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Graded rings and birational geometry

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Abstract

This paper is a written version of my lecture "Rings and varieties" at the Kinosaki algebraic geometry workshop in Oct 2000, and a series of two lectures at Tokyo University in Dec 2000. It is intended to be informative and attractive, rather than strictly accurate, and I expect it to stimulate work in a rapidly developing field (as did its predecessor Reid [R3]). The paper was prepared in a hurry to meet a deadline, and one or two sections remain in first draft. I apologise to the reader and the referee for any inconvenience caused.

The canonical ring of a regular algebraic surface of general type, the graded ring over a K3 surface with Du Val singularities polarised by an ample Weil divisor, or the anticanonical ring of a Fano variety is a Gorenstein ring. In simple cases, a Gorenstein ring is a hypersurface, a codimension 2 complete intersection, or a codimension 3 Pfaffian. We now have additional techniques based on the idea of projection in birational geometry that produce results in codimension 4 (and 5, etc.), even though there is at present no useable structure theory for the graded ring.

This paper applies graded ring methods, especially *unprojection*, to the existence of Fano 3-folds and of Sarkisov birational links between them. The 3-fold technology applies also to some extent to construct canonical surfaces. A recurring theme is that unprojection often acts as a working substitute for a structure theory of Gorenstein rings in low codimension. I discuss what little I understand of the structure of codimension 4 Gorenstein rings, and present a general and entirely useless structure theorem. The final section of the paper contains a brief outline of forthcoming joint work with Gavin Brown on $\mathbb{C}^*$ covers of Mori flips of Type A, intended to illustrate the use of serial unprojection.

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1 Introduction

On the geometric side, I am interested in the following problems:

1. Existence of Fano 3-folds
2. Sarkisov birational links between Fano 3-folds
3. Applications of 3-fold technology to canonical surfaces
4. Structure theory of Gorenstein rings in low codimension
5. $\mathbb{C}^*$ covers of Mori flips.

Question 1 is biregular as stated, but is frequently studied in birational terms, notably by *projection* methods. As a first introduction to this idea, I spend some time in Section 2 below on the trivial algebraic trick

$$(Bx - Ay = 0) \quad \mapsto \quad (xs = A, ys = B)$$

that goes from a hypersurface to a codimension 2 complete intersection (c.i.), contracting a divisor $x = y = 0$. This has many applications to constructing new Fano varieties, and links between them. As described in Papadakis–Reid [PR] and in 2.1–2.5, all the quadratic involutions of Corti–Pukhlikov–Reid [CPR] and most of the construction of links in Corti and Mella [CM] are here.

The main methods of constructing Fano 3-folds are:

(a) Graded ring methods,

(b) Birational methods,

(c) Embedding a variety in a symmetric space in the style of Mukai.

Method (a) is closely related to the question of projective embeddings. On the algebraic side, the simplest cases are graded rings in low codimension with a known structure: hypersurfaces, codimension 2 c.i.s, and codimension 3 Pfaffians.

Method (c) is currently a distantly perceived aspiration: we hope that we can eventually understand the usually complicated system of equations defining a variety in geometric terms, for example, as a section of a *key variety* having an interpretation, say in terms of linear algebra or algebraic groups. The key variety is often simpler and has more structure than its lower dimensional sections. In this vein, Examples 7.1–7.6 obtain several classic and modern constructions of surfaces and 3-folds as general sections of bigger "key varieties". This is perhaps a model for applications of 3-fold techniques to older branches of geometry, such as canonical surfaces.

Definition 1.1 A *Fano 3-fold* (also $\mathbb{Q}$-Fano 3-fold) is a variety X in the Mori category (that is, X is projective and has at worst $\mathbb{Q}$-factorial terminal singularities) with $\rho = \operatorname{rank} \operatorname{Pic} X = 1$ and $-K_X$ ample. The *anticanonical ring* of X is

$$R = R(X, -K_X) = \bigoplus_{n \geq 0} H^0(\mathcal{O}_X(-nK_X)).$$

It is known to be a Gorenstein ring (see for example [GW]).

An important case is when $-K_X$ generates the class group of X, that is, $\operatorname{Cl} X = \mathbb{Z} \cdot (-K_X)$, corresponding to Fano's *varietà di prima specie.* The alternative is that $-K_X$ is divisible in $\operatorname{Cl} X$, or that $\operatorname{Cl} X$ has a finite torsion subgroup (for example, if X is an Enriques–Fano variety); it is then normally more efficient to work with the slightly bigger ring $\bigoplus_{D\in\operatorname{Cl} X} H^0(\mathcal{O}_X(D))$, which is graded by $\mathbb{N} \oplus$ torsion.

See [CPR], 3.1 for the definition of Sarkisov link of type II and Corti [Co] for more general Sarkisov links. As explained there, a link $X \dashrightarrow X'$ of type II between Fano 3-folds involves first making an extremal extraction $Y \to X$ (usually a point blowup), then running a 2-ray game or minimal model program on Y until it finally contracts a divisor back down. All the links constructed in [CPR] are made by calculating the anticanonical ring of Y. That is, the birational question is attacked by biregular or graded ring methods.

1.2 Projection through the ages

It is interesting that successive generations of algebraic geometers interpret *projection* in several remarkably different ways.

(i) Historically, projection always means linear projection of a variety in (unweighted) projective space $\mathbb{P}^n$ to a smaller $\mathbb{P}^{n'}$. Projection from a general linear centre disjoint from the variety is used in proving foundational results such as Noether normalisation, or the existence of a birational projection to a hypersurface, or to define the dimension, function field or canonical divisor of a variety.

(ii) Generic projection allows us to assert that any variety has a birational morphism to a hypersurface with ordinary singularities. *Italian projection* is a technique for studying a canonical surface S that goes back to Enriques and was later developed by Ciliberto [Ci] and Catanese [Ca]–[Ca2] and others. In modern terms, it consists of analysing the canonical ring $R(S, K_S)$ of S as a module over a polynomial ring $k[x_1, \dots, x_n]$ (preferably with $n = 4, 5$, etc.), where $x_i \in H^0(K_S)$ or $H^0(2K_S)$ correspond to some initial set of generators that define a generic projection $X \to \overline{X} \subset \mathbb{P}^3$ or $\mathbb{P}^4$. For surfaces with $p_g = 4$, taking a basis of $H^0(K_S)$ as generators (the 1-*canonical map*) is so natural and instinctive that it is often not perceived as a choice. If X is a canonical surface then $K_X = \mathcal{O}_X(1)$, and the image $\overline{X}$ of a projection $X \to \overline{X} \subset \mathbb{P}^3$ has $K_{\overline{X}} = \mathcal{O}(d-4)$, so that the difference between K_X and $K_{\overline{X}}$ has to be accounted for by the normalisation of $\overline{X}$ along its double locus (called "subadjunction"), the intersection of $\overline{X}$ with the adjoint of smallest degree. See [Ca]–[Ca2] for details.

While in the hands of the maestri this method gives very interesting examples and results, it is conceptually messy and computationally unpleasant, and probably intractable. See Problem 7.7 for a comparison between Italian and Gorenstein projection.

(iii) Meanwhile, del Pezzo exploited linear projections $S_d \dashrightarrow S_{d-1}$ between del Pezzo surfaces from a point $P \in S_d$, and Fano and later Iskovskikh worked with linear projection of a Fano 3-fold V from a centre contained in V, notably projection from a line $\pi_L \colon V \dashrightarrow V'$. These take one anticanonical variety to another, that is,

$$K_V = \mathcal{O}_V(-1) \quad \text{and} \quad K_{V'} = \mathcal{O}_{V'}(-1)$$

and are cases of *Gorenstein projection*: for this to work involves the discrepancy of the blowup coinciding exactly with the multiplicity of the centre subtracted from the linear system (compare [PR], 2.7).

(iv) Mori and his followers (notably Takeuchi and Takagi) reworked Fano and Iskovskikh's study of Fano 3-folds in terms of extremal rays or MMP. Instead of just doing the linear projection that comes instinctively to someone versed in projective geometry, Mori views Fano's projection $V \dashrightarrow V'$ as first the blowup of a line $\widetilde{V} \to V$, followed by a MMP or 2-ray game in the Mori cone of $\widetilde{V}$, that finds and contracts extremal rays to obtain first a flop, then a divisorial contraction to V'.

(v) My view of projection is based on the work on Sarkisov links in [CPR]: if X is a Fano 3-fold, and $X_1 \to X$ a Mori extraction (usually a point blowup), say with exceptional divisor E of discrepancy $\frac{1}{r}$, the anticanonical ring $R_1 = R(X_1, -K_{X_1})$ is a subring of $R = R(X, -K_X)$, consisting of forms of degree d vanishing $\frac{d}{r}$ times on E (see Example 9.13 for a particular case, and compare [CPR], 3.4). This ties in closely both with Fano projections and with the Mori 2-ray game, but in general, it does not directly predict anything about the algebra of

R_1 or the geometry of $Y = \operatorname{Proj} R_1$, or the rational map $X \dashrightarrow Y$. In good cases, $X \dashrightarrow Y$ may be a projection from one weighted projective space (w.p.s.) to another, obtained by eliminating a single generator of R of high weight; but we do not start out by assuming that, and more complicated things happen in applications (see Examples 9.13–9.14).

In higher codimension, the idea of Kustin–Miller unprojection [KM], [PR] often acts as a workable substitute for a structure theorem. I discuss this in Sections 5–8 with some pretty applications. More complicated unprojections not of Kustin–Miller type, with exceptional divisor that is not projectively Gorenstein, can be used to similar effect (see Section 9), even when the algebra is complicated and not really properly understood. The examples of Type II unprojections discussed in Section 9 arising from Selma Altınok's work [A] are really nontrivial applications of these methods.

In Section 10, I explain an application of geometric ideas to the structure theory of rings in codimension 4. Although I state a "structure theorem", the answer is still elusive, and my result is not yet explicit enough to have any predictive power.

The idea of unprojection is just made for serial use. That is, it can be used many times over in an inductive way to produce Gorenstein rings of arbitrary codimension, whose properties are nevertheless controlled by just a few equations as a new unprojection variable is adjoined. Section 11 discusses briefly how this applies to the $\mathbb{Z}$-graded rings over Mori flips (forthcoming joint work with Gavin Brown).

1.3 Acknowledgments

Several items in what follows are derived from conversations with Selma Altınok, Gavin Brown, Alessio Corti, Mori Shigefumi, Mukai Shigeru, Stavros Papadakis and Takagi Hiromichi, and I refer in several places to results from Papadakis' forthcoming thesis [P]. I thank Takagi for providing me with excellent lecture notes. My stay in Japan was generously supported by Kyoto Univ., RIMS, and I am extremely grateful to Professors Kawamata, Miyaoka, Mori and Saito Kyoji for invaluable assistance and friendly hospitality. This paper was written during a short summer solstice visit to John Cannon's Magma group at the University of Sydney; I thank them for the invitation, and for all the wonderful meals.

