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ON CONJECTURES OF BEILINSON-BLOCH-KATO AND FINITENESS OF ALGEBRAIC CYCLES

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ABSTRACT. In this article, we introduce a result on conjectures of Beilinson and Bloch-Kato on zeros of L-functions of motives, and its application to the finiteness of codimension-two torsion algebraic cycles.

1. CHOW GROUP

Let us start with the definition and a conjectual picture of algebraic cycles. For an algebraic variety X over a field k and an integer $d \geq 0$, the Chow group of algebraic cycles of codimension d modulo rational equivalence is defined by

$$CH^d(X) = \operatorname{Coker}\left(\bigoplus_{x \in X^{(d-1)}} \kappa(x)^* \xrightarrow{\text{div}} \bigoplus_{y \in X^{(d)}} \mathbb{Z}\right)$$

where $X^{(i)}$ is the set of points of codimension i (equivalently, the set of irreducible closed subvariety of codimension i) and div is the divisor map.

If $d = 1$, $CH^1(X)$ coincides with the Picard group, and its structure is well-known; for a projective smooth X we have an exact sequence

$$0 \to \underline{\mathrm{Pic}}^0_X(k) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0.$$

Here, $\underline{\mathrm{Pic}}^0_X$ is an abelian variety called the Picard variety, and $\mathrm{NS}(X)$ is the Néron-Severi group which is known to be finitely generated. Therefore, if for example k is a finite field or a number field, $\mathrm{Pic}(X)$ is a finitely generated abelian group.

The structure of Chow groups in general is highly unknown. Over number fields, however, we have the following conjectures. Let $\overline{k}$ be the algebraic closure of k, $\overline{X} = X \otimes_k \overline{k}$, and let

$$\mathrm{cl}^d_X : CH^d(X) \longrightarrow H^{2d}_{\text{ét}}(\overline{X}, \mathbb{Q}_p(d))$$

be the étale cycle class map and $CH^d(X)_{\text{hom}}$ be its kernel. For X over a number field and an integer $n \geq 0$, the Hasse-Weil L-function $L(H^n(X), s)$ is defined and expected to have meromorphic continuation to the whole complex plane . We simply denote by $H^*(k, -)$ the Galois cohomology group $H^*(\mathrm{Gal}(\bar{k}/k), -)$.

Conjecture 1.1 (Tate [T1], Bloch [Bl2], Beilinson [Be], Bloch-Kato [B-K]). Let X be a projective smooth variety over a number field k.

(i) $CH^d(X)$ is a finitely generated abelian group.

(ii)

$$\begin{aligned}&\mathrm{rank} CH^d(X)/CH^d(X)_{\text{hom}} \\ &\qquad = \dim_{\mathbb{Q}_p} H^0(k, H^{2d}(\overline{X}, \mathbb{Q}_p(d))) = -\mathrm{ord}_{s=d+1} L(H^{2d}(X), s).\end{aligned}$$

(iii)

$$\begin{aligned}&\mathrm{rank} CH^d(X)_{\text{hom}} \\ &\qquad = \dim_{\mathbb{Q}_p} H^1_f(k, H^{2d-1}_{\text{ét}}(\overline{X}, \mathbb{Q}_p(d))) = \mathrm{ord}_{s=d} L(H^{2d-1}(X), s).\end{aligned}$$

Here $H^1_f(k, -)$ is the Selmer group of Bloch-Kato (see (3.1)). Note that (iii) for an elliptic curve ($d = 1$) (or an abelian variety of dimension d) is nothing but the Birch-Swinnerton-Dyer conjecture.

2. TORSION CYCLES OF CODIMENSION TWO

Though we know very little about the above conjectures for $d \geq 2$, torsion of Chow groups of codimension two has been studied after Bloch's program [Bl1] (cf. [CT]). Let $\mathcal{K}_n$ be the Zariski sheafification of algebraic K-theory functor $U \mapsto K_n(\Gamma(U, \mathcal{O}_X))$ (e.g. $\mathcal{K}_0 = \mathbb{Z}$, $\mathcal{K}_1 = \mathcal{O}_X^*$). For a prime number p invertible in k, let $CH^2(X)\{p\}$ denote the p-primary torsion subgroup. Then we have an exact sequence

$$0 \longrightarrow H^1_{\text{Zar}}(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow NH^3_{\text{ét}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow CH^2(X)\{p\} \longrightarrow 0 \tag{2.1}$$

where N denotes the first step of the coniveau filtration on the étale cohomology group:

$$NH^3_{\text{ét}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) = \operatorname{Ker}(H^3_{\text{ét}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to H^3_{\text{ét}}(k(X), \mathbb{Q}_p/\mathbb{Z}_p(2))).$$

