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ON CLASSIFICATION OF WEAKENED FANO 3-FOLDS WITH $B_2 = 2$

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1. Introduction

We will work over $\mathbb{C}$ in this talk.

Definition 1.1. Let X be a 3-dimensional smooth projective variety and $(\Delta, 0)$ a germ of the 1-dimensional disk.

(1) We call X a Fano 3-fold when its anti-canonical divisor $-K_X$ is ample.
(2) We call X a weak Fano 3-fold when its anti-canonical divisor $-K_X$ is nef and big.
(3) Let X be a weak Fano 3-fold, we call X a weakened Fano 3-fold when X is not a Fano 3-fold and there exists a small deformation $\mathfrak{f}\colon \mathscr{X} \to (\Delta, 0)$ of X such that the fiber $\mathscr{X}_s = \mathfrak{f}^{-1}(s)$ is a Fano 3-fold for any $s \in (\Delta, 0) \setminus \{0\}$.

This article contains the classification of weakened Fano 3-folds with $B_2 = 2$. The i-th Betti number of a manifold X will be denoted by $B_i(X)$. Let X be a weak Fano 3-fold. We remark that $B_2(X) \geq 2$ because X is a weak Fano which is not a Fano 3-fold.

Fano 3-folds with $B_2 \geq 2$ are classified by Mori and Mukai (cf. [M-M 1], [M-M 2]). The classification of Fano 3-folds with $B_2 = 2$ is useful for the classification of weakened Fano 3-folds with $B_2 = 2$.

Example 1.2. Let $F \cong \mathbb{F}_0$ be a smooth quadric surface in $\mathbb{P}^3$, H a hyperplane in $\mathbb{P}^3$ and Γ a non-singular curve of bi-degree $(2,4)$ on F which is a hyperelliptic curve of degree 6 and genus 3. Let $\psi\colon X \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ along Γ, E the strict transform of F, f_1 a curve of bi-degree $(1,0)$ on E, f_2 a curve of bi-degree $(0,1)$ on E, and D the exceptional divisor of ψ. We have that $\psi^*F = E + D$. Then X is a weak Fano 3-fold with $B_2(X)$ which is not a Fano 3-fold. In fact $(-K_X \cdot f_1) = 0$ and $(-K_X \cdot f_2) = 2$, thus $-K_X\,|_E$ is a divisor of bi-degree $(2,0)$ on E, and $(-K_X)^3 = (-K_{\mathbb{P}^3})^3 - 2\{(-K_{\mathbb{P}^3} \cdot \Gamma) - g(\Gamma) + 1\} = 20$. Thus it is enough to show that $(-K_X \cdot Z) > 0$ for every irreducible and reduced curve Z on X with $Z \not\subset E$. Case in which $\psi(Z)$ is a point, Z is a exceptional line and $(D \cdot Z) = -1$. Hence $(-K_X \cdot Z) = (\psi^*(-K_{\mathbb{P}^3}) - D \cdot Z) = 1$. Case in which $\psi(Z)$ is not a point, Since $-K_X \sim 4\psi^*H - D \sim_{\mathbb{Q}} 4\psi^*H - (2\psi^*H - E) = 2\psi^*H + E$, Hence $(-K_X \cdot Z) \geq (2\psi^*H \cdot Z) = (2H \cdot \psi_* Z) > 0$.

Let $\mathscr{C} \hookrightarrow \mathbb{P}^3 \times (\Delta, 0)$ be a family of curves of genus 3 and degree 6 which is a deformation of Γ to non-hyperelliptic curves. Let $\mathscr{X} \to \mathbb{P}^3 \times \Delta$ be the blow-up along $\mathscr{C}$. Then $\mathscr{X}_s$ is a Fano 3-fold of No.12 in Table 2 of [M-M 1] for any $s \in (\Delta, 0) \setminus \{0\}$. Thus X is a weakened Fano 3-fold.

Example 1.3. Let Q be a smooth quadric 3-fold, $F \cong \mathbb{F}_0$ a smooth quadric surface in Q, Γ be a non-singular curve on F of bi-degree $(1,3)$ which is a curve of genus 0 and degree 4. Let $\psi\colon X \to Q$ be the blow-up of Q along Γ, E the strict transform of F, f_1 a curve of bi-degree $(1,0)$ on E, f_2 a curve of bi-degree $(0,1)$ on E. Then X is a weak Fano 3-fold with $B_2(X)$ which is not a Fano 3-fold. We can show it by the similar way as above. We remark that $(-K_X)^3 = 28$, $(-K_X \cdot f_1) = 0$, and $(-K_X \cdot f_2) = 2$.

