代数幾何学シンポジウム記録
2000年度 pp.112-116

WEIGHT FILTRATION AND MIXED TATE MOTIVES

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This talk reported on a slight improvement of a result of Marc Levine on the existence of a weight filtration on a suitable category of mixed Tate motives [10], by Annette Huber and the author. The reader will realise that this text is little more than a survey summarising the main results of [10], that he is warmly encouraged to consult for details and proofs.

According to [10], a $\mathbf{Q}$-linear tensor triangulated category T is *of Tate type* if it is generated by objects $\mathbf{Q}(n)$, $n \in \mathbf{Z}$, together with isomorphisms

$$\mathbf{Q}(n) \otimes \mathbf{Q}(m) \to \mathbf{Q}(n+m)$$

satisfying the usual associativity and commutativity constraints, and such that

$$\operatorname{Hom}_T(\mathbf{Q}(n)[a], \mathbf{Q}(m)[b]) = \begin{cases} 0 & \text{if } n > m \\ 0 & \text{if } n = m \text{ and } a \neq b \\ \mathbf{Q}Id & \text{if } n = m \text{ and } a = b. \end{cases}$$

For $-\infty \leq a \leq b \leq +\infty$, let $T_{[a,b]}$ denote the strictly full triangulated subcategory of T generated by the $\mathbf{Q}(n)$ for $a \leq -2n \leq b$; denote $T_{[a,a]}$ simply by T_a. Then $T_a = 0$ for a odd, while for a even T_a is equivalent to the derived category of the category of finite dimensional $\mathbf{Q}$-vector spaces.

In [10], Levine constructs exact functors

$$W_{\leq b} : T \to T_{[-\infty,b]}$$
$$W^{>b} : T \to T_{[b+1,+\infty]}$$

which are respectively left and right adjoint to the corresponding inclusions of categories[1] and such that, for any $X \in T$, there is a functorial exact triangle

$$W_{\leq b}X \to X \to W^{>b}X \to W_{\leq b}X[1].$$

(This comes from the fact that $T_{[-\infty,0]}$ and $T_{[1,+\infty]}$ define a t-structure on T.) Set $\mathrm{gr}_a^W X = W^{>a-1}W_{\leq a}X$: This defines an exact functor $T \to T_a$. The collection of these data is the *weight filtration* on T.

[1] This is not explicitly stated in [10].

There are at least three examples of $\mathbf{Q}$-linear tensor triangulated categories of Tate type in the literature, relative to a fixed base field k: those obtained by taking the strictly full subcategory generated by Tate objects in the categories of motives (with rational coefficients) defined respectively by Hanamura [5], Levine [11] and Voevodsky [12]. The above construction applies to each of them.

Our contribution is to refine the definition of a weight filtration to the strictly full subcategory $TDM_{\mathrm{gm}}(k)$ of Voevodsky's category $DM_{\mathrm{gm}}(k)$ [12] generated by the Tate objects[2]. Since Voevodsky's motives are with integral coefficients, this provides an integral version of Levine's weight filtration. However, Levine's methods are sufficient to provide such a weight filtration integrally, with a suitable modification of his axioms. Our real contribution is therefore rather to realise this weight filtration with an explicit formula, which may be useful in computations.

Recall that the subcategory $DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \subset DM_{\mathrm{gm}}(k)$ of effective geometrical motives is fully embedded into a sheaf-theoretic category $DM^{\mathrm{eff}}_{-}(k)$. There is a partially defined internal Hom:

$$\underline{\mathrm{Hom}}_{\mathrm{eff}} : DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \times DM^{\mathrm{eff}}_{-}(k) \to DM^{\mathrm{eff}}_{-}(k)$$

which is a partial right adjoint of the tensor product on $DM^{\mathrm{eff}}_{-}(k)$. It is a very interesting open question whether $\underline{\mathrm{Hom}}_{\mathrm{eff}}(X,Y) \in DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ for all $X, Y \in DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$; this is obvious if $X, Y \in TDM^{\mathrm{eff}}_{\mathrm{gm}}(k)$, the effective part of $TDM_{\mathrm{gm}}(k)$, and then $\underline{\mathrm{Hom}}_{\mathrm{eff}}(X,Y) \in TDM^{\mathrm{eff}}_{\mathrm{gm}}(k)$. Hence the category $TDM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ enjoys an everywhere-defined internal Hom.

Our formula for $W_{\leq 2n}X$ ($X \in TDM^{\mathrm{eff}}_{\mathrm{gm}}(k)$, $n \geq 0$) is then

$$W_{\leq 2n}X = \underline{\mathrm{Hom}}_{\mathrm{eff}}(\mathbf{Z}(n), X)(n).$$

This extends to $X \in TDM_{\mathrm{gm}}(k)$ and $n \in \mathbf{Z}$ without difficulty, using the fact that the functor $TDM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to TDM_{\mathrm{gm}}(k)$ is a full embedding. The corresponding result for DM would require resolution of singularities for the ground field k, but in the case of mixed Tate motives it is not necessary [13].