2 The $Bx - Ay$ argument

The most basic example of unprojection consists simply of replacing a hypersurface $Bx - Ay = 0$ that contains a codimension 2 c.i. $x = y = 0$ by the codimension 2 c.i. $xs = A, ys = B$. Despite its trivial appearance, this trick has many applications.

2.1 The unprojection variable $s = A/x = B/y$

Write $\mathbb{P} = \mathbb{P}^n(a_0, \dots, a_n) = \operatorname{Proj} k[x_0, \dots, x_n]$ for the w.p.s. with weights $\operatorname{wt} x_i = a_i$. Let $D : (x = y = 0) \subset \mathbb{P}^n$ be a codimension 2 c.i.; here x, y could be two of the coordinates x_i, x_j, or any two hypersurfaces with no common components. Then any hypersurface containing D is of the form

$$X : (Bx - Ay = 0) \subset \mathbb{P}^n(a_0, \dots, a_n).$$

Assume that $\deg A > \operatorname{wt} x$. Now define

$$Y : (xs = A, ys = B) \subset \mathbb{P}^{n+1} = \operatorname{Proj} k[x_0, \dots, x_n, s],$$

where $\operatorname{wt} s = \deg A - \operatorname{wt} x$. Then Y contains the point "at infinity" of the w.p.s. $P_s = (0 : \cdots : 0 : 1)$, where $x_i = 0$ for all i, but $s \neq 0$.

There are two inverse birational maps: $X \dashrightarrow Y$ is the unprojection, or the graph of s, obtained by adjoining the unprojection variable

$$s = \frac{A}{x} = \frac{B}{y}. \tag{2.2}$$

The inverse $Y \dashrightarrow X$ corresponds algebraically to eliminating s. In terms of geometry, it blows P_s up to a divisor $D \subset X$.

The following familiar setup is a special case of the $Bx - Ay$ trick: let

$$L \subset S_3 \subset \mathbb{P}^3$$

be a cubic del Pezzo surface containing the line $L : (x = y = 0)$. Then the defining equation of S_3 is $Bx - Ay$, where A, B are quadratic polynomials in $\mathbb{P}^3$. The condition for S_3 to be nonsingular along L is that A, B have no common zeros on L, so that s given by (2.2) is well defined, and defines a morphism $S_3 \to T_4 = Q_1 \cap Q_2 \subset \mathbb{P}^4$ to a del Pezzo surface of degree 4. This is the contraction morphism of L provided by Castelnuovo's criterion.

However, the same equations apply much more generally: the hypersurface $X : (Bx - Ay = 0)$ can be of any degree in a w.p.s. of any dimension, and can be arbitrarily singular, provided only that x, y remains a regular sequence. If A, B do not both vanish along any component of $x = y = 0$ (that is, if $D : (x = y = 0) \subset X$ is a Weil divisor, or a Cartier divisor at every generic point), then $X \dashrightarrow Y$ is birational.

2.3 Application to Sarkisov links

Consider an anticanonically embedded hypersurface

$$\mathbb{P}(1, a_1, a_2) \subset X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$$

of degree $d = \deg X = a_1 + \cdots + a_4$ containing a plane $\mathbb{P}(1, a_1, a_2)$. Here X is one of the "famous 95", but *is not in the Mori category*: it has equation $Bx_3 - Ax_4 = 0$, and is not $\mathbb{Q}$-factorial at points with $A = B = x_3 = x_4$.

Assume that $a_4 > a_3$. Then X is the *midpoint* of a Sarkisov link of type II

$$Z \dashleftarrow X \dashrightarrow Y, \tag{2.4}$$

which is either one of the quadratic involutions of [CPR], 4.4–4.9, or of the type studied by Corti and Mella [CM]. Both broken arrows are given by the $Bx - Ay$ trick: suppose that X is the hypersurface $X : (Bx_3 - Ax_4 = 0)$. The rational map $X \dashrightarrow Y$ contracts the plane $\mathbb{P}(1, a_1, a_2)$ to the point $P_s \in Y$, with Y the graph of $s = \frac{A}{x_3} = \frac{B}{x_4}$, and

$$Y_{d-a_3, d-a_4} : (sx_3 = A, sx_4 = B) \subset \mathbb{P}(1, a_1, \ldots, a_4, a_1 + a_2).$$

This is a *general* codimension 2 c.i. of the stated degrees. If $A = x_4$, then the first equation $sx_3 = A$ eliminates x_4, and Y is a general hypersurface in $\mathbb{P}(1, a_1, \ldots, a_3, a_1 + a_2)$. In this case $X : (x_4^2 + \cdots = 0)$ has a biregular involution, $Y \cong Z$, and the link (2.4) is one of the quadratic involutions of [CPR], 4.4–4.9.

On the other hand, Z is the graph of $t = \frac{x_4}{x_3} = \frac{B}{A}$ (recall that $a_4 > a_3$). Then $X \dashrightarrow Z$ contracts the divisor $D : (x_3 = A = 0)$, and Z is defined by the equations $x_4 = tx_3$, $At = B$. Because of the first equation, Z is still a hypersurface

$$Z_{d-a_3} \subset \mathbb{P}(1, a_1, a_2, a_3, a_4 - a_3)$$

with defining equation $F = A(x_0, \dots, x_3, tx_3)t - B(x_0, \dots, x_3, tx_3)$, that is, $At - B$ after the substitution $x_4 \mapsto tx_3$. Because of this, Z_{d-a_3} is not a *general* hypersurface of the stated degree. It is a Fano 3-fold in the Mori category, but has a funny terminal singularity at the point P_t. At this point, the classification of Sarkisov links gets tangled up with the classification of divisorial extractions in the Mori category, on which there has been considerable recent progress; see Corti–Mella [CM], Kawakita [Ka]–[Ka2] and Takagi [T].

2.5 Corti–Mella

The typical case, and the starting point of [CM], is when $Z = Z_4 \subset \mathbb{P}^4$ is a quartic hypersurface with a singularity of analytic type $xy = z^3 + t^3$. Then Z is algebraically factorial, so in the Mori category. Corti and Mella prove that the $(2,1,1,1)$ and $(1,2,1,1)$ weighted blowups of the singular point are divisorial extractions. Each of these blowups leads to a Sarkisov link of type II as just described:

$$\begin{array}{ccccc} & \operatorname{Bl} Z_4 & & & \\ \swarrow & & \searrow & & \\ Z_4 & & (X_5 \subset \mathbb{P}(1^4, 2)) & \dashrightarrow & Y_{3,4} \subset \mathbb{P}(1^4, 2^2) \\ & & \text{cont'g } \mathbb{P}^2 : (x_0 = y = 0) & & \text{general element} \end{array} \tag{2.6}$$

I have only described the easy part of Corti and Mella's argument, constructing the link (2.6) as an application of a fairly trivial piece of algebra. The hard part of their work is to show that Z_4 and $Y_{3,4}$ are a *birationally rigid pair*: that is, any Mori fibre space birational to them is biregular to Z_4 or $Y_{3,4}$. This is the problem of excluding links to any other Mori fibre spaces. For this, in addition to the technology of [CPR] and Corti [Co2], they need to prove that the *only* extremal extractions from the singular point $xy = z^3 + t^3$ are the $(2,1,1,1)$ and $(1,2,1,1)$ weighted blowups.

3 Varieties and graded rings, $\operatorname{Proj} R$, Hilbert series

Everyone knows the correspondence

$$X = \operatorname{Proj} R, \mathcal{O}_X(1) \longleftrightarrow R = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(n)) \tag{3.1}$$

between projective varieties and graded rings. See for example [EGA2] or [Hartshorne], Chapter II. With the exception of Section 11, I assume that the ring is $\mathbb{N}$-graded, that is, $R_n \neq 0$ only for $n \geq 0$, and $R_0 = k$ (the ground field $k = \mathbb{C}$). The ring R is almost never generated in degree 1, so that $\mathcal{O}_X(k)$ is not necessarily determined by $\mathcal{O}_X(1)$, and I should really specify $(X, \bigoplus_{k\in\mathbb{Z}} \mathcal{O}_X(k))$ on the l-h.s. of (3.1); for our purposes it is usually enough to take $\mathcal{O}_X(k) = \mathcal{O}_X(kD)$ for some ample Weil divisor.

3.2 Tutorial on Hilbert series

One of the standard applications of graded rings is when the Hilbert[1] series $P(t) = \sum P_n t^n$ is known, where $P_n = \dim R_n$ (typically, by the Riemann–Roch formula), and we can use it to guess a plausible form of R by generators and equations, and hence a plausible model of X as a variety in a w.p.s. with those generators and defining equations.

Example 3.3 X is a surface of general type with invariants $p_g = h^0(K_X)$, $q = h^1(\mathcal{O}_X)$ and K^2. I assume that $q = 0$, so that X is regular; using Kodaira vanishing, this implies that $H^1(X, nK_X) = 0$ for all n, so that the graded ring $R(X, K_X)$ is Gorenstein by [GW]. Then by Riemann–Roch

$$P_n = \begin{cases} 1 \\ p_g \\ p_g + 1 + \binom{n}{2} K^2 & \text{for } n \geq 2. \end{cases}$$

The Hilbert series $P(t) = \sum P_n t^n$ is thus

$$P(t) = 1 + p_g t + (p_g + 1 + K^2)t^2 + \cdots + \left(p_g + \binom{n}{2} K^2\right) t^n + \cdots$$

I calculate $(1-t)P(t)$ by long multiplication; this amounts simply to differencing the coefficients of the power series:

$$(1-t)P(t) = 1 + (p_g - 1)t + (1 + K^2)t^2 + \cdots + nK^2 t^n + \cdots$$

[1]The letter P stands for Poincaré. The technique is so called because it was first used systematically by Cayley and Sylvester in the context of invariant theory. I recently asked a couple of math historians where to find Cayley and Sylvester's treatment, and I am indebted to them for the handy tip: read their collected works in the library.

Again multiply by $1-t$:

$$(1-t)^2P(t) = 1 + (p_g - 2)t + (K^2 - p_g + 2)t^2 + \cdots + K^2t^n + \cdots ,$$

and again, to get

$$(1-t)^3P(t) = 1 + (p_g - 3)t + (K^2 - 2p_g + 4)t^2 + (p_g - 3)t^3 + t^4. \qquad (3.4)$$

Notice that the polynomial is symmetric ("Gorenstein symmetry"), and the sum of the coefficients is $K^2 = \deg X$.