X is a weakened Fano 3-fold. For example, let x, y a homogeneous coordinate on $\mathbb{P}^1$, z_0, z_1, z_2, z_3, z_4 a homogeneous coordinate on $\mathbb{P}^4$, and $(\Delta, 0)$ a germ of the 1-dimensional disk with parameter t, we assume that Γ is given by a embedding $\mathbb{P}^1$ into $\mathbb{P}^4$ defined by x^4, x^3y, 0, xy^3, y^4 and Q is a smooth quadric 3-fold containing Γ defined by $z_0z_4 + z_2^2 - z_1z_3 = 0$. We consider a family of embeddings of $\mathbb{P}^1$ into $\mathbb{P}^4$ over $(\Delta, 0)$ defined by x^4, x^3y, tx^2y^2, xy^3, y^4, $\mathscr{C} \to \mathbb{P}^4 \times (\Delta, 0)$, which is a family of curves of genus 0 and degree 4, and is a deformation of Γ to a curve not contained in any hyperplane in $\mathbb{P}^4$. Let $\mathscr{Q} \to \mathbb{P}^4 \times (\Delta, 0)$ be a deformation of Q in $\mathbb{P}^4$ defined by $z_0z_4 + z_2^2 - (1+t^2)z_1z_3 = 0$. We consider the family of embeddings $\mathscr{C} \hookrightarrow \mathscr{Q}$. Let $\mathscr{X} \to \mathscr{Q}$ be the blow-up of $\mathscr{Q}$ along $\mathscr{C}$. Then $\mathscr{X}_s$ is a Fano 3-fold of No.21 in Table 2 of [M-M 1] for any $s \in (\Delta, 0) \setminus \{0\}$. Thus X is a weakened Fano 3-fold.

Example 1.4. Let $M = \mathcal{O}_{\mathbb{P}^2}(2) \oplus \Omega^1_{\mathbb{P}^2}(2)$, $\pi_Z\colon Z = \mathbb{P}(M) \to \mathbb{P}^2 = Y$ the $\mathbb{P}^2$-bundle over $\mathbb{P}^2$ associated to M, and L_Z the tautological line bundle (that is, L_Z is $\mathcal{O}_{\mathbb{P}^2}(1)$ on each fiber, and $(\pi_Z)_*L_Z = M$). We have that $K_Z = -3L$. By the trivial surjection $M \to \Omega^1_{\mathbb{P}^2}(2)$, we have the following commutative diagram,

$$\begin{tikzcd} W := \mathbb{P}(\Omega^1_{\mathbb{P}^2}(2)) \arrow[rr, hook] \arrow[dr, "\pi_W"'] & & Z \arrow[dl, "\pi_Z"] \\ & \mathbb{P}^2. & \end{tikzcd}$$

Let L_W be the tautological line bundle of $\pi_W\colon W \to \mathbb{P}^2$. Then $L_W = L_Z\,|_W$. Let $l \subset \mathbb{P}^2$ be a line $H_Z = (\pi_Z)_*l$, and $H_W = (\pi_W)_*l$. Since W is a divisor of $\mathbb{P}^2 \times \mathbb{P}^2$ of bi-degree $(1,1)$, we may assume that π_W is the second projection. We remark that the restriction $\mathrm{Pic}\,(\mathbb{P}^2 \times \mathbb{P}^2) \to \mathrm{Pic}\,(W)$ is an isomorphism. Since $(\pi_W)_*L_W = \Omega^1_{\mathbb{P}^2}(2)$, we have that $K_W = -2H_W + (-2L_W)$ which is bi-degree $(-2,-2)$. Thus L_W is bi-degree $(1,0)$ and is base-point free. On the other hand, $K_W = K_Z + W\,|_W = -3L_Z + W\,|_W = -2H_Z - 2L_Z\,|_W$. Since the restriction $\mathrm{Pic}\,(Z) \to \mathrm{Pic}\,(W)$ is isomorphism, $W \in |\, L_Z - 2H_Z \,|$. Thus $H^1(Z, \mathcal{O}_Z(L_Z - W)) = H^1(Z, \mathcal{O}_Z(-2H_Z))$. By the Leray spectral sequence, we have the exact sequence:

$$0 \to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-2)) \to H^1(Z, \mathcal{O}(-2H_Z)) \to R^1(\pi_Z)_*\mathcal{O}_Z \otimes \mathcal{O}_{\mathbb{P}^2}(-2).$$