The first K-cohomology group has an expression

$$H^1_{\text{Zar}}(X, \mathcal{K}_2) \simeq H\left[K_2(k(X)) \to \bigoplus_{x \in X^{(1)}} \kappa(x)^* \to \bigoplus_{y \in X^{(2)}} \mathbb{Z}\right].$$

Thus the problem of showing the finiteness of $CH^2(X)\{p\}$ becomes the problem of constructing elements in the K-cohomology group. In this line of study we have the following results (as far as the auther knows):

Theorem 2.1. *Let X be a projective smooth variety over a number field k. Then the p-primary torsion subgroup $CH^2(X)\{p\}$ is finite in the following cases:*

(i) $H^2(X, \mathcal{O}_X) = 0$ ([CT-R] *and* [Sa]).

(ii) $X = E \times E$ *where E is a semi-stable ellipptic curve over $\mathbb{Q}$, and $p \nmid 6 \cdot \operatorname{cond}(E)$* ([L-S]).

(iii) $X = E \times E$ *where E is an elliptic curve over $\mathbb{Q}$ with complex multiplication by the ring of integers of an imaginary quadratic field K, satisfying a technical assumption which is satisfied if* $\operatorname{cond}(E)$ *is a power of a prime, or $X = (E \times E) \otimes_{\mathbb{Q}} K$, and $p \nmid 6 \cdot \operatorname{cond}(E)$* ([L1] [L-R] [O1]).

(iv) $X = \operatorname{Km}(E \times E)$ *the Kummer $K3$-surface associated to $E \times E$, for E and p as in* (ii) *and* (iii), *or $X = \operatorname{Km}(E \times E)_K$ for E as in* (iii) ([L1] [O1]).

(v) X = *Fermat quartic surface over $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{-1})$ defined by the equation $x_0^4 + x_1^4 = x_2^4 + x_3^4$ in $\mathbb{P}^3$* ([O1]).

The method of proof used in [L-S] [L1] [O1] will be reviewed in Section 4.

Remark 2.2. The structure of codimension-two Chow group is believed to be very different whether $H^2_{\text{Zar}}(X, \mathcal{O}_X) = 0$ or not (Mumford's theorem, Bloch's conjecture, see [Bl1]). For example, to show (i) of the above thoerem using (2.1), we only need decomposable elements, i.e. the image of

$$\operatorname{Pic}(X') \otimes (k')^* = H^1_{\text{Zar}}(X', \mathcal{K}_1) \otimes H^0_{\text{Zar}}(X', \mathcal{K}_1)$$
$$\xrightarrow{\cup} H^1_{\text{Zar}}(X', \mathcal{K}_2) \xrightarrow{\operatorname{Norm}_{k'/k}} H^1_{\text{Zar}}(X, \mathcal{K}_2)$$

where $X' := X \otimes_k k'$ for a finite base extension k'/k. In other cases we also need "indecomposable elements" .

Over p-adic fields (finite extensions of $\mathbb{Q}_p$), we have similar results. For the varieties in the above theorem considered over p-adic fields, we know the finiteness of prime-to-p torsion of $CH^2(X)$. In this situation, we have results on the product of possibly different two elliptic curves [Sp] and a class of Hilbert-Blumenthal surfaces [L2]. The study of the p-part of the Chow group of varieties over p-adic fields is related to the surjectivity of a syntomic regulator.

Remark 2.3. The author noticed after the simposium that we can extend the result (ii) and (iii) to any n-fold product of an elliptic curve E (and E_K) as above. In a similar way, using the inductive structure of Fermat varieties [Ka-Sh], we can show the similar result for the Fermat variety of degree 4 and arbitrary dimension d of the form

$$x_0^4 + \cdots + x_r^4 = x_{r+1}^4 + \cdots + x_{d+1}^4, \quad (r \geq 1, d \geq 2)$$

over $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{-1})$.

3. BEILINSON-BLOCH-KATO CONJECTURES

Now we introduce the result of [O2] on conjectures Beilinson [Be] and Bloch-Kato [B-K]. Let us recall a part of their conjectures which describe the order of zeros of L-functions of motives in terms of motivic chohomology groups (K-groups) and Selmer groups. For precise definitions and statements, see the original articles or, for exapmle, [Sc] [N] [F-PR].