An important case is when $p_g \geq 3$ and $|K_X|$ is free; then there are elements $x_1, x_2, x_3 \in H^0(K_X)$ that form a regular sequence for $R(X, K_X)$, and (3.4) is the Hilbert function of the Artinian quotient ring $R(X, K_X)/(x_1, x_2, x_3)$. In particular, all the coefficients of (3.4) are ≥ 0. However, (3.4) holds without any assumption on $|K_X|$, for example, even if $p_g = 0$.

I gave the above treatment of Hilbert series in a very simple case to illustrate the method, but there are similar formulas and methods much more generally. There is already, for example, quite a lot of experience of working with Hilbert series on surfaces with quotient singularities or 3-folds with canonical singularities; compare Altınok [Al] or Kawakita [Ka]–[Ka2].

Example 3.5 In Reid [R], I considered the algebraic surface X with $p_g = 3$, $q = 0$ and $K^2 = 4$ arising as the universal cover of a $\mathbb{Z}/4$ Godeaux surface. Write $R(X, K_X)$ for the canonical ring of X. Its multiplied out Hilbert polynomial[2] is

$$\begin{aligned}(1-t)^3P(t) &= 1 + (p_g - 3)t + (K^2 - 2p_g + 4)t^2 + (p_g - 3)t^3 + t^4 \\ &= 1 + 2t^2 + t^4.\end{aligned}$$

Thus the ring needs 3 generators x_1, x_2, x_3 in degree 1, and 2 generators y_1, y_3 in degree 2 (at least). Putting in these generators gives

$$(1-t)^3(1-t^2)^2P(t) = 1 - 2t^4 + t^8 = (1-t^4)^2.$$

[2] I apologise for this unconventional use of terminology. *Hilbert polynomial* traditionally means the polynomial $P_{\mathcal{F}}(n) = \chi(X, \mathcal{F}(n))$, which coincides with $h^0(X, \mathcal{F}(n))$ after all the cohomology has died out, when $n \gg 0$. Here I am using *multiplied out Hilbert polynomial* for the numerator of the Hilbert series $P(t) = \sum P(n)t^n$ after a denominator $\prod(1 - t^{a_i})$ has been chosen, corresponding to a choice of generators $(x_1, \ldots, x_n)$. Maybe it would be better to say *Hilbert numerator*, or *Cayley–Sylvester polynomial.*

In most cases of interest for w.p.s., $\mathcal{O}(1)$ is not a line bundle, so $P_{\mathcal{F}}(n)$ is usually not a polynomial, but one of a choice of polynomials depending on n modulo the index.

We note that this coincides with the multiplied up Hilbert polynomial of a c.i. of two hypersurfaces[3] of degree 4:

$$k[x_1, x_2, x_3, y_1, y_3]/(f_0, f_2).$$

Thus a plausible model for $R(X, K_X)$ is $X = X_{4,4} \subset \mathbb{P}(1^3, 2^2)$. One sees that a suitable choice of the two relations makes X nonsingular, and setting

$$x_i \mapsto \varepsilon^i x_i \text{ for } i = 1, 2, 3 \quad \text{and} \quad y_i \mapsto \varepsilon^i y_i \text{ for } i = 1, 3$$

defines a fixed point free action of $\mathbb{Z}/4$ on X, where $\varepsilon = \exp(2\pi i/4)$ is a primitive 4th root of 1. In [R], I showed that every $\mathbb{Z}/4$ Godeaux surface is obtained in this way by dividing a surface $X = X_{4,4} \subset \mathbb{P}(1^3, 2^2)$ by this group action.

Remark 3.6 I conclude this brief tutorial on Hilbert series with the relation between the multiplied out Hilbert polynomial $\prod_{i=0}^{n}(1-t^{a_i})P_R(t) = Q(t)$ and the free resolution of the graded ring $R = R(X, \mathcal{O}_X(1))$ over the polynomial ring $A = k[x_0, \dots, x_n]$. The generators x_i are always chosen so that R is a finite module over A. Geometrically, this means that the x_i have no common zeros on X and define a finite morphism $\pi\colon X \to \overline{X} \subset \mathbb{P}(a_0, \dots, a_n)$. Then $\pi_*\mathcal{O}_X$ is a sheaf on $\mathbb{P}$ or on the image $\overline{X}$ whose Serre module is the ring $R = \bigoplus H^0(\mathcal{O}_X(n))$. I write the sheaf $\pi_*\mathcal{O}_X$ even when I mean the ring R. (As explained in [PR], 2.4, the rigorous algebraic treatment works via the coherent Lefschetz principle with the vertex of the affine cone over X, that is, R localised at the "irrelevant" maximal ideal, but I don't want to spend time on this.) By the Hilbert syzygies theorem, there exists a finite free resolution

$$0 \leftarrow \pi_*\mathcal{O}_X \leftarrow \mathcal{L}_0 \leftarrow \mathcal{L}_1 \leftarrow \cdots \leftarrow \mathcal{L}_m \leftarrow 0, \tag{3.7}$$

where each $\mathcal{L}_i$ is a free graded module, that is, $\mathcal{L}_i = \bigoplus \mathcal{O}_{\mathbb{P}}(-b_{i,j})$. Here $\mathcal{L}_0 = \mathcal{O}_{\mathbb{P}}$ if and only if $X = \overline{X}$ is embedded as a projectively normal subvariety, that is, $k[x_0, \dots, x_n] \quad R(X, \mathcal{O}_X(1))$. Each homomorphism $\mathcal{L}_{i+1} \to \mathcal{L}_i$ is

[3]This is a basic exercise. [Hint: expand $\prod \frac{1}{1-x_i}$ as the sum of all monomials in $k[x_1, x_2, \dots]$, each with coefficient 1. Substitute $x_i \mapsto t^{a_i}$, where $\operatorname{wt} x = a_i$ to prove that the Hilbert series of the weighted polynomial ring is $\prod \frac{1}{1-t^{a_i}}$. Cutting by a regular element of degree d multiplies by $(1-t^d)$, so a weighted c.i. has Hilbert series $\frac{\prod(1-t^{d^j})}{\prod(1-t^{a_i})}$.] For more practice, do the [Homework].

a matrix whose entries are homogeneous of degrees $b_{i+1,j} - b_{i,k}$, so that the homomorphism can be considered to be homogeneous of degree 0. Then

$$Q(t) = \prod_{i=0}^{n}(1 - t^{a_i})P_R(t) = \sum(-1)^i t^{b_{i,j}}.$$

In other words, each direct summand $\mathcal{O}_{\mathbb{P}}(-b_{i,j})$ contributes a term $t^{b_{i,j}}$, with the generators of $\mathcal{O}_X$ (that is, $\mathcal{L}_0$) counting positively, the relations $\mathcal{L}_1$ negatively, the first syzygies positively, second syzygies negatively and so on. Unfortunately, the polynomial expression $Q(t)$ does not entirely determine the shape of the resolution (3.7). For example, a positive term may mean a new generator, or a first syzygy between the relations, etc. See the sidestep in Example 7.1 for a typical instance.

The really useful thing is *Gorenstein symmetry*. If R is Gorenstein, the resolution (3.7) has length equal to the codimension $m = c$. Moreover, $\mathcal{L}_c \cong (\mathcal{L}_0)^\vee \otimes \mathcal{O}_{\mathbb{P}}(-k)$, where k is the *adjunction number*, that is,

$$\omega_X = \omega_{\mathbb{P}^n} \otimes \mathcal{O}_X(k) = \mathcal{O}_X\Big(k - \sum a_i\Big),$$

and $\mathcal{L}_{c-i} \cong \mathcal{L}_i^\vee \otimes \mathcal{O}(-k)$. In particular, the polynomial $Q(t)$ is symmetric: t^m and $(-1)^c t^{k-m}$ appear with the same coefficient. In writing out $Q(t)$, I usually indicate the final term $(-1)^c t^k$, but only write out the terms up to the centre of Gorenstein symmetry, say something like $1-t^3-3t^4+12t^4-\cdots-t^9$.

Example 3.8 If $X \subset \mathbb{P}(a_0, \dots, a_n)$ is a 5×5 Pfaffian then

$$\prod_{i=0}^{n}(1 - t^{a_i})P_X(t) = 1 - \sum_{i=1}^{5} t_i^{b_i} + \sum_{i=1}^{5} t_i^{k-b_i} - t^k$$

where $b_l = \deg \mathrm{Pf}_l$. For example, if X is a K3 surface in weighted $\mathbb{P}^5$ then $K_X = 0$, so that $k = \sum a_i$, and the entries in the skew matrix are $k - b_i - b_j$. Thus the Pfaffian $\mathrm{Pf}_l = \mathrm{Pf}_{ij.i'j'}$ (where $\{l, i, j, i', j'\} = \{1,2,3,4,5\}$) has degree

$$b_l = 2k - b_i - b_j - b_{i'} - b_{j'},$$

and hence $\sum_{i=1}^{5} b_i = 2k = 2\sum_{i=1}^{6} a_i$.

4 Constructing Fano 3-folds by unprojection, first examples

Example 4.1 Given $\Pi : (x_1 = x_2 = x_3 = 0) \subset \mathbb{P}^5$, construct a c.i. $X_{2,3} \subset \mathbb{P}^5$ containing Π and as general as possible (preferably nonsingular, but see below). Suppose that

$$X_{2,3} : \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0, \tag{4.2}$$

with $\deg a_i = 2, \deg b_i = 1$.

To contract Π to a point, I construct a function (homogeneous form) with pole on Π. There is a clever way of doing this (see Section 5), but I want to start by explaining a stupid way. If I view (4.2) as 2 linear equations in 3 variables, they have a unique solution up to proportionality

$$x_1 \sim a_2b_3 - a_3b_2, \quad \text{etc.},$$

by Cramer's rule. This suggests setting $yx_i = A_i$, with A_i the 2×2 minors of (4.2), so that

$$y = A_i/x_i \quad \text{for } i = 1, 2, 3$$

gives the required rational homogeneous form of degree 2 with ideal of denominators (x_1, x_2, x_3), the ideal of Π.