Hence $H^1(Z, \mathcal{O}_Z(L_Z - W)) = 0$. Thus $|\, L_Z \,|$ is base-point free, because $W + 2H_Z \in |\, L_Z \,|$. Moreover we have that $(L_Z)^4 = 6$ by easy calculations. We remark that the birational contraction $\phi_Z\colon Z \to \bar{Z}$ defined by L_Z is primitive (i.e. $\rho(Z/\bar{Z}) = 1$), the exceptional locus is W, and $\phi_W := \phi_Z\,|_W\colon W \to \mathbb{P}^2$ is the first projection. Let $X \in |\, 2L_Z \,|$ be a general member, then X is a weak Fano 3-fold which is not a Fano 3-fold because $(-K_X)^3 = 2(L_Z)^4 = 12$, and we may assume that $E := X \cap W$ is the pull back of a smooth quadric curve C in $\mathbb{P}^2$ by ϕ_W. By the exact sequence

$$0 \to \mathcal{O}_C \to \Omega^1_{\mathbb{P}^2}(2) \mid_C \to \mathcal{O}_C \to 0,$$

we have that $E \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $\phi := \phi_Z \mid_X : X \to \phi_Z(X) =: \bar{X}$. Then $(-K_{\bar{X}} \cdot C) = 2$.

X is a weakened Fano 3-fold with $B_2(X) = 2$. In fact, let $\mathscr{M}$ be a defomation of locally free sheaves of rank 3 over $(\Delta, 0)$ from $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \Omega^1_{\mathbb{P}^2}(2)$ to $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$. Let $\mathbb{Z} := \mathbb{P}(\mathscr{M}) \to \mathbb{P}^2 \times (\Delta, 0)$, $\mathscr{L}_{\mathfrak{X}}$ be the tautological line bundle. $\mathscr{X} \in \mid 2\mathscr{L}_{\mathfrak{X}} \mid$ is a deformation of X to $\mathscr{X}_t \subset \mathbb{P}^2 \times \mathbb{P}^2$ which is a divisor of bi-degree $(2, 2)$.

Similarly, we have that a member $X \in \mid L_Z \mid$ is a weakened Fano 3-fold with $(-K_X)^3 = 48$ which will deform to a divisor of $\mathbb{P}^2 \times \mathbb{P}^2$ of bi-degree $(1, 1)$.

Theorem 1.5. *Let X be a weakened Fano 3-fold with $B_2(X) = 2$. Then $(-K_X)^3 = 12, 20, 28$ or 48. Moreover,*

(1) if $(-K_X)^3 = 12$ or 48, X is a conic bundle over $\mathbb{P}^2$.
(2) if $(-K_X)^3 = 20$, X is isomorphic to Example (1.2).
(3) if $(-K_X)^3 = 28$, X is isomorphic to Example (1.3).

One of the key points of the classification of weakened Fano 3-folds with $B_2 = 2$ is the following theorem.

Theorem 1.6. (cf. [Pa] and [Mi]) *Let X be a weak Fano 3-fold with $B_2(X) = 2$ which is not a Fano 3-fold, $\phi_{ac}\colon X \to X_{ac}$ a multi-anti canonical model, E the exceptional locus of ϕ_{ac}, and $C = \phi(E)$.*

X is a weakened Fano 3-fold if and only if $\phi \mid_E\colon E \to C$ is a $\mathbb{P}^1$-bundle structure of E over $C \cong \mathbb{P}^1$ and $(-K_{X_{ac}} \cdot C) = 2$.

Let $Z_1(X)$ be the set of numerically equivalence class of 1-cycles on X. Let $N_1(X) = Z_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\overline{NE}(X)$ the closed convex cone in $N(X)$ generated by effective 1-cycles on X.

Since $N_1(X) \cong \mathbb{R}^2$, $\overline{NE}(X)$ has just two edges, corresponding to ϕ_{ac} and the unique extremal contraction $\psi\colon X \to Y$. (cf. [K-M-M]) We study possibilities of ψ using the divisors $-K_X$ and E.

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Notations. (1) "$\sim$" means linearly equivalence.
(2) "$\sim_{\mathbb{Q}}$" means $\mathbb{Q}$-linearly equivalence.
(3) The $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ over $\mathbb{P}^1$, a Hirzebruch surface of degree n, is denoted by $\mathbb{F}_n$, and the surface which is obtained by the contraction of the negative section of $\mathbb{F}_n$ is denoted by $\mathbb{F}_{n,0}$.
(4) For a projective variety X, $(\cdot)_X$ means the intersection number on X and we will denote it by $(\cdot)$, if admissible.