They state that a regulator (resp. p-adic regulator) map induces an isnomorphism from a motivic cohomology group defined by algebraic K-theory to a Beilinson-Deligne cohomology group (resp. Selmer group) of Hodge (p-adic Hodge) theoretic nature. We define $H^m_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}}$, the "integral part" of a motivic cohomology group by the image of

$$K'_{2n-m}(\mathfrak{X})_{\mathbb{Q}} \longrightarrow K_{2n-m}(X)_{\mathbb{Q}} = \bigoplus_t K_{2n-m}(X)^{(t)}_{\mathbb{Q}}$$

$$\xrightarrow{pr_n} K_{2n-m}(X)^{(n)}_{\mathbb{Q}} =: H^m_{\mathcal{M}}(X, \mathbb{Q}(n))$$

where $\mathfrak{X}$ is a (regular) proper flat model of X over $\mathcal{O}_k$. This is conjectured to be a finite dimensional $\mathbb{Q}$-vector space. For example, we have

$$H^{2d}_{\mathcal{M}}(X, \mathbb{Q}(d))_{\mathbf{Z}} = H^{2d}_{\mathcal{M}}(X, \mathbb{Q}(d)) \simeq CH^d(X)_{\mathbb{Q}},$$
$$H^3_{\mathcal{M}}(X, \mathbb{Q}(2)) \simeq H^1_{\mathrm{Zar}}(X, \mathcal{K}_2)_{\mathbb{Q}}.$$

The Beilinson-Deligne cohomology group is defined by

$$H^m_{\mathcal{D}}(X_{/\mathbb{C}}, \mathbb{R}(n)) = \mathbf{H}^m(X(\mathbb{C}), [(2\pi i)^n \mathbb{R} \to \Omega_X^{\bullet < n}])$$

and $H^m_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n))$ is the part fixed by compex conjugations both on $X(\mathbb{C})$ and the coefficient. Then the Beilinson conjecture claims that the regulator map r_∞ induces an isomorphism

$$r_\infty \otimes_{\mathbb{Q}} \mathbb{R} : H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbf{Z}} \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} H^{i+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n))$$

for $i - 2n \leq -3$. If $i - 2n = -2$ it takes the form

$$((H^{2n-1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbf{Z}} \oplus (CH^{n-1}(X)/\mathrm{hom})_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} H^{2n-1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n)).$$

An important property of Beilinson-Deligne cohomology is that if we assume the fundamental property of the L-function of X, we have

$$\dim_{\mathbb{R}} H^{i+1}_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n)) = \mathrm{ord}_{s=i+1-n} L(H^i(X), s)$$
$$(-\mathrm{ord}_{s=n} L(H^i(X), s) \text{ if } i - 2n = -2).$$

The ($\mathbb{Q}_p$-vector-space analog of) Selmer group associated to the p-adic representation $V = H^i(\overline{X}, \mathbb{Q}_p(n))$ is a subgroup of the first Galois cohomology group $H^1(k, V)$ defined by certain local conditions:

$$H^1_f(k, V) := \mathrm{Ker}\left(H^1(k, V) \to \bigoplus_{v:\ \text{place of } k} \frac{H^1(k_v, V)}{H^1_f(k_v, V)}\right) \tag{3.1}$$

where k_v is the completion of k at v. Then Bloch-Kato conjectures that the p-adic regulator map r_p induces an isomorphism

$$r_p \otimes_{\mathbb{Q}} \mathbb{Q}_p : H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbf{Z}} \otimes \mathbb{Q}_p \xrightarrow{\sim} H^1_f(k, V)$$

if $i - 2n \neq -1$. Note that it is not a priori clear that the image is contained in $H^1_f(k, V)$.

Combining both conjectures (and the Tate conjecture) we obtain

$$\mathrm{ord}_{s=i+1-n} L(H^i(X), s) = \dim_{\mathbb{Q}} H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbf{Z}} = \dim_{\mathbb{Q}_p} H^1_f(k, V)$$

for $i-2n \le -2$.