Adjoining y with the new equations $yx_i = A_i$ gives rise to a new variety $Y \subset \mathbb{P}(1^6, 2)$ defined by the 5 Pfaffians of

$$\begin{pmatrix} y & a_1 & a_2 & a_3 \\ & b_1 & b_2 & b_3 \\ & & x_3 & -x_2 \\ & & & x_1 \end{pmatrix} \quad \text{of degrees} \quad \begin{pmatrix} 2 & 2 & 2 & 2 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}. \tag{4.3}$$

4.4 Notation

I write

$$M = \begin{pmatrix} m_{12} & m_{13} & m_{14} & m_{15} \\ & m_{23} & m_{24} & m_{25} \\ & & m_{34} & m_{35} \\ & & & m_{45} \end{pmatrix}$$

for a skew 5×5 matrix. That is, I omit the diagonal terms (which are zero) and the $m_{ji} = -m_{ij}$ with $i < j$. If you are a beginner, you may prefer to write out the diagonal zeros for a while until you get used to it. The 4×4 Pfaffians are say, deleting the 5th row and column,

$$\mathrm{Pf}_5 = \mathrm{Pf}_{12.34} = m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23}.$$

In the above construction, y appears linearly in 3 of the Pfaffians (4.3), such as $\mathrm{Pf}_{12.45} = yx_1 - a_2b_3 + a_3b_2$, as the constant of proportionality in Cramer's rule, and the 2 Pfaffians not involving y are the equations (4.2).

Of course, I lied about $X_{2,3}$ being nonsingular. In fact, since X contains the plane Π, it has a number of singularities and cannot be factorial. Generically, the singularities are 7 nodes at the points $A_1 = A_2 = A_3 = 0$. We do not admit X as a Fano 3-fold in the Mori category because it is not $\mathbb{Q}$-factorial (compare [CPR], 4.1). At these points every numerator and denominator of $y = A_i/x_i$ vanishes, so that y is not defined, and the rational map $X \dashrightarrow Y$ involves first blowing up Π to make it a Cartier divisor before contracting it.

As in 2.3, the nonfactorial variety $X_{2,3}$ is the midpoint of a Sarkisov link:

$$\mathbb{P}^2 \dashleftarrow X_{2,3} \dashrightarrow Y. \tag{4.5}$$

Here the left-hand map is obtained by restricting the linear projection map $\mathbb{P}^5 \dashrightarrow \mathbb{P}^2$ given by x_1, x_2, x_3. Since over $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{P}^2$ the two equations

$$\sum \lambda_i a_i = \sum \lambda_i b_i = 0$$

are linear and quadratic in x_i, this is a conic bundle.

To be polite, I write out the broken arrows in the Sarkisov link (4.5):

$$\begin{array}{ccccccc} & & Z & \dashleftarrow & \widetilde{Y} & & \\ & \swarrow & & \searrow\ \swarrow & & \searrow & \\ \mathbb{P}^2 & & & X_{2,3} & & & Y_{3,4} \end{array}$$

Here $\widetilde{Y} \to X$ is the blowup of the plane Π, a flopping extraction that makes Π Cartier. $\widetilde{Y} \to Y$ is just the contraction of $\mathbb{P}^2$ in the nonsingular $\widetilde{Y}$ to a singularity of type $\frac{1}{2}(1,1,1)$. The map $\widetilde{Y} \dashrightarrow Z$ is the flop of the curves over the nonfactorial points of Π, and $Z \to \mathbb{P}^2$ is the conic bundle.

Example 4.6 This example starts similarly, but with the gradings slightly changed. However, the different grading makes a crucial difference to the left hand side of the link. Let $x_1, \dots, x_4, y_1, y_2$ be homogeneous coordinates on the w.p.s. $\mathbb{P}(1^4, 2^2)$, and consider the plane $\Pi = \mathbb{P}^2(x_2, x_3, x_4)$ defined by $(x_1 = y_1 = y_2 = 0)$. A general c.i. $X_{3,4} \subset \mathbb{P}^5(1^4, 2^2)$ containing Π has equations

$$X_{3,4} : \begin{pmatrix} a & b_1 & b_2 \\ c & d_1 & d_2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ y_2 \end{pmatrix} = 0, \tag{4.7}$$

with $\deg a = 3, \deg b_i = \deg c = 2$ and $\deg d_i = 1$. As before, I introduce a new variable

$$y_3 = \frac{b_1 d_2 - b_2 d_1}{x_1} = \text{etc.}$$

as the constant of proportionality in Cramer's rule. This gives the new variety $Y \subset \mathbb{P}(1^4, 2^3)$ defined by the Pfaffians of

$$\begin{pmatrix} a & c & -y_2 & y_1 \\ & y_3 & b_1 & b_2 \\ & & d_1 & d_2 \\ & & & x_1 \end{pmatrix} \quad \text{of degrees} \quad \begin{pmatrix} 3 & 2 & 2 & 2 \\ & 2 & 2 & 2 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}.$$

As before, the rational map $X \dashrightarrow Y \subset \mathbb{P}(1^4, 2^3)$ contracts the plane Π to a singularity of type $\frac{1}{2}$. (I write $\frac{1}{2}$ as an abbreviation for the index 2 singularity $\frac{1}{2}(1,1,1)$ since no ambiguity is possible. The degrees can easily be predicted from the multiplied out Hilbert polynomial $1 - 2t^3 - 3t^4 + 3t^5 + 2t^6 - t^9$, compare [CPR], 7.2.2.)

So far, this is exactly the same as Example 4.1. However, the other side of the link

$$Z \dashleftarrow \left(\Pi \subset X_{3,4} \subset \mathbb{P}(1^4, 2^2)\right) \dashrightarrow Y \subset \mathbb{P}(1^4, 2^3)$$

is completely different, mainly because the plane Π has defining equations $x_1 = y_1 = y_2 = 0$ of different weights: x_1 has degree 1 and vanishes once on Π, whereas the y_i have degree 2 and still vanish only once on Π. Weight-for-weight, x_1 vanishes more. This is crucial in the strategy explained in 1.2, (v) of constructing links via graded rings.

What happens is that the hyperplane $x_1 = 0$ cuts $X_{3,4}$ in $\Pi \cup F$, where the residual surface F is defined by

$$x_1 = 0, \quad \operatorname{rank}\begin{pmatrix} y_2 & b_1 & d_1 \\ -y_1 & b_2 & d_2 \end{pmatrix} \le 1.$$

This is a scroll passing through the two $\frac{1}{2}$ singularities of X, with $(y_1 : y_2)$ the coordinate in the fibre. The other side of the link $Z \dashleftarrow X$ contracts F to a line on a Fano 3-fold $Z_{2,3} \subset \mathbb{P}(1^6)$ passing through 2 ordinary double points (Takagi, [T], Case 2.1). I take up this story again in Example 9.16 as an example of an unprojection of Type III.

5 Kustin–Miller unprojection

The common theme of all the examples of Sections 2–4 was to make a new variety as the graph of a homogeneous form s with pole along a divisor. *Unprojection* does this systematically. The simplest case of unprojection ("Type I") is due to Kustin–Miller [KM] in the early 1980s and Papadakis–Reid [PR].

5.1 The unprojection variable $s \in \mathcal{H}om(\mathcal{I}_D, \omega_X)$

The main idea is as follows: suppose that X and $D \subset X$ are projectively Gorenstein varieties, D has codimension 1 in X and $\dim X \ge 2$. Then the adjunction formula for the Grothendieck dualising sheaf ω_D automatically provides a homogeneous form on X with pole along D. More precisely, assume that

$$\omega_X = \mathcal{O}_X(k_X) \quad \text{and} \quad \omega_D = \mathcal{O}_D(k_D), \quad \text{with} \quad k_X > k_D.$$

Theorem 5.2 ([KM], [PR]) *There is a rational section s of $\mathcal{O}_X(k_X - k_D)$ with pole on D that defines a rational map*

$$X \dashrightarrow Y \subset \mathbb{P}^n[s] = \operatorname{Proj} k[x_0, \dots, x_n, s]$$

taking D to $P_s = (0 : \cdots : 0 : 1)$. Moreover, Y is again projectively Gorenstein.

Sketch proof The dualising sheaf of D is given by the adjunction formula:

$$\omega_D = \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_D, \omega_X).$$

Here the $\mathcal{E}xt$ is calculated by applying the derived functor of $\mathcal{H}om$ to the short exact sequence $\mathcal{I}_D \to \mathcal{O}_X \to \mathcal{O}_D$. Clearly $\mathcal{H}om(\mathcal{O}_D, \omega_X) = 0$ and $\mathcal{E}xt^1(\mathcal{O}_X, \omega_X) = 0$ because $\mathcal{O}_X$ is locally projective, so that we obtain the exact sequence

$$0 \to \omega_X \to \mathcal{H}om(\mathcal{I}_D, \omega_X) \to \omega_D \to 0,$$

where the last map is the Poincaré residue map: if D is a Weil divisor, it can be written in the vulgar form $\mathcal{O}_X(K+D) \to \mathcal{O}_D(K_D)$. Now since $\omega_D = \mathcal{O}_D(k_D)$, we can twist back to obtain

$$\mathcal{H}om(\mathcal{I}_D, \mathcal{O}_X(k_X - k_D)) \to \mathcal{O}_D \to 0.$$

Since also $H^1(\omega_X(i)) = 0$ for all i by the projectively Gorenstein assumption, we deduce that there exists an element

$$s \in \mathcal{H}om(\mathcal{I}_D, \mathcal{O}_X(k_X - k_D))$$

that has residue $1 \in \mathcal{O}_D$. Thus s has divisor of poles exactly D. It is our unprojection variable; it is the same thing as the elements s calculated in an ad hoc way in Sections 2–4, but here it is derived in a systematic way from Grothendieck duality, without any direct calculation.

See [PR] for the proof that Y is projectively Gorenstein. Note that if we write $\mathcal{I}_N(k_X - k_D) = s(\mathcal{I}_D) \subset \mathcal{O}_X(k_X - k_D)$ then N is the divisor of zeros of s. Under $X \dashrightarrow Y$, D is contracted to a point, and N maps isomorphically to the hypersurface section $s = 0$ of Y. The point of the proof in [PR] is that the isomorphism $s\colon \mathcal{I}_{D,X} \cong \mathcal{I}_{N,X}(k_X - k_D)$ (as ideals in the Gorenstein scheme X) implies that D is projectively Gorenstein if and only if N is, and then Y is projectively Gorenstein because its hypersurface section $s = 0$ is isomorphic to N, hence projectively Gorenstein.

Remark 5.3 When contracting a divisor D in a normal variety X, it is traditional to assume that $\mathcal{O}_D(-D)$ is positive in some sense; here I express this as the comparison $k_X > k_D$ between ω_X and ω_D. This comes to the same thing for a Cartier divisor D by the adjunction formula, but is much more powerful in general. We do not need to assume that X is normal (or even reduced), or that D is even a Weil divisor (see [PR], Example 2.2).

The standard proof of Castelnuovo's contractibility criterion consists of persuading a line bundle $\mathcal{L}$ to be very ample outside D but trivial on D, and to have a section that restricts to the generator $1 \in \mathcal{O}_D$. This involves nonsingularity, intersection numbers and cohomology vanishing.