2. PRELIMINARY

We devide types of ψ into 3 cases to treat it.

Definition 2.1. Let $\psi\colon X \to Y$ be an extremal contraction from a smooth weak Fano 3-fold X.

(1) ψ is called type R when $\dim(Y) = 3$,
(2) ψ is called type C when $\dim(Y) = 2$, and
(3) ψ is called type D when $\dim(Y) = 1$,

Let X be a weakened Fano 3-fold with $B_2(X) = 2$, $\phi_{ac}\colon X \to X_{ac}$ a multi-anti canonical model, E the exceptional locus of ϕ_{ac}, and $C = \phi(E)$. By Theorem 1.6, $\phi\mid_E\colon E \to C$ is a $\mathbb{P}^1$-bundle structure of E over $C \cong \mathbb{P}^1$ and $(-K_{X_{ac}} \cdot C) = 2$. Thus E is a Hirzebruch surface of degree n. Let f be a reduced fiber of $\phi\mid_E\colon E \to C$ and h a section of $\phi\mid_E\colon E \to C$ with $(h^2)_E = n$. We have the following informations under this setting.

Lemma 2.2. *Notations are as above.*

(1) $-K_X\mid_E \sim 2f$, *in particular* $(-K_X \cdot f) = 0$ *and* $(-K_X \cdot h) = 2$,
(2) $E\mid_E \sim -2h + nf$, *in particular* $(E \cdot f) = -2$ *and* $(E \cdot h) = -n$,
(3) $(K_X^2 \cdot E) = 0$,
(4) $(K_X \cdot E^2) = 4$, *and*
(5) $(E^3) = 0$.

Proof. (1) is because $(-K_{X_{ac}} \cdot C) = 2$. (2) is because $K_X + E\mid_E \sim K_E \sim -2h + (n-2)f$. For (3), $(K_X^2 \cdot E) = (K_X\mid_E)_E^2 = (2f)_E^2 = 0$. For (4), there exists a member S_{ac} of $\mid -K_{X_{ac}}\mid$ such that $S_{ac} \cap C = \{p_1, p_2\}$, S_{ac} is smooth away from $\{p_1, p_2\}$, (S_{ac}, p_i) is the ordinary double point for $i = 1, 2$, and $S := \phi_{ac}^* S_{ac} \to S_{ac}$ is a minimal resolution of S (Cf. [Mi] and [Shin]). Hence $E\mid_S$ is disjoint two (-2)-curves on S. Thus $(-K_X \cdot E^2) = (E\mid_S)_S^2 = -4$. (5) is because $8 = (K_E^2)_E = ((K_X + E)^2 \cdot E) = (K_X^2 \cdot E) + 2(K_X \cdot E^2) + (E^3) = 8 + (E^3)$ by (3) and (4). □

Lemma 2.3. *If $n = 0$, then $\psi(h)$ is not a point.*

Proof. Let L be a divisor on X such that the complete linear system of mL defines ψ for sufficiently large m. Let $L \sim_{\mathbb{Q}} -aK_X - bE$. Then $(L \cdot f) = 2b > 0$ and $(L \cdot h) = 2a + nb$ by Lemma 2.2 (1) and (2). If $n = 0$ and $(L \cdot h) = 0$ then $L \sim_{\mathbb{Q}} -bE$. Let $S \in \mid -K_X\mid$, we have that $(L\mid_S)_S^2 \geq 0$ because L is nef, but $(L\mid_S)_S^2 = (L^2 \cdot (-K_X)) = -b^2(K_X \cdot E^2) = -4b^2 < 0$. It is a contradiction. □

This lemma leads us to the following proposition.

Proposition 2.4. *ψ is not of type D.*

Proof. Let L be a divisor on X such that the complete linear system of mL defines ψ for sufficiently large m. If ψ is of type D, then $\dim \psi(E) = 1$ because $(L \cdot f) > 0$. Hence $L\mid_E \sim ch$ for some $c > 0$ and $n = 0$ because of a property of Hirzebruch surface E. Then $(L \cdot h) = (ch \cdot h)_E = 0$. It contradicts to Lemma 2.3. □

3. The case ψ is of type C

The notations X, $\phi_{ac}: X \to X_{ac}$, E, $\psi: X \to Y$, f, h, d, n are as in Section 1. We will show the following proposition in this section.