Our result concernes mostives of even degree generated by cycles. For a projcvtive smooth variety X over a number field k and an integer $d \ge 0$, let Λ be the image of $\mathrm{cl}^d_{\overline{X}} : CH^d(\overline{X}) \to H^{2d}_{\text{ét}}(\overline{X}, \mathbb{Q}_p(d))$, and T, V, A be $\Lambda \otimes \mathbb{Z}_p$, $\Lambda \otimes \mathbb{Q}_p$, $\Lambda \otimes \mathbb{Q}_p/\mathbb{Z}_p$, respectively. On these modules the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ acts continuously and in fact gives an Artin representation

$$\rho : \mathrm{Gal}(k'/k) \longrightarrow GL(V)$$

for a finite extension k' of k where all the generators of Λ are defined (note that Λ is a free abelian group of finite type by definition). Let $V(r) = V \otimes \mathbb{Q}_p(1)^{\otimes r}$ be the Tate twist where $\mathbb{Q}_p(1) = (\varprojlim \mu_{p^n}(\bar{k})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Theorem 3.1. [O1] *Let r_1 and r_2 be the number of the real and complex places of k, respectively, and τ_v be the complex conjugate with respect to a real place v. For an integer $r \ge 1$, put*

$$N(r) = \begin{cases} (r_1+r_2)d - \sum_{v:\text{real}} \mathrm{rank}_{\mathbb{Z}} \Lambda^{\tau_v = (-1)^r} & \text{if } r \ge 2, \\ (r_1+r_2)d - \sum_{v:\text{real}} \mathrm{rank}_{\mathbb{Z}} \Lambda^{\tau_v = -1} - \mathrm{rank}_{\mathbb{Z}} \Lambda^{\mathrm{Gal}(k'/k)} & \text{if } r = 1. \end{cases}$$

Then we have:

(i)

$$\mathrm{ord}_{s=1-r} L(\rho, s) = \dim_{\mathbb{Q}_p} H^1_f(k, V(r)) = N(r).$$

(ii) *There exist $N(r)$ elements in $H^{2m+1}_{\mathcal{M}}(X, \mathbb{Q}(m+r))_{\mathbb{Z}}$, independent of p, whose image by r_p lie in and generate $H^1_f(k, V(r))$.*

Remark 3.2. The computation of $\mathrm{ord}_{s=1-r} L(\rho, s)$ is classical (cf. [T2]).

The proof of this theorem is reduced to the case of $\mathrm{Spec}(k)$, for which Beilinson's conjecture is a theorem of Borel [Bo1] [Bo2] (for $r = 1$ this is the classical Dirichlet unit theorem), and Bloch-Kato's conjecture is a theorem of Soulé [So1] [So2]. Note that if $H^2_{\mathrm{Zar}}(X, \mathcal{O}_X) = 0$ then V is the whole $H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_p(1))$.

4. Application

In this section, we review briefly the method of Langer-Saito for the finiteness of $CH^2(X)\{p\}$, and the technique of [O1] which applies Theorem 3.1 to cases when the Selmer group is not finite. See also Langer's article [L3].

In Theorem 2.1, the Selmer group is finite for (ii) or (iii). For (v), it is not finite and has $\mathbb{Z}_p$-corank 2 (resp. 4) over $\mathbb{Q}$ (resp. $\mathbb{Q}(\sqrt{-1})$). For (iv), although the papers [L1] [O1] treated cases when the Selmer group is finite, in general it can be infinite if the 2-torsion points of E are defined only over a number field with infinite unit group.

Let X/k be as before. Let N be an integer such that $\mathfrak{X}[1/N] := \mathfrak{X} \otimes_{\mathcal{O}_k} \mathcal{O}_k[1/N]$ is smooth over $\mathcal{O}_k[1/N]$, and assume $p \nmid 6N$. The first step is to show:

$$\text{(A)} \qquad \operatorname{Ker}(CH^2(\mathfrak{X}[1/N]) \to CH^2(X)) \ \text{ is torsion}$$

By the localization exact sequence ($X_v := \mathfrak{X} \otimes \mathbb{F}_v$ where $\mathbb{F}_v$ is the residue field)

$$H^1_{\mathrm{Zar}}(X, \mathcal{K}_2) \to \bigoplus_{v \nmid N} \operatorname{Pic}(X_v) \to CH^2(\mathfrak{X}[1/N]) \to CH^2(X) \to 0,$$

this is again a problem of constructing elements in $H^1_{\mathrm{Zar}}(X, \mathcal{K}_2)$.

From (A) it follows that $CH^2(X)\{p\}$ is $\mathbb{Z}_p$-module of cofinite type (i.e. of the form $\mathbb{Q}_p/\mathbb{Z}_p{}^{\oplus n} \oplus$ (finite p-group)), since $CH^2(\mathfrak{X}[1/N])$ is such (cf. [CT-R]). Let us denote the first and the second group in (2.1) by M and N, respectively. Then it suffices to show

$$\text{(B)} \qquad M = N_{\mathrm{div}}.$$

Using the Hochschild-Serre spectral sequence we can embed these groups (after a slight modification of N) into $H^1(k, A)$ where $A = H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$. Further we take localizations with local conditions:

$$\alpha : H^1(k, A) \to \bigoplus_{v:\ \text{place of } k} \frac{H^1(k_v, A)}{H^1_f(k_v, A)}.$$

The following (B1) and (B2) imply (B):

(B1) The image of $M =$ the image of N_{div},

(B2) The Selmer group $H^1_f(k, A) := \operatorname{Ker}(\alpha)$ is finite.