The construction of Theorem 5.2 works instead by finding a section

$$s \in \operatorname{Hom}(\mathcal{I}_D, \omega_X \otimes \mathcal{O}_X(-k_D))$$

whose residue on D generates $\omega_D \otimes \mathcal{O}_X(-k_D)$. There are no considerations of nonsingularity, intersection numbers or cohomology vanishing, just direct use of the projectively Gorenstein assumption on X. The extra power, as so often in my experience, comes from using the raw form of Grothendieck–Serre duality, without trying to interpret ω_X and ω_D in terms of differentials Ω^n_X or line bundles $\mathcal{O}_X(K_X + D)$ as we used to do in centuries past with nonsingular varieties. See Section 9 for generalisations.

6 Applications to Fano 3-folds

Iano-Fletcher [Fl] lists the K3s and anticanonical Fano varieties whose graded rings are hypersurfaces or codimension 2 c.i.s. There are the "famous 95" families of hypersurfaces, and 84 (respectively 85) families of codimension 2 c.i.s. The odd one out here is the remarkable codimension 2 Fano 3-fold

$$X_{12,14} \subset \mathbb{P}(2,3,4,5,6,7)$$

that does not correspond to a family of K3s, because $H^0(-K_X) = 0$.

Examples 4.1 and 4.6 above are typical cases of unprojection from codimension 2 c.i. to codimension 3 Pfaffian. Several dozen more can be found by choosing a codimension 2 c.i. from [Fl], 16.7, Table 6 containing a suitable plane $\mathbb{P}(a_1, a_2, a_3)$ as a divisor. Altınok [A] lists the K3s whose graded rings are codimension 3 Pfaffians (69 families) and codimension 4 rings (115 confirmed families, and another 23 plausible candidates that still require more work). Many of her cases of codimension 4 K3s extend to Fano 3-folds (with some effort). All but a handful of her codimension 3 and confirmed codimension 4 cases are obtained as Type I unprojections.

Remark 6.1 Both Fletcher's and Altınok's lists contain an implicit generality assumption that exclude, for example, the monogonal and hyperelliptic degenerations such as $X_{2,6} \subset \mathbb{P}(1^3, 2, 3)$ and $X_{2,4} \subset \mathbb{P}(1^4, 2)$. It would be interesting to plug this gap; linear systems on K3s are well behaved with a small number of exceptions that are themselves clear-cut dichotomies, so there should only be a couple of dozen new cases.

While not so interesting in themselves, these huge lists of Fano 3-folds are now acquiring some importance, and we search them repeatedly to discover regular patterns (for example, codimension 2 c.i.s that contain a plane $\mathbb{P}(1, a_1, a_2)$ and have only terminal singularities, generalising Examples 4.1–4.6), and then to find the first few cases where that pattern breaks down. Compare Altınok's Example 9.13, which looks like a weighted 5×5 Pfaffian on the basis of its Hilbert polynomial, but fails one little test; all of this can easily be automated. It is a really worthwhile project to make a computer database containing all the known information about the lists in searchable form – at present, it might take several hours' search and calculation to find, say, a codimension 4 example $X \subset \mathbb{P}(1^2, 2^4, a_7, a_8)$ having $7 \times \frac{1}{2}$ and some singularity of index ≥ 5, with multiplied out Hilbert polynomial starting in $1 - 3t^2 - 4t^5 + \cdots$, and expensive taste in cigars. A working first version of this database, programmed by Gavin Brown but based largely on Altınok's thesis [A], will be included in the next export of Magma [Ma] in early summer 2001.

6.2 Takagi's lists

In his Tokyo thesis [T], Takagi Hiromichi gives a systematic treatment of Fano 3-folds with singularities of Gorenstein index 2 and $h^0(-K_X) = g + 2$ with genus $g \geq 2$. This is a major achievement, comparable to the work of Fano, Iskovskikh and others over several decades in the nonsingular case. To simplify, assume that the only singularities are quotient singularities $\frac{1}{2}$. Takagi's lists include several cases of anticanonical 3-folds X embedded in $\mathbb{P}^7(1^a, 2^b)$ with $a + b = 8$ as projectively Gorenstein codimension 4 subvarieties. Consider in particular the following numerical types:

genus	singularities	embedding	Hilbert polynomial
$g=4$	$2 \times \frac{1}{2}$	$X \subset \mathbb{P}(1^6, 2^2)$	$1 - t^2 - 7t^3 + 7t^4 - \cdots + t^9$
$g=3$	$3 \times \frac{1}{2}$	$X \subset \mathbb{P}(1^5, 2^3)$	$1 - 6t^3 - t^4 + 12t^5 - \cdots + t^{10}$
$g=2$	$4 \times \frac{1}{2}$	$X \subset \mathbb{P}(1^4, 2^4)$	$1 - 3t^3 - 6t^4 + 8t^5 + \cdots + t^{11}$

Takagi gives a rigorous geometric treatment of every variety with these invariants. His work is in terms of Mori theory, so that, for example, he constructs varieties, their blowups and morphisms between them using the MMP, rather than by calculating graded rings. However, he pointed out to me that each of these numerical cases in codimension 4 gives rise to 2 different types of variety, and made the beautiful and almost certainly correct prediction that these probably correspond to the two families of unprojection treated by Papadakis [P], that we call *Tom* and *Jerry*. I verify this here in the $g = 4$ cases (the $g = 2$ and $g = 3$ cases would make fun exercises). In this case, projecting from either of the $\frac{1}{2}$ singularities gives a Fano 3-fold $X \subset \mathbb{P}(1^6, 2)$ with multiplied out Hilbert polynomial $1-t^2-4t^3+4t^4+t^5-t^7$, which is a 5×5 Pfaffian given by a matrix of degrees

$$\begin{pmatrix} 2 & 2 & 2 & 2 \\ & 1 & 1 & 1 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}. \tag{6.3}$$

This family of 3-folds X was constructed in Example 4.1 by unprojecting a plane, but now I require that it *contains another plane* Π. By choosing coordinates, I assume $\Pi = \mathbb{P}^2(x_4, x_5, x_6)$, defined by $x_1 = x_2 = x_3 = y_1 = 0$.

The point of Tom and Jerry is this:

There are two quite different ways of putting Π *inside* X.

Example 6.4 (Takagi, No. 4.4) The first method is to assume that the bottom right 4×4 block of the 5×5 matrix consists of linear combinations of the given regular sequence x_1, x_2, x_3, y_1:

$$M = \begin{pmatrix} x_4 & x_5 & x_6 & p \\ & x_1 & x_2 & y_1 \\ & & x_3 & ax_2 \\ & & & bx_1 \end{pmatrix}, \quad \text{with } \deg p = 2,\ \deg a, b = 1. \tag{6.5}$$

The Pfaffians of M then clearly belong to the ideal generated by x_1, x_2, x_3, y_1. One sees in this case that (6.5) is the *general* solution: x_1, x_2, x_3 and y must appear with unit coefficients for reasons of degree, and any other terms can be eliminated by row and column operations. (The terms that survive cannot be eliminated: $m_{24} = x_2$ and $m_{35} = ax_2$ are "Pfaffian partners", with no row or column in common, and the same for $m_{23} = x_1$ and $m_{45} = bx_1$.)

This is the setup for a *Tom unprojection.* Theorem 5.2 asserts that there exists an unprojection variable y_2, a rational homogeneous form of degree 2 with Π as its divisor of poles; however, it does not say how to construct y_2. This problem is solved for the general Tom unprojection in Papadakis' thesis [P]. Here the answer specialises to

$$\begin{aligned} y_2x_1 &= ax_4x_6 - px_5, \\ y_2x_2 &= bx_4x_5 - px_6, \\ y_2x_3 &= bx_5^2 + ax_6^2, \\ y_2y_1 &= abx_4^2 - p^2. \end{aligned}$$

Remark 6.6 In this case, the whole set of 9 equations can be given as the 4×4 Pfaffians of the following 6×6 extrasymmetric matrix

$$\begin{pmatrix} x_4 & x_5 & x_6 & p & y_2 \\ & x_1 & x_2 & y_1 & p \\ & & x_3 & ax_2 & ax_6 \\ & & & bx_1 & bx_5 \\ & & & & abx_4 \end{pmatrix}.$$

The unprojection variable y_2 goes in the top right-hand corner, from whence it multiplies the 4×4 block containing the regular sequence x_1, x_2, x_3, y_1. The matrix is symmetric about the antidiagonal, except that the 356 triangle of entries m_{35}, m_{36}, m_{56} is multiplied by a and the 456 triangle by b. Of its 15 Pfaffians, the last 6 are just repetitions or simple multiples of the first 9.

The mechanism in geometry is that the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ is a (nongeneral) linear section of Grass$(2,6) \subset \mathbb{P}^{14}$; it is a Schubert cycle, the lines of $\mathbb{P}^5$ meeting two copies of $\mathbb{P}^2$ spanning $\mathbb{P}^5$. Thus $\mathbb{P}^2 \times \mathbb{P}^2$ is defined by the Pfaffians of a 6×6 (nongeneral) skew matrix. In algebra, if N is a generic 3×3 matrix, and we write $N = A+B$ with A symmetric and B skew, then the 2×2 minors of N generate the same ideal as the 4×4 Pfaffians of the skew matrix $\left(\begin{smallmatrix} B & A \\ -A & B \end{smallmatrix}\right)$. Multiplying a triangle such as the bottom right triangle 456 by an indeterminate is a flat deformation.

An extrasymmetric matrix of this type appears fairly often with Tom unprojections. To the best of my knowledge, it appeared first in Duncan Dicks' thesis [D] (see also [R1]). However, the general Tom unprojection treated in Papadakis [P] is more general than this 6×6 extrasymmetric format, so don't waste too much time looking for the matrix if it does not want to come out. (Compare the end of Example 7.1, which just fails to have an extrasymmetric format.)

Proposition 6.7 *For general a, b, p, the variety X defined by the Pfaffians of (6.5) is the midpoint of a link*

$$\mathbb{P}^2 \dashleftarrow X \dashrightarrow Y,$$

where $X \dashrightarrow Y$ is the unprojection discussed above that contracts Π to a point of $Y \subset \mathbb{P}(1^6, 2^2)$, and $X \dashrightarrow \mathbb{P}^2$ is a conic bundle defined by the linear system $|-K_X - \Pi|$, or equivalently, the ratio $x_1 : x_2 : x_3$.

