisms of minimal varieties of general type (see [KMM] for terminologies).

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Kollár [Kol, §7] proved the statement for algebraic fundamental groups: $\widehat{\pi}_1$ and some special cases of 1.1, and conjectured 1.1.

Remark 1.4. A KLT singularity is a rational singularity [KMM, §1-3], however rational sigularities do not have a property as in Theorem 1.1. For example, we let S be a fake projective plane (Mumford, Ishida-Kato, ...), i.e., $p_g = q = 0$, $c_1^2 = 3c_2 = 9$, K_S ample. By Yau, S is a ball quotient. By a general theory of surfaces of general type, we have an embedding by $|3K_S| : S \hookrightarrow \mathbb{P}^{27}$ with degree 81. We let X be a cone over S in $\mathbb{P}^{27}$. Since $p_g(S) = q(S) = 0$, X has a rational singularity. Since X is a cone, we see $\pi_1(X) = 1$. By blowing-up at the vertex, we have a resolution $f : Y \longrightarrow X$. Since Y has a $\mathbb{P}^1$-bundle structure over S, we see $\pi_1(Y) = \pi_1(S)$. Thus the kernel of $\mu_* : \pi_1(Y) \longrightarrow \pi_1(X)$ is quite large.

2. Motivation and some background

My first result in this direction was Corollay 1.3 which was motivated by the following

Theorem 2.1. (*S. Kobayashi* [Kob]) *Let (X, ω) be a compact Kähler manifold with positive Ricci curvature, i.e., X is Fano; $-K_X$ is ample. Then X is simply connected.*

I wanted to generalize Theorem 2.1 for singular Fano varieties in view of the classification theory of algebraic varieties. One of the striking result of the study of Fano manifolds is its rational connectedness due to Kollár-Miyaoka-Mori [KMMo1]. For Fano varieties, one also expects such a property. Our Corollary 1.3 is a small evidence for this, which I mean Corollary 1.3 fit in the following diagram:

$$\begin{array}{ccccc} & & \text{“Fano varieties”} & & \\ & \text{Miyaoka-Mori} \swarrow & \downarrow ? & \searrow \text{Cor. 1.3} & \\ \text{uniruled} & \longleftarrow & \begin{array}{c}\text{rationally (chain)}\\ \text{connectedness}\end{array} & \overset{\mathbb{Q}\text{-Fano}}{\longrightarrow} & \pi_1 = 1 \end{array}$$

The middle downarrow is unknown. Furthermore

Question 2.2. Is every fiber of morphisms in Theorem 1.1 and 1.2 rationally (chain) connected ?

Theorem 1.1 and 1.2 are also supporting evidences that the question is affirmative.

This is just a background, we return to our problem. We next explain the methods of proof of Theorem 2.1 and some notions which we will use later in our proof.

Method of Proof (Theorem 2.1). Let (X, ω) be a compact Kähler manifold with positive Ricci curvature; X is Fano, and let $\pi : \widetilde{X} \longrightarrow X$ be the universal cover.

- $\sharp\pi_1(X) < +\infty \Longrightarrow \pi_1(X) = 1$

  This follows from quite standard argument

$$\begin{array}{ccll} \chi(X,\mathcal{O}) & = & (\deg\pi)^{-1}\chi(\widetilde{X},\mathcal{O}) & \longleftarrow \text{multiplicativity of } \chi \\ \| & & \| & \longleftarrow \text{Kodaira vanishing} \\ h^0(X,\mathcal{O}) & & (\deg\pi)^{-1}h^0(\widetilde{X},\mathcal{O}) & \end{array}$$

  Since both X and $\widetilde{X}$ are compact, $h^0(X,\mathcal{O}) = h^0(\widetilde{X},\mathcal{O}) = 1$, and therefore $\deg\pi = 1$.

- $\sharp\pi_1(X) < +\infty$

  The following three methods of proof are known.

(1) Original: differential geometric

We apply the following theorem for our universal cover $(\widetilde{X}, \widetilde{\omega})$.

Myers theorem: Let (M, g) be a complete Riemannian manifold with Ric $g >$ const. > 0. Then one can bound its diameter, in particular M is compact.

If we want to study of fundamental groups of singular varieties with $K_X \equiv 0$ (related to the so-called generalized Bogomolov decomposition), this kind of differential geometric method should be generalized.

(2) Use of Atiyah's L^2-index theorem [A], '76 (see §3)

Our proof is a generalization of this method.