Proposition 3.1. *If ψ is of type C, then $(-K_X)^3 = 12$ or 48.*

Proof. Since $\rho(Y) = 1$, and Y is rational by Proposition 3.5 of [M-M 2], $Y \cong \mathbb{P}^2$. Let L be a divisor on X such that the complete linear system of mL defines ψ for sufficiently large m, $\delta_X = (-K_X)^3$, and $L \sim_{\mathbb{Q}} -aK_X - bE$. Then $(L \cdot f) = 2b > 0$ and $(L \cdot h) = 2a + nb$, and $(L^3) = (-aK_X - bE)^3 = \delta_X a^3 - 3ab^2(K_X \cdot E^2) = \delta_X a^3 - 12ab^2$ by Lemma 2.2. Since $(L \cdot h - nf) = 2a - nb \geq 0$, $a \geq \frac{nb}{2}$. If $n = 0$, then $(L \cdot h) = a$. Hence $a > 0$ by Lemma 2.3. Thus $a > 0$ in any case. We have an equation

$$a = \frac{\sqrt{12}}{\sqrt{\delta_X}} b \in \mathbb{Q}. \tag{3.1}$$

Using the classification of Mori and Mukai (cf. Table 2 of [M-M 1]), we have that $\delta_X = 12$ or 48 because δ_X is a deformation invariant.

When $\delta_X = 12$, we have that $a = b$ and $a \geq \frac{nb}{2} = \frac{na}{2}$. Thus $n = 0$, 1, or 2.

When $\delta_X = 48$, we have that $2a = b$ and $a \geq \frac{nb}{2} = na$. Thus $n = 0$ or 1. $\square$

4. The case ψ is of type R

The notations X, $\phi_{ac}: X \to X_{ac}$, E, $\psi: X \to Y$, f, h, d, n are as in Section 1. We will show the following proposition first in this section.

Proposition 4.1. *If ψ is of type R, then the pair $((-K_X)^3, (-K_Y)^3) = (20, 64)$ or $(28, 54)$*

Proof. Since $\rho(Y) = 1$, Y is a Fano 3-fold. Let D be the exceptional locus of ψ, $\delta_X = (-K_X)^3$, $\delta_Y = (-K_Y)^3$, and $\psi^*(-K_Y) \sim_{\mathbb{Q}} -aK_X - bE$. Then $(\psi^*(-K_Y) \cdot f) = 2b > 0$ and $\delta_Y = a^3\delta_X - 12ab^2$ by Lemma 2.2. Thus $a \neq 0$ because $\delta_Y > 0$ and

$$b^2 = \frac{a^3\delta_X - \delta_Y}{12a}. \tag{4.1}$$

If $\psi(D\,|_E)$ is 0-dimensional, then $D\,|_E \sim m(h - nf)$ for some natural number m and $\psi^*(-K_Y)\,|_E \sim 2bh$ because $\dim \psi(f) = 1$. Thus $D \sim_{\mathbb{Q}} \frac{m}{4}(nK_X - 2E)$ and $\psi^*(-K_Y) \sim_{\mathbb{Q}} \frac{b}{2}(-nK_X - 2E)$ by Lemma 2.2. Since $(\psi^*(-K_Y)\,|_D)^2_D = 0$, $((-nK_X - 2E)^2 \cdot (nK_X - 2E)) = n^3(K_X)^3 + (4n - 8n)(K_X \cdot E^2) = -\delta_X n^3 - 16n = 0$. On the other hand, we have that $n \neq 0$ by Lemma 2.3, and that $\delta_X > 0$, it is a contradiction. Thus $\psi(D\,|_E)$ is 1-dimensional, and we have that $\psi: X \to Y$ is the blow-up of Y along the nonsingular curve $\Gamma := \psi(D)$ and Y is smooth by Section 3 of [Mo]. We have that $D \sim_{\mathbb{Q}} \psi^*(-K_Y) + K_X = -(a-1)K_X - bE$. Since $(\psi^*(-K_Y)^2 \cdot D) = 0$, we have that $((-aK_X - bE)^2 \cdot (-(a-1)K_X - bE)) = a^2(a-1)\delta_X + 4(-3a+1)b^2 = 0$. Combining with (4.1), we have an equation

$$2\delta_X a^3 - 3\delta_Y a + \delta_Y = 0. \tag{4.2}$$

Let $x = 4a$, then x is an integer. Thus the equation

$$\delta_X x^3 - 24\delta_Y x + 32\delta_Y = 0 \tag{4.3}$$

has an integral root. Since δ_X and δ_Y is a deformation invariant, using the list in Table 2 in [M-M 1], we have the following 2 possibilities:

(4.1.1) $\delta_X = 20$, $\delta_Y = 64$, and $x = 8$.
(4.1.2) $\delta_X = 28$, $\delta_Y = 54$, and $x = 6$.