In other words, the Fano 3-fold Y and its link $Y \dashrightarrow \mathbb{P}^2$ are in Takagi [T], Case 4.4. I omit the proof. The main point to note is simply that the Pfaffian equations

$$\begin{aligned} \mathrm{Pf}_3: \quad & x_6 y_1 = b x_1 x_4 + x_2 p \\ \mathrm{Pf}_4: \quad & x_5 y_1 = a x_2 x_4 + x_1 p \end{aligned}$$

imply that y_1 vanishes only once on Π so $y_1 \notin H^0(-2K_X - 2\Pi)$. Thus the ring $R(X, -K_X - \Pi)$ is the polynomial ring $k[x_1, x_2, x_3]$.

Example 6.8 (Takagi, No. 1.1) The other way of imposing the plane Π on X is to assume that the 5×5 matrix M has first two rows with all entries in the ideal (x_1, x_2, x_3, y_1); the general solution with degrees (6.3) is

$$M = \begin{pmatrix} y_1 & a_1 & a_2 & a_3 \\ & x_1 & x_2 & x_3 \\ \hline & & x_6 & -x_5 \\ & & & x_4 \end{pmatrix} \quad \text{with} \quad (a_1, a_2, a_3) = (x_1, x_2, x_3)A, \qquad (6.9)$$

where A is a 3×3 matrix with linear entries. In other words, a_1, a_2, a_3 are linear combinations of x_1, x_2, x_3 with coefficients of degree 1. Clearly all the

Pfaffians of M belong to (x_1, x_2, x_3, y_1), so that Theorem 5.2 again implies that there exists an unprojection variable y_2 with poles along Π.

This is the setup for a *Jerry unprojection*. The equations involving y_2 are treated in Papadakis [P]; in general they are much more complicated than those for Tom, but they simplify considerably in the present case. The two Pfaffians of (6.9) not involving y_1 are bilinear in x_1, x_2, x_3 and x_4, x_5, x_6:

$$\mathrm{Pf}_{23.45} = (x_1, x_2, x_3)\begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = 0, \quad \text{and} \quad \mathrm{Pf}_{13.45} = (x_1, x_2, x_3)A\begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} = 0.$$

The equations (6.9) can be obtained from these two linear equations in (x_4, x_5, x_6) by solving by Cramer's rule, with unprojection variable y_1 as constant of proportionality (that is, $y_1 x_4 = a_2 x_3 - a_3 x_2$, etc.). On the other hand, I can view them also as two linear equations for 3 unknowns (x_1, x_2, x_3), and solve them with y_2 as constant of proportionality. As usual, this can be written as a 5×5 Pfaffian:

$$\left(\begin{array}{cc|cc} x_3 & -x_2 & x_4 & a_1' \\ & x_1 & x_5 & a_2' \\ & & x_6 & a_3' \\ & & & y_2 \end{array}\right), \quad \text{where} \quad \begin{pmatrix} a_1' \\ a_2' \\ a_3' \end{pmatrix} = A\begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix}.$$

The equation for $y_1 y_2$ turns out to be

$$y_1 y_2 = (x_1, x_2, x_3)A^{\dagger}\begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix},$$

where $A^{\dagger}$ is the adjoint matrix of A.

In this case, the Pfaffian equations say that $y_1 \cdot (x_4, x_5, x_6)$ are quadratics in x_1, x_2, x_3, so that $y_1 \in H^0(-2K_X - 2\Pi)$. Thus the ring $R(X, -K_X - \Pi)$ is the graded ring $k[x_1, x_2, x_3, y_1]$, and X is the midpoint of a link

$$Z \dashleftarrow X \dashrightarrow Y$$

where $Z = \mathbb{P}(1^3, 2)$ is the Veronese cone. Thus Y is Takagi's Case 1.1.

Example 6.10 A more general Jerry unprojection (but still not the most general, see Papadakis [P]) comes from the Pfaffian form:

$$M = \begin{pmatrix} x & a_1 & a_2 & a_3 \\ & b_1 & b_2 & b_3 \\ \hline & & z_3 & -z_2 \\ & & & z_1 \end{pmatrix}, \quad \text{where} \quad \begin{matrix} (a_1, a_2, a_3) = (y_1, y_2, y_3)A, \\ (b_1, b_2, b_3) = (y_1, y_2, y_3)B, \end{matrix} \tag{6.11}$$

with A, B generic 3×3 matrixes. This defines a codimension 3 Gorenstein variety containing the codimension 4 c.i. (x, y_1, y_2, y_3). The same bilinear trick as in Example 6.8 puts the unprojection variable t into a set of Pfaffian equations

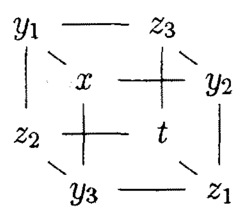
$$\left(\begin{array}{cc|cc} y_3 & -y_2 & b'_1 & a'_1 \\ & y_1 & b'_2 & a'_2 \\ & & b'_3 & a'_3 \\ & & & t \end{array}\right), \quad \text{where} \quad \begin{pmatrix} a'_1 \\ a'_2 \\ a'_3 \end{pmatrix} = A \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b'_1 \\ b'_2 \\ b'_3 \end{pmatrix} = B \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

The "long equation" for xt turns out to be

$$xt = (y_1, y_2, y_3) N(A, B) \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

where $N(A, B)$ is a biquadratic expression[4] in the entries of A, B, and is a moderately horrible mess (although presumably a covariant of the two bilinear forms).

These equations define a flat deformation of the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. To see this, I write the equations of the latter in terms of a little 2×2 cube labelled with the variables



[4]Papadakis has calculated this more accurately, obtaining:

$$xt = \sum (\pm 1) y_{i_1} C_{i_2, j_2} D_{i_3, j_3} z_{j_1} \quad \text{summed over } \{i_1, i_2, i_3\}, \{j_1, j_2, j_3\} = \{1, 2, 3\},$$

where $C = \bigwedge^2 A$ and $D = \bigwedge^2 B$. Compare (11.4.3).

Then the equations of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ are

$$\begin{aligned} xz_i = y_jy_k, \quad ty_i = z_jz_k \quad &\text{for } \{i,j,k\} = \{1,2,3\}, \\ \text{and} \quad xt = y_iz_i \quad &\text{for } i = 1,2,3. \end{aligned}$$

Projecting from t gives 5 equations in the Pfaffian form

$$\begin{pmatrix} x & y_1 & y_2 & 0 \\ & 0 & y_2 & y_3 \\ & & z_3 & z_2 \\ & & & z_1 \end{pmatrix},$$

which is a specialisation of (6.11).

7 Applications to surfaces of general type

Example 7.1 In Reid [R], I calculated the canonical ring of the universal cover Y of a $\mathbb{Z}/3$ Godeaux surface. This is a regular surface with $p_g = 2$, $K^2 = 3$, so that, using the Hilbert series as explained in 3.2, you see that its canonical ring $R(Y, K_Y)$ needs at least

2 generators x_1, x_2 in degree 1,
3 generators y_0, y_1, y_2 in degree 2, and
2 generators z_1, z_2 in degree 3.

Then

$$(1-t)^2(1-t^2)^3(1-t^3)^2P(t) = 1 - 3t^4 - 3t^5 - 3t^6 \\ + 2t^6 + 6t^7 + \cdots + t^{15}$$

(by Gorenstein symmetry, t^{15-k} appears together with t^k). The curious sidestep $-3t^6 + 2t^6$ in this expression is explained as follows: in constructing a plausible model, we expect (or can prove, see [R]) that $|2K_Y|$ is free, and so $R(Y, K_Y)$ is a finite module over the polynomial ring $A = k[x_1, x_2, y_0, y_1, y_2]$, generated by $1, z_1, z_2$. Therefore there must be at least 3 equations in degree 6, expressing z_1^2, z_1z_2, z_2^2 in terms of this basis.

The same ring can be obtained much more simply as a Tom unprojection. Rather amazingly, it is then a deformation of the graded ring over the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$. For this, I start from the equations of $\mathbb{P}^2 \times \mathbb{P}^2$ in a slightly idiosyncratic form

$$\operatorname{rank}\begin{pmatrix} x_0 & y_2 & z_2 \\ z_1 & x_1 & y_0 \\ y_1 & z_0 & x_2 \end{pmatrix} \le 1, \quad \text{that is,} \quad \begin{aligned} x_iz_i &= y_jy_k \\ y_iz_i &= x_jx_k \\ z_jz_k &= x_iy_i \end{aligned} \quad \text{for } \{i,j,k\} = \{1,2,3\}.$$

The first step is to make these equations weighted homogeneous with $\operatorname{wt} x_i, y_i, z_i = 1, 2, 3$. For this, introduce a new variable S with $\operatorname{wt} S = 3$, and modify the equations to

$$\begin{aligned} x_iz_i &= y_jy_k, \\ y_iz_i &= Sx_jx_k, \\ z_jz_k &= Sx_iy_i. \end{aligned} \tag{7.2}$$

Now *project away from* z_0; in other words, separate the 9 equations (7.2) into 4 equations linear in z_0, of the form

$$x_0z_0 = \text{something}, \quad y_0z_0 = \cdots, \quad z_0z_1 = \cdots, \quad z_0z_2 = \cdots,$$

and 5 equations not involving z_0. It is easy to mount the latter as the Pfaffians of the 5×5 skew matrix:

$$\begin{pmatrix} x_1 & x_2 & y_1 & -y_2 \\ & y_0 & z_2 & 0 \\ & & 0 & z_1 \\ & & & Sx_0 \end{pmatrix}$$

I now vary the entries in the bottom right 4×4 block to make them into general linear combinations of x_0, y_0, z_1, z_2:

$$M_0 = \left(\begin{array}{c|ccc} x_1 & x_2 & y_1 & -y_2 \\ \hline & y_0 & z_2 & r_1x_0 \\ & & r_2x_0 & z_1 \\ & & & Sx_0 - r_0y_0 \end{array}\right), \quad \text{with } \operatorname{wt} r_i = 2. \tag{7.3}$$

This is the data for a Tom unprojection, as in Example 6.4. The Pfaffians of M_0 are clearly contained in the ideal generated by the regular sequence x_0, y_0, z_1, z_2, so they define a codimension 3 Gorenstein variety X in the ambient affine space with coordinates x_i, y_i, z_i, S, r_i such that X contains the codimension 4 c.i. $D : (x_0 = y_0 = z_1 = z_2 = 0)$. This means that we can unproject D in X by Theorem 5.2. As before, the explicit equations of the unprojection can be read from Papadakis' thesis:

$$\begin{aligned} x_0z_0 &= y_1y_2 + r_0x_1x_2, \\ y_0z_0 &= Sx_1x_2 - r_2x_1y_2 - r_1x_2y_1, \\ z_0z_1 &= Sx_2y_2 + r_0r_1x_2^2 - r_2y_2^2, \\ z_0z_2 &= Sx_1y_1 + r_0r_2x_1^2 - r_1y_1^2. \end{aligned} \tag{7.4}$$

It is easy to see that the set of 9 equations (7.3–7.4) is symmetric under permuting $\{0,1,2\}$, so that they could be written in terms of the Pfaffians of 3 matrixes like M_0. You can also try to mount them as a 6×6 extrasymmetric Pfaffian (compare Remark 6.6):

$$\begin{pmatrix} x_1 & x_2 & -y_2 & y_1 & z_0 \\ & y_0 & r_1x_0 & z_2 & Sx_1 - r_1y_1 \\ & & z_1 & r_2x_0 & Sx_2 - r_2y_2 \\ & & & Sx_0 - r_0y_0 & r_0r_1x_2 \\ & & & & r_0r_2x_1 \end{pmatrix}$$

This matrix just fails to give the full set of equations: it only gives the equation for x_0z_0 multiplied by S, r_1, r_2.