(3) Use of Mori's theory. Kollár-Miyaoka-Mori, Campana '92

The rational-connectedness of Fano manifolds implies $\pi_1(X)$ can not be large.

3. Method (2)

We will prove the finiteness of $\pi_1(X)$ for (X,ω) as in Theorem 2.1 by contradiction. Atiyah's L^2-index theorem [A] asserts that the holomorphic Euler character $\chi(X,\mathcal{O})$ has multiplicativity, as in the case of finite étale covering, even if $\pi : \widetilde{X} \longrightarrow X$ is infinite.

Atiyah's L^2-index theorem

$$1 \underset{\uparrow}{=} \chi(X,\mathcal{O}) \overset{\downarrow}{=} \chi_{(2)}(\widetilde{X},\mathcal{O})$$

vanishing $:= \sum_{q\geq 0}(-1)^q \dim_{\pi_1(X)} H^q_{(2)}(\widetilde{X},\mathcal{O})$

$$\searrow = \dim_{\pi_1(X)} H^0_{(2)}(\widetilde{X},\mathcal{O}).$$

Then it follows that there exists

$$0 \neq \sigma \in H^0_{(2)}(\widetilde{X},\mathcal{O}) : \quad \text{an } L^2\text{-holomorphic function on } \widetilde{X}.$$

Where $H^q_{(2)}(\widetilde{X},\mathcal{O})$ is the L^2-**cohomology group** with respect to the pull-backed metric $\widetilde{\omega} = \pi^*\omega$, and

$$h^q_{(2)}(\widetilde{X},\mathcal{O}) = \dim_{\pi_1(X)} H^q_{(2)}(\widetilde{X},\mathcal{O}) := \sum \int_{X_0} |s_i|^2 dV_{\widetilde{\omega}} \quad \in \mathbb{R}_{\geq 0},$$

$\{s_i\}_{i\in\mathbb{N}} \subset H^q_{(2)}(\widetilde{X},\mathcal{O})$ an orthogonal basis, X_0 a fundamental domain; it is the so-called *von-Neumann dimension* (which is finite, non-negative and well-defined; see [Ko2, Chapter 6] for expository notes). Atiyah's L^2-index theorem is also hold for every Hermitian vector bundle over a compact Hermitian manifold and its unramified Galois covering, moreover it is recently generalized by Eyssidieux [E], Campana-Demailly [CD] for every coherent analytic sheaf over a compact complex analytic space, which we will use later.

Since $\widetilde{X}$ covers the compact X and since σ is holomorphic, we see that $\sigma^k \in H^0_{(1)}(\widetilde{X},\mathcal{O})$: L^1-holomorphic for every $k \geq 2$. Then we can consider **Poincaré series**:

$$P(\sigma^k) := \sum_{\gamma\in\pi_1(X)} \gamma^*\sigma^k \quad \in H^0(X,\mathcal{O}) = \mathbb{C}$$

which is convergent on every relatively compact domain in $\widetilde{X}$, and is invariant under the action of $\pi_1(X)$, and therefore we can regard it as an object on X.

Classical Fact 3.1. *Assume* $\sharp\pi_1(X) = +\infty$. *If* $P(\sigma^k) \equiv$ *const. for every* $k \gg 1$ *(this condition is automatically satisfied since we are considering Poincaré series of holomorphic functions and* X *is compact), then* $\sigma \equiv$ *const.*

Therefore, if $\sharp\pi_1(X) = +\infty$, the L^2-integrability of σ implies that $\sigma \equiv$ const. $= 0$. This leads a contradiction, and the finiteness of $\pi_1(X)$.

4. Proof of Theorem (Finiteness)

Here we explain the method of proof of the following which is a mixture of Theorem 1.1 and 1.2:

Theorem 4.1. *Let* $f : X \longrightarrow S$ *be a proper surjective morphism of normal varieties with connected fibres. Assume that there exists an effective* $\mathbb{Q}$-*divisor* Δ *such that the pair* (X, Δ) *is KLT and that* $-(K_X + \Delta)$ *is* f-*nef and* f-*big. Let* $\mu : Y \longrightarrow X$ *be a resolution of singularities. Then for every point* $0 \in S$, *there exists a contractible neighbourhood* $0 \in U \subset S$ *such that* $\pi_1((f \circ \mu)^{-1}(U)) = 1$. *In particular one has isomorphisms* $\pi_1(Y) \cong \pi_1(X) \cong \pi_1(S)$.