For any $n \in \mathbf{Z}$, the functor

$$\begin{aligned} D^b_c(Ab) &\to TDM_{\mathrm{gm}}(k)_{2n} \\ C &\mapsto C \otimes \mathbf{Z}(n)[2n] \end{aligned}$$

is an equivalence of triangulated categories, where $D^b_c(Ab)$ is the bounded derived category of finitely generated $\mathbf{Z}$-modules. In this way, one may

[2]Note that Voevodsky's category is homological rather than cohomological: the functor associating its motive to a variety is covariant rather than contravariant. For this reason, the Tate object which would be denoted by $\mathbf{Z}(-n)$ in Levine's setting is denoted here by $\mathbf{Z}(n)$.

translate the functors gr_W^{2n} into a collection of triangulated functors

$$(c_n) : TDM_{\mathrm{gm}}(k) \to D^b_f(\mathrm{gr}\, Ab)$$

where $\mathrm{gr}\, Ab$ denotes the category of $\mathbf{Z}$-graded abelian groups. This total functor is a tensor functor and is conservative (but not an equivalence of categories!) This formalism also yields spectral sequences

$$E_2^{p,q}(M,n) = H^{p-q}(k, c_q(M)^* \otimes \mathbf{Z}(n-q)) \Rightarrow H^{p+q}(M, \mathbf{Z}(n))$$

for $M \in TDM_{\mathrm{gm}}(k)$. Here $c_q(M)^*$ denotes the dual of $c_q(M)$ in $D^b_f(Ab)$ and $H^{p+q}(M, \mathbf{Z}(n)) := \mathrm{Hom}_{DM_{\mathrm{gm}}(k)}(M, \mathbf{Z}(n)[p+q])$. This is one of the *weight spectral sequences for motivic cohomology* for mixed Tate motives. In [9], we give variants of these spectral sequences for étale motivic cohomology of geometrically cellular varieties over a field of characteristic 0: they turn out to be very useful for the computation of the unramified cohomology of projective homogeneous varieties.

A t-structure on a tensor triangulated category of Tate type T is *motivic* if it is nondegenerate and if

$$X \in A \iff X \otimes \mathbf{Q}(1) \in A$$

where A is its heart. Levine shows that there is a (unique) motivic t-structure on T provided the following condition holds:

(1) $\mathrm{Hom}_T(\mathbf{Q}(n), \mathbf{Q}(m)[a]) = 0$ for $n < m$ and $a \leq 0$.

The heart A of this t-structure inherits a weight filtration and admits a perfect duality which is compatible with this weight filtration; in particular, it is a Tannakian category.

In the case where T is a category of (triangulated) mixed Tate motives defined from [5], [11] or [12], there are isomorphisms

$$\mathrm{Hom}_T(\mathbf{Q}(n), \mathbf{Q}(m)[a]) \simeq K_{2(m-n)-a}(k)^{(m-n)}$$

for $m \geq n$, where $K_i(k)$ is the i-th algebraic K-group of k and $K_i(k)^{(r)}$ is the weight r part of $K_i(k) \otimes \mathbf{Q}$ for the action of the Adams operations. (In fact, in the cases of [10] and [12], one even has $\mathrm{Hom}_T(\mathbf{Z}(n), \mathbf{Z}(m)[a]) \simeq CH^{m-n}(k, 2(m-n)-a)$ where $CH^p(k,q)$ denotes Bloch's q-th p-codimensional higher Chow group of k [2], and the latter is known to coincide with the weight $m-n$ part of K-theory after tensoring with $\mathbf{Q}$.) In this case, condition (1) is the *Beilinson-Soulé conjecture.*

The Beilinson-Soulé conjecture is known for global fields, in particular for number fields. Hence Levine's theorem applies in this case. Moreover, Levine then constructs isomorphisms

$$\mathrm{Ext}^p_A(M, N) \xrightarrow{\sim} \mathrm{Hom}_T(M, N[p])$$

for $M, N \in A$.

One knows a lot more on the groups of (1) however, thanks to Borel's computation of the algebraic K-theory of number fields [3]: the group $\mathrm{Ext}^p_{\mathcal{A}}(\mathbf{Q}(n), \mathbf{Q}(m))$ is 0 for $p > 1$, and the dimension of $\mathrm{Ext}^1_{\mathcal{A}}(\mathbf{Q}(n), \mathbf{Q}(m))$ is

- 0 for $m = n$
- $r_1 + r_2$ for $m - n$ odd > 1
- r_2 for $m - n$ even > 0

where r_1 and r_2 are respectively the number of real and complex embeddings of k. Unfortunately, for $m-n = 1$, we have $\mathrm{Ext}^1_{\mathcal{A}}(\mathbf{Q}(n), \mathbf{Q}(m)) \simeq k^* \otimes \mathbf{Q}$, which is infinite-dimensional. To have a workable theory of extensions of mixed Tate motives, one should have a definition of them over a ring of S-integers – something which is lacking at the present stage.

Finally, let us mention that there are now realisation functors from the various triangulated categories of motives to suitable categories, via cohomological functors. In the case of Voevodsky's category, see [6] for the construction of a tensor realisation functor from $DM_{\mathrm{gm}}(k)$ to Huber's category of mixed realisations, when k is of characteristic 0.

For the relationships between polylogarithms and extensions of Tate motives over $\mathbf{Q}$, see [4], [1], [8], [11, pp. 303–310] and [7, Appendix].

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