(B1) can be deduced from (A) using p-adic Hodge theory and some technical argument for bad primes if we assume the Tate conjecture for divisors on the special fibers which is known for the cases of Theorem 2.1. We just remark that there is an isomorphism [L-S]

$$\mathrm{Pic}(X_v)\otimes\mathbb{Q}_p \xrightarrow{\sim} \frac{H^1(k_v,V)}{H^1_f(k_v,V)}$$

for $v \nmid 6N$ such that the boundary map in (2.1) and the localized p-adic regulator map are compatible. Theorem 2.1 (ii)-(iv) were proved by showing (A), (B1) and (B2) For (ii) and (iii), (A) is due to Mildenhall [M].

In general, (B2) is false as we mentioned in the previous section, in which case we replace (B2) by the following axiom arising from the Bloch-Kato conjecture:

(B3) There is a subgroup $M_{\mathbb{Z}}$ of M which surjects onto $H^1_f(k,A)_{\mathrm{div}}$.

In the case (v) of Theorem 2.1, (B3) was shown by constructing very explicitly elements in $M_{\mathbb{Z}}$. In a general situation we can formulate as follows. Define a sub-$\mathrm{Gal}(\overline{k}/k)$-representation $V_c \subset V$ by

$$\mathrm{Im}\left(\mathrm{cl}^1_{\overline{X}}:\mathrm{Pic}(\overline{X})\otimes\mathbb{Q}_p \to H^2_{\text{ét}}(\overline{X},\mathbb{Q}_p(1))\right)(1),$$

and T_c, A_c similarly. Then, Theorem 3.1 (ii) for $d=1$, $r=1$ states

(B4) There is a subgroup $M_{\mathbb{Z}}$ of M which surjects onto $H^1_f(k,A_c)_{\mathrm{div}}$.

Therefore, for (B3) it suffices to show

(B5) The Selmer group $H^1_f(k,A/A_c)$ is finite.

As we have seen, the essencial part of the proof is construction of elements in $H^1_{\mathrm{Zar}}(X,\mathcal{K}_2)$ (A) and the finiteness of a Selmer group (B5). The proof of (B5) in the known cases required Iwasawa theoretic results involving "Euler systems". If $H^2_{\mathrm{Zar}}(X,\mathcal{O}_X)=0$, (A) is easy (Remark 2.2) and (B5) is trivial since $A_c=A$. However, we cannot expect that (B5) is true in general.

Remark 4.1. We explain roughly how the results of Remark 2.3 are deduced from the known results. Let $X=E\times E\times\cdots\times E$ (n times). Then by Künneth decomposition we have

$$V=H^2_{\text{ét}}(\overline{X},\mathbb{Q}_p(2))\simeq \bigoplus_{i_1+\cdots+i_n=2} H^{i_1}_{\text{ét}}(\overline{E},\mathbb{Q}_p(i_1))\otimes\cdots\otimes H^{i_n}_{\text{ét}}(\overline{E},\mathbb{Q}_p(i_n)).$$

Note that $H^0_{\text{ét}}(\overline{E},\mathbb{Q}_p)\simeq\mathbb{Q}_p$ and $H^2_{\text{ét}}(\overline{E},\mathbb{Q}_p(2))\simeq\mathbb{Q}_p(1)$. This means that V is a direct sum of the direct summands of $H^2_{\text{ét}}(\overline{E\times E},\mathbb{Q}_p(2))$ via the pull-backs by the projections $pr_{i,j} : X \to E \times E$. Therefore we obrain $H^1_f(k, V) = 0$ ($k = \mathbb{Q}$ or K). For the part (A), we have a similar expression of $\mathrm{NS}(X_v)$ by $\mathrm{NS}(E_v \times E_v)$ and we can obtain enough elements in M by the pull-backs $pr^*_{i,j}(H^1_{\mathrm{Zar}}(E \times E, \mathcal{K}_2))$.

A Fermat varity of dimension n is constructed from n-fold product of the Fermat curve of the same degree by blowing-up, taking quotient by a finite group, and blowing-down [Ka-Sh]. This enables us to use the similar method as the product of an elliptic curve.

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