□

In the following, we treat the above 2 cases in details.

The case $\delta_X = 20$, $\delta_Y = 64$, and $x = 8$

By [Is 1], and [Is 2], Y is isomorphic to $\mathbb{P}^3$. By the computation in the above proof, we have that $a = 2$ and $b = 2$. Thus we have that

$$\psi^*(-K_X) \sim -2K_X - 2E.$$

Since $-2K_X - 2 \mid_E \sim 4h + (4-2n)f$ is a nef divisor, thus we have that $n = 0$, 1, or 2. For $m > 0$, Since $m\psi^*(-K_Y) - E - K_X \sim_{\mathbb{Q}} (m + \frac{1}{2})\psi^*(-K_Y)$ is nef and big, $H^1(X, \mathcal{O}(m\psi^*(-K_Y) - E)) = 0$ by Kawamata-Viehweg vanishing theorem. Thus we may assume that $F: = \psi(E)$ is normal. Since $\mathbb{F}_n \not\subset \mathbb{P}^3$ for $n > 0$, we have that $E \cong \mathbb{F}_0$ or $\mathbb{F}_{2,0}$. Since $D \sim -K_X - 2E$, as in Section 4 of [M-M 2] using Lemma 2.2, we have that $(D^2 \cdot (-K_X)) = 2g(\Gamma) - 2 = 4$. Thus $g(\Gamma) = 3$. Moreover we have that $(-K_Y \cdot \Gamma) = 24$ by the formula $\delta_X = \delta_Y - 2\{(-K_Y \cdot \Gamma) - g(\Gamma) + 1\}$. Thus we have that $(H \cdot \Gamma) = 6$ for a hyperplane H in $\mathbb{P}^3$.

$$\Gamma \subset F \subset \mathbb{P}^3$$

By III. Ex 5.4, IV. Remark 6.4.1, and V. Ex 2.9 of [Ha], we have that $n = 0$ and Γ is a non-singular curve of bi-degree (2.4) or (4.2) on $F \cong \mathbb{F}_0$.

The case $\delta_X = 28$, $\delta_Y = 54$, and $x = 6$

By [Is 1], and [Is 2], Y is isomorphic to $Q \subset \mathbb{P}^4$ a non-singular quadric 3-fold. By the computation in the above proof, we have that $a = \frac{3}{2}$ and $b = \frac{3}{2}$. Thus we have that

$$\psi^*(-K_X) \sim_{\mathbb{Q}} -\frac{3}{2}K_X - \frac{3}{2}E.$$

Since $-\frac{3}{2}K_X - \frac{3}{2}E$ is a nef $\mathbb{Q}$-divisor, thus we have that $n = 0$, 1, or 2. For $m > 0$, since $m\psi^*(-K_Y) - E - K_X \sim_{\mathbb{Q}} (m + \frac{2}{3})\psi^*(-K_Y)$ is nef and big, $H^1(X, \mathcal{O}(m\psi^*(-K_Y) - E)) = 0$ by Kawamata-Viehweg vanishing theorem. Thus we may assume that $F: = \psi(E)$ is normal and we have that $E \cong \mathbb{F}_0$, $\mathbb{F}_1$, or $\mathbb{F}_{2,0}$. If $n = 1$, using Lemma 2.2, $(\psi^*(-K_Y) \cdot f) = (-\frac{3}{2}K_X - \frac{3}{2}E \cdot h) = 3 + \frac{3}{2} = \frac{9}{2}$ is not a integer. Hence it is a contradiction and we have that $E \cong F \cong \mathbb{F}_0$ or $\mathbb{F}_{2,0}$. Let H be a hyperplane in $\mathbb{P}^4$ and H_Q the complete intersection of Q and H. We have that $F \in \mid H_Q \mid$. By the same calculation in the first case we have that $g(\Gamma) = 0$ and $(H_Q \cdot \Gamma) = 4$.

$$\Gamma \subset F \subset Q \subset \mathbb{P}^4$$

By III. Ex 5.4, IV. Remark 6.4.1, and V. Ex 2.9 of [Ha], we have that $n = 0$ and Γ is a non-singular curve of bi-degree (1.3) or (3.1) on $F \cong \mathbb{F}_0$.

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