In case $S = X$, this is nothing but Theorem 1.1.

Proof of finiteness. We may assume without loss of generalities that S is Stein, contractible and $\dim S > 0$ so that there exists a deformation retract $Y \longrightarrow Y_0 := (f \circ \mu)^{-1}(0)$.

We take a general holomorphic function $a \in H^0(S, \mathfrak{m}_{S,0}^k)$ with a sufficiently high multiplicity k at $0 \in S$. Since the fundamental group is a birational invariant of smooth varieties, by taking a modification of Y, we also may assume that $\mu : Y \longrightarrow X$ is the so-called *log-resolution* of (X, Δ) and $\operatorname{div}(f^*a)$. Namely, $\mu : Y \longrightarrow X$ is a projective birational morphism such that the exceptional divisor $\sum E_i$ of μ plus the strict transform Δ_1 of Δ plus $D := \operatorname{div}((f \circ \mu)^*a)$ is supported by a divisor with simple normal crossings only, and such that

$$K_Y + \Delta_1 = \mu^*(K_X + \Delta) + \sum e_i E_i$$

holds with $e_i > -1$ for all i. Then we set

$$F := \sum \ulcorner e_i \urcorner E_i \quad \text{effective } \mu\text{-exceptional divisor;}$$

$$\Delta_2 := \sum (\ulcorner e_i \urcorner - e_i) E_i \quad \text{effective fractional } \mathbb{Q}\text{-divisor.}$$

Since $-(K_X + \Delta)$ is f-nef and f-big, we may also assume that there exist an $(f \circ \mu)$-ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor Δ_3 with very small coefficients such that

$$-\mu^*(K_X + \Delta) = A + \Delta_3$$

and that $(Y, \Delta_Y := \Delta_1 + \Delta_2 + \Delta_3)$ is KLT (cf. [KMM, 0-3-6]). Then by definition, we have

$$F = K_Y + \Delta_Y + A.$$

So we can apply the vanishing theorem of Kawamata-Viehweg [KMM, §1-2], Demailly-Nadel [D2, §5] for F and some sheaves related to F.

Since we took a general $a \in H^0(S, \mathfrak{m}_{S,0}^k)$ with $k \gg 1$, there exists a rational number $0 \ll t_0 < 1$ with the following two properties:

(1) The non-KLT locus of $(Y, \Delta_Y + t_0 D)$ is Y_0.

We note that t_0 may *not* be the so-called log-canonical threshold, namely $(Y, \Delta_Y + t_0 D)$ may not be LC along Y_0, and that $D = \operatorname{div}((f \circ \mu)^* a)$ is linearly equivalent to 0. We let $\mathcal{I}_W$ the multiplier ideal sheaf of $(Y, \Delta_Y + t_0 D)$, and W the complex subspace of Y defined by $\mathcal{I}_W$ ($\operatorname{supp} W = Y_0$).

(2) The natural injection $H^0(Y, F \otimes \mathcal{I}_W) \longrightarrow H^0(Y, F)$ is not surjective.

By the vanishing theorem of Kawamata-Viehweg, Demailly-Nadel, we have

$$H^q(Y, F) = H^q(Y, F \otimes \mathcal{I}_W) = 0 \quad \text{for } q > 0.$$

These imply, by a long exact sequence argument, we have vanishings and a non-vanishing:

$$H^q(W, F) = 0 \quad \text{for } q > 0, \quad \text{and} \quad H^0(W, F) \neq 0 \quad \text{by (2)}.$$

In particular $\chi(W, F) \neq 0$.

Let $\pi : \widetilde{Y} \longrightarrow Y$ be the universal cover, and set $\widetilde{F} := \pi^* F$, etc. We also have vanishings of L^2-cohomology groups:

$$H^q_{(2)}(\widetilde{Y}, \widetilde{F}) = H^q_{(2)}(\widetilde{Y}, \widetilde{F} \otimes \mathcal{I}_{\widetilde{W}}) = 0 \quad \text{for } q > 0.$$

(Please be carefull! I did explain nothing about L^2-cohomology groups in this section, which is slightly different from that in §3.) Then by a long exact sequence of L^2-cohomology groups, we have

$$H^q_{(2)}(\widetilde{W}, \widetilde{F}) = 0 \quad \text{for } q > 0.$$

Then by the L^2-index theorem for $\pi|_{\widetilde{W}} : \widetilde{W} \longrightarrow W$ with $F|_W$ [CD, 5.1] [E, 6.2], and by vanishings and a non-vanishing, we have

$$0 \neq \sum_{q \geq 0} (-1)^q h^q(W, F) = \sum_{q \geq 0} (-1)^q h^q_{(2)}(\widetilde{W}, \tilde{F}) = h^0_{(2)}(\widetilde{W}, \tilde{F}).$$