Note that these equations define a flat deformation of the affine cone over the Segre variety, because they specialise to it on setting $S = 1$ and $r_i = 0$. My surfaces Y from [R] are obtained by setting

$$x_0 + x_1 + x_2 = z_0 + z_1 + z_2 = 0,$$

and r_i = quadratic, S = cubic expressions in x_i, y_i, and the $\mathbb{Z}/3$ action by cyclic permutation of $(0,1,2)$. There is a little cyclotomic change of coordinates to go from the eigencoordinates of [R] to the cyclic permutation coordinates here. The treatment of this example originated in the observation that the equations written out in [R], pp. 86–87 as

$$\begin{array}{llcll} R_0 & x_2z_1 + x_1z_2 & = & y_1y_2 - y_0^2 & + \cdots \\ R_1 & \qquad\quad x_2z_2 & = & y_0y_1 - y_2^2 & + \cdots \\ R_2 & x_1z_1 & = & y_0y_2 - y_1^2 & + \cdots \\ S_0 & y_2z_1 + y_1z_2 & = & (x_1x_2 - x_0^2)s & + \cdots \\ S_1 & \qquad\quad y_2z_2 & = & (x_0x_1 - x_2^2)s & + \cdots \\ S_2 & y_1z_1 & = & (x_0x_2 - x_1^2)s & + \cdots \end{array} \tag{7.5}$$

take the much nicer form (7.2) if you replace them by their cyclotomic combinations $R_0+\varepsilon R_1+\varepsilon^2R_2$ and $S_0+\varepsilon S_1+\varepsilon^2S_2$ (taken over the 3 roots of $\varepsilon^3=1$), and change coordinates to $x_0+\varepsilon x_1+\varepsilon^2x_2$, etc.

Example 7.6 Takagi's list of Fano 3-folds includes

2 codim 4 families 2.2 and 3.3	$X\subset\mathbb{P}(1^4,2^4)$ of degree $2+4\times\frac{1}{2}=4$,
3 codim 5 families 2.3, 3.4, 5.1	$X\subset\mathbb{P}(1^4,2^5)$ of degree $2+5\times\frac{1}{2}=9/2$,
1 codim 6 family 2.4	$X\subset\mathbb{P}(1^4,2^6)$ of degree $2+6\times\frac{1}{2}=5$.

There are almost certainly rather simple unprojection constructions for each of these varieties. They also have sections $S\in|-2K_X|$ that are canonical surfaces with $p_g=4$ and $K_S^2=8,9,10$.

Problem 7.7 Canonical surfaces with invariants in this range have been studied by Ciliberto [Ci] and Catanese [Ca]–[Ca2] from the point of view of generic or "Italian" projection discussed in 1.2, (ii). These examples are interesting test cases to compare the methods and results of Italian versus Gorenstein projection. Thus the treatment of Example 7.1 by Gorenstein projection can be compared to the original treatment of [R], which is a kind of Italian projection: it treats the canonical ring $R(X,K_X)$ as a module over the subring $k[x_1,x_2,y_0,y_1,y_2]$ generated by 1 and the z_i, with the equations (7.5) as defining relations. As another example, it seems clear that Ciliberto's surfaces with $p_g=4$, $K^2=8$ must either form two families, sections of Takagi's Cases 2.2 and 3.3, or only one family which is a section of both. I would very much like to know which of these holds. Compare the del Pezzo surface of degree 6, which is a linear section of *both* $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ and $\mathbb{P}^2\times\mathbb{P}^2$. This bifurcation of cases in going from surfaces to 3-folds seems to be at the heart of the codimension 4 Gorenstein problem. Compare Problem 8.5.

8 Tom and Jerry: who are they?

Many examples of codimension 4 Gorenstein rings with a 9×16 resolution seem to relate to $\mathbb{P}^2\times\mathbb{P}^2$ or $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, although it seems hard at present to say anything precise and general along these lines. Tom unprojections often relate to $\mathbb{P}^2\times\mathbb{P}^2$ and Jerry unprojections to $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$, but the short names have the advantage that they do not imply any immodest claim concerning our current understanding of Gorenstein codimension 4.

Problem 8.1 Give an intrinsic treatment of Tom and Jerry.

Write $C\,\mathrm{Grass}(2,5) \subset \bigwedge^2 \mathbb{C}^5$ for the affine cone over $\mathrm{Grass}(2,5)$, that is, the generic 5×5 Pfaffian variety. It is an almost homogeneous space under $\mathrm{GL}(5,\mathbb{C})$, and in particular has an action of the centre $(\mathbb{C}^*)^5$, which gives many choices of gradings.

A Pfaffian subvariety $X \subset A$ in a regular local scheme $A = \operatorname{Spec} \mathcal{O}$ is the inverse image $X = \varphi^{-1}(C\,\mathrm{Grass}(2,5))$ of $C\,\mathrm{Grass}(2,5)$ under a morphism $\varphi\colon A \to \bigwedge^2 \mathbb{C}^5$. To set up unprojection data, we want X to contain a given codimension 4 c.i. $D : (x_1 = \cdots = x_4 = 0) \subset A$. There is presumably no loss of generality in taking the regular sequence $x_1, \ldots, x_4 \in \mathcal{O}$ as part of a regular system of parameters of $\mathcal{O}$.

Tom and Jerry each achieve $X \supset D$ by requiring that φ take D to a Schubert cell:

Tom The condition on φ is that $\varphi(D)$ consists of 2-dimensional subspaces containing $e_1 = (1,0,0,0,0)$; or $\varphi(D) \subset e_1 \wedge \mathbb{C}^5$. Algebraically, the skew matrix defining X has bottom right 4×4 block contained in the ideal of D:

$$\varphi^*(a_{ij}) \in (x_1, \ldots, x_4) \quad \text{for } i, j \geq 2.$$

Jerry In this case, $\varphi(D)$ must consist of 2-dimensional subspaces contained in $\mathbb{C}^3 = \langle e_3, e_4, e_5 \rangle \subset \mathbb{C}^5$; or $\varphi(D) \subset \mathrm{Grass}(2, \mathbb{C}^3)$. That is, two rows and columns of the matrix are contained in the ideal of D:

$$\varphi^*(a_{ij}) \in (x_1, \ldots, x_4) \quad \text{for } i \leq 2 \text{ or } j \leq 2.$$

The point of the problem, however, is to give also a description in intrinsic terms of the unprojected variety and its equations. Compare Papadakis [P].

Problem 8.2 Do Tom and Jerry account for *every* set of unprojection data $D \subset X \subset A$ where D is a codimension 4 c.i. and X is a 5×5 Pfaffian?

The cone $C\,\mathrm{Grass}(2,5)$ over the whole Grassmann variety does not have any divisors to unproject, so we are going to cut it down a bit by equations forming a regular sequence, but probably not very general, until we get an X with some interesting class group. But it is then a very strong restriction to ask an effective divisor D in X to be a codimension 4 c.i. in the ambient space.

Problem 8.3 Can all the currently known Gorenstein codimension 4 rings with 9×16 resolution be accommodated within Tom and Jerry unprojection structures?

Altınok's treatment of codimension 4 K3 surfaces includes 23 candidates that cannot be obtained as Type I unprojections from codimension 3 (see Example 9.13). In Example 9.14, I discuss a Fano 3-fold, also derived from Altınok's work, that has a Type II projection, but no Type I projection. However, it is quite conceivable that these cases could be part of a bigger variety that does project nicely, by analogy with Example 7.6.

The case that I really do not know how to do at present is Duncan Dicks' "rolling factors format" of Example 10.8 (see also Dicks [D] and Reid [R2], Section 5). If this can't be done, it possibly casts doubt on the whole sentiment of Problem 8.3.

One unresolved issue is whether Jerry (say) is a *structure* in its own right, or a link or *relation* between two structures. Example 10.8 is a kind of structure ("fat $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$") that at present I don't know how to relate to 5×5 Pfaffians by a Jerry unprojection.

Remark 8.4 Kustin and Miller remark that the generic $(2k+1) \times (2k+1)$ Pfaffian is an unprojection (see [KM], p. 311): you can separate the variables into m_{12} and the remaining m_{ij}, and view the two Pfaffians not involving m_{12} as defining a codimension 2 c.i., and those involving m_{12} linearly as unprojection equations. (See Example 4.1 for a 5×5 case.)

I want to stress that this only works as stated for a *sufficiently general* matrix. In fact, the generic $(2k+1) \times (2k+1)$ Pfaffian variety is the $(k-1)$st secant variety of $\mathrm{Grass}(2, 2k+1)$, because a skew form of rank $\le 2k-2$ can be written as a sum $\sum e_i \wedge f_i$ of $k-1$ forms of rank 2. The projection eliminates the variable m_{12}; for it to work, m_{12} must be algebraically independent of the other m_{ij}. Geometrically, the point $P_{m_{12}} = (1, 0, \dots, 0)$ must be in the variety in order to act as a centre of a projection. In other words, a (possibly nongeneric) $(2k+1) \times (2k+1)$ Pfaffian variety X can only have a projection of Type I if it has a point of the smallest possible rank 2, that is, a point of $\mathrm{Grass}(2, 2k+1)$.

More generally, to see a variety as an unprojection, you must first find a suitable centre of projection, and you may well have to put your variety in a bigger one first before this is possible. This happened in both Example 7.1 and Example 7.6. Thus in Example 7.6, if you only consider the surface S, you cannot see the $\frac{1}{2}$ singularities of the 3-fold X, and thus its unprojection structure. For example, in the codimension 4 cases, the 4 new generators y_i in degree 2 can be thought of as a dual basis to the $4 \times \frac{1}{2}$ singularities. If we take $\sum_{i=1}^{4} y_i = 0$, we have lost all the possible centres of Gorenstein projection.