Moreover by the vanishing: $H^1_{(2)}(\tilde{Y}, \tilde{F} \otimes \mathcal{I}_{\widetilde{W}}) = 0$, we have

$$0 \neq \sigma \in H^0_{(2)}(\tilde{Y}, \tilde{F})$$

as an extension of $H^0_{(2)}(\widetilde{W}, \tilde{F})$.

Let us denote $P^{(k)}$ the subspace of $H^0(Y, F^{\otimes k})$ which is generated by products of Poincaré series:

$$P^{(k)} := \left\langle \left\{ \bigotimes P(\sigma^{\otimes k_i});\ P(\sigma^{\otimes k_i}) := \sum_{\gamma \in \pi_1(Y)} \gamma^* \sigma^{\otimes k_i}, \sum k_i = k, k_i \geq 2 \right\} \right\rangle.$$

Then we qoute the following lemma of Gromov [Gr, 3.2.A] ([Ko2, Chapter 13] for expository):

Sublemma 4.2. *Assume $\pi_1(Y)$ is infinite. Then there exist $k \gg 1$ and $\mathfrak{p}$, $\mathfrak{p}' \in P^{(k)}$ such that $(\mathfrak{p}/\mathfrak{p}')|_{Y_0}$ is a non-constant meromorphic function on Y_0.*

On the other hand, since F is μ-exceptional and $f : X \longrightarrow S$ has connected fibres, we have natural isomorphisms: $H^0(Y, F^{\otimes k}) \cong H^0(X, \mathcal{O}) \cong H^0(S, \mathcal{O})$ for every positive k. Therefore such a quotient $\mathfrak{p}/\mathfrak{p}'$ must be a constant on Y_0. This is a contradiction. Thus $\pi_1(Y)$ must be finite.

REFERENCES

[A] Atiyah M., *Elliptic operators, discrete groups and von Neumann algebras*, Astérisque **32-33** (1976) 43–72.

[CD] Campana F.-Demailly J.-P., *Cohomologie L^2 sur les revêtements d'une variété complexe compacte*, math.AG/0002074.

[D1] Demailly J.-P., *Estimations L^2 pour l'opérateur $\overline{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète*, Ann. Sic. Éc. Norm. Sup. **15** (1982) 457–511.

[D2] Demailly J.-P., *L^2 vanishing theorem for positive line bundles and adjunction theory*, Transcendental Methods in Algebraic Geometry, Lecture Notes in Math., vol. 1646, Springer-Verlag, New York, 1996, pp. 1–97.

[E] Eyssidieux P., *Invariants de Von Neumann des faisceaux cohérents*, Math. Ann. **317** (2000) 527–566.

[Gr] Gromov M., *Kähler hyperbolicity and L_2-Hodge theory*, J. Diff. Geom. **33** (1991) 263–292.

[KMM] Kawamata Y., Matsuda K.-Matsuki K., *Introduction to the Minimal Model Problem*, Algebraic Geometry (Sendai 1985), Advanced Studies in Pure Math. **10** (1987), North-Holland, 283–360.

[Kob] Kobayashi S., *On compact Kähler manifolds with positive definite Ricci tensor*, Ann. Math. **74** (1961) 570–574.

[Ko1] Kollár J., *Shafarevich maps and plurigenera of algebraic varieties*, Invent. math. **113** (1993) 177–215.

[Ko2] Kollár J., *Shafarevich Maps and Automorphic Forms*, 1995, Princeton University Press.

[KMMo1] Kollár J., Miyaoka Y.-Mori S., *Rationally connected varieties*, J. Alg. Geom. **1** (1992) 429–448.

[KMMo2] Kollár J., Miyaoka Y.-Mori S., *Rational connectedness and boundedness of Fano manifolds*, J. Diff. Geom. **36** (1992) 765–779.

[T1] Takayama S., *Simple connectedness of weak Fano varieties*, J. Alg. Geom. **9** (2000) 403–407.

[T2] Takayama S., *On the fundamental groups associated to contractions of extremal rays*, to appear in J. Alg. Geom.

[T3] Takayama S., *Local simple connectedness of resolutions of log-terminal singularities*, preprint (2000).

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