A reasonable conclusion is that the dimension of a variety is not a very significant invariant in these constructions, and it is a mistake to concentrate only on curves or surfaces or 3-folds. Instead, one should work with notions such as codimension, coindex, genus, multiplied out Hilbert polynomial, homological properties, etc., that are invariant or transform in a simple way on taking a hyperplane section, and work for preference with a "key variety" that is as fat as possible.

Problem 8.5 How do Tom and Jerry intersect? As with their celluloid namesakes, scenes in which Tom and Jerry appear together are on the whole more interesting than their solo performances. As mentioned at the end of Problem 7.7, this includes the famous deformation theory of the del Pezzo surface of degree 6.

8.6 Why Pfaffians?

As I tell my students, mounting a set of half-understood equations in the form of Pfaffians is much more fun than doing crosswords, and moreover, has some intellectual content. Apart from personal addiction, there are several other reasons why Pfaffians turn up throughout this kind of calculation:

1. They are an effective and simple way of handling syzygies. If you have written down two or three equations, and suspect that you have probably missed one or two more, you have to do things like $x_1 f_2 - x_2 f_1$ to cancel some leading term and make the combination a pure multiple of x_3. At the end of it, you have f_3 and f_4 with some simple linear identities. Most frequently, the equations themselves can be written as 2×2 minors or 4×4 Pfaffians in a way that gives the 3-term or 4-term syzygies in an automatic way.

   As we see in Section 11, there are serial unprojection rings of arbitrary codimension determined by a representative set of equations and syzygies given as 5×5 Pfaffians.

2. Most current questions on Gorenstein rings are concerned with small codimension, meaning $3, 4, 5, 6$, and in particular with unprojecting

from codimension 2 or 3 to codimension 4 or 5 or 6. The prominence of Pfaffians in this study is not surprising in view of the Buchsbaum–Eisenbud theorem. Pfaffians bigger than 5×5 tend not to appear in this study because they give varieties of high coindex. The $(2k+1)\times(2k+1)$ unweighted case already has coindex $2k-2$; for example, the simplest Gorenstein graded ring over a surface with 7×7 Pfaffian structure is the canonical surface $S_{14}\subset\mathbb{P}^5$.

3. The 4×4 Pfaffians are the Plücker equations defining $\mathrm{Grass}(2,n)$. There is a natural progression 2×2 minors $\to$ Pfaffians $\to$ the quadratic equations defining the codimension 5 spinor variety $\mathrm{Spin}(5,10)\subset\mathbb{P}^{15}$ (or orthogonal Grassmann variety, see Mukai [Mu]). Just as a 4×4 Pfaffian is a trinomial that you can think of as $m_{12}m_{34}-\left|\begin{smallmatrix} m_{13} & m_{14}\\ m_{23} & m_{24}\end{smallmatrix}\right|$, each spinor equation is a 4-nomial that you can think of as

$$x_1x_2 - 4\times 4 \text{ Pfaffian}.$$

Problem 8.7 As a Pfaffian addict, I can't wait to start on the codimension 5 rings, where the spinor equations play a similarly prominent role. According to Mukai [Mu], the spinor coordinates ξ_I on the spinor space (= even Clifford algebra) $\mathbb{C}^{16}=\bigwedge^0(\mathbb{C}^5)\oplus\bigwedge^2(\mathbb{C}^5)\oplus\bigwedge^4(\mathbb{C}^5)$ are indexed by even subsets of $\{1,2,3,4,5\}$. The equations defining the spinor variety $\mathrm{Spin}(5,10)\subset\mathbb{P}^{15}$ are the 10 spinor equations $N_{\pm i}$, typically

$$\begin{aligned} N_1 &= \xi_\phi\xi_{2345}-\xi_{23}\xi_{45}+\xi_{24}\xi_{35}-\xi_{25}\xi_{34}\\ N_{-1} &= \xi_{12}\xi_{1345}-\xi_{13}\xi_{1245}+\xi_{14}\xi_{1235}-\xi_{15}\xi_{1234}.\end{aligned}$$

$\mathrm{Spin}(5,10)\subset\mathbb{P}^{15}$ has coindex 3 and the multiplied out Hilbert polynomial $1-10t^2+16t^3-16t^5+10t^6-t^8$ (the same as for a canonical curve of genus 7, or a nonsingular Fano 3-fold of genus 7). In the unweighted case, as for a nonsingular Fano 3-fold, a point projection has the wrong discrepancy (see [PR], 2.7); a Fano style projection from a line should go to a 5×5 Pfaffian containing a cubic scroll, providing the first case of a Type III unprojection from codimension 3 to codimension 5.

That was the unweighted form. We can find many weighted homogeneous forms, because the affine cone $C\,\mathrm{Spin}(5,10)\subset\mathbb{C}^{16}$ has an action of $\mathrm{GL}(5)$ and of its centre $(\mathbb{C}^*)^5$. It would be interesting to find weighted K3s and Fanos as sections of the weighted spinor varieties. This is the simplest structure for codimension 5 Gorenstein rings, analogous to the codimension 4 structure

5×5 Pfaffian intersect a hypersurface (think of the del Pezzo surface of degree 5). The next problem is then to find some nice examples of links by projection from these varieties to lower codimension.

9 Harder unprojections

We can relax the assumption in Theorem 5.2: there is no special need for the divisor D to be Gorenstein in order to unproject it. The best way to think of this is from the top: Fano and the generations following him project nonsingular Fano 3-folds from a line, a conic, or project doubly from a point (compare the discussion in 1.2). From [CPR] and Takagi [T] onwards, we can do more exotic projections from points or curves on Fanos in the Mori category. The exceptional divisor is usually not projectively Gorenstein.

I discuss here two families of examples:

Type II In this case D is not projectively Cohen–Macaulay, because it is not projectively normal, but the normalisation $\mathcal{O}_{\widetilde{D}} = \mathcal{O}_D \oplus \mathcal{O}_D \cdot t$ needs only one module generator, and moreover, $\mathcal{O}_{\widetilde{D}}$ is Gorenstein. This arises in connection with the elliptic involution of [CPR], 4.10–4.12 and 7.3, and with several interesting codimension 4 K3s and Fano 3-folds from Altınok's lists [A]. See Examples 9.5–9.14. It is really a generic phenomenon for slightly nonnormal embeddings $\mathbb{P}(a_1, a_2, b) \hookrightarrow \mathbb{P}(a_1, a_2, a_3, \dots, a_n)$ between w.p.s.s.

Type III In this case, D is projectively Cohen–Macaulay, but $\omega_D(n_0)$ is generated by *two* sections that define a fibre space structure $\varphi \colon D \to \mathbb{P}^1$. That is, D is homologically like a cubic scroll, so Cohen–Macaulay but not Gorenstein. The typical case is Fano's projection of a Fano 3-fold from a line, with the cubic scroll as exceptional divisor. Corti suggested treating the inverse rational map as a new type of unprojection, and calculated the first cases himself. See Example 9.16, where I conclude the story begun in Example 4.6 based on Takagi, Case 2.1.

Remark 9.1 By assigning roman numerals, I am certainly not suggesting a case division or classification. Rather, these are certain pathologies that turn up frequently, and that we can begin to handle alongside the Kustin and Miller Type I cases. Based on experience of projecting Fano 3-folds from different centres (and seeking the unprojection giving the left-hand side of the corresponding link), I believe that D can be really very bad from the point of view of commutative and homological algebra. There are certainly cases when D is a badly nonnormal scroll, or when $\mathcal{O}_{\tilde{D}}$ is Gorenstein but needs many generators as an $\mathcal{O}_D$ module.

Problem 9.2 Find the best theorem of the following shape. I state the problem in the local setup. Compare [PR], 2.4 for the translation from local to projective.

> *X is a local Gorenstein scheme and $D \subset X$ a subscheme of pure codimension 1. The adjunction formula for ω_D gives the usual exact sequence*
>
> $$0 \to \omega_X \to \mathcal{H}om(\mathcal{I}_D, \omega_X) \to \omega_D \to 0.$$
>
> *Identifying $\omega_X = \mathcal{O}_X$ interprets elements of the $\mathcal{H}om$ as rational functions on X with poles along D. Pick a set of generators $s_i \in \mathcal{H}om(\mathcal{I}_D, \omega_X)$, say with $s_0 = \mathrm{id}\colon \mathcal{I}_D \subset \mathcal{O}_X = \omega_X$. As in [PR], Lemma 1.1, assume without loss of generality that the s_i are injective and have divisor of poles exactly D.*
>
> *Define the unprojection (ring) of D in X by*
>
> $$\mathcal{O}_Y = \mathcal{O}_X[s_1, \dots, s_n]/(\text{relations}) \quad \textit{and} \quad Y = \operatorname{Spec} \mathcal{O}_Y. \tag{9.3}$$
>
> *Then under suitable (fairly mild) conditions, Y is a Gorenstein scheme.*

It is part of the problem to say what the ideal of relations in (9.3) should be. When it turns out that Y is birational to X, we could just take the relations between the s_i holding in the total ring of fractions $k(X)$, but in general Y may have new components.

Maybe we should find the relations by studying a few examples; it would be really cool if all the relations were determined by linear ones. Conjecture 9.12 suggests that in some easy cases, we should look for linear relations in the s_i and certain fairly simple and predictable quadratic relations yoked to them by Pfaffians.

In the projective setup, in view of the applications, I want to assume that D is a codimension 1 subscheme of a projectively Gorenstein scheme X, and that there is a threshold value $k_D \in \mathbb{Z}$ with $k_X > k_D$ for which $\omega_D(-k_D)$ is still generated by its H^0, but the resulting linear system is not big, so that the morphism $\varphi_{\omega_D(-k_D)}$ contracts D to a smaller dimensional variety.

As before, this means that the elements $s_i \in \mathcal{H}om(\mathcal{I}_D, \omega_X(-k_D))$ whose residues generate $\omega_D(-k_D)$ become homogeneous forms with poles along D, and have positive degree $k_X - k_D$ under the identification $\omega_X = \mathcal{O}_X(k_X)$.

Remark 9.4 As in Remark 5.3, the assumption $k_X > k_D$ is a negativity condition on $D \subset X$. Note the fortunate circumstance that the s_i correspond to generators of $\omega_D(-k_D) = \omega_{\widetilde{D}}(-k_D)$, which is good even if D is not normal. A condition expressed in terms of $\mathcal{O}_D$ would be much worse in this respect. This is another advantage of the approach via Grothendieck–Serre duality.

In the modern view, we usually expect this kind of canonical threshold to be a rational number. But $k_D \in \mathbb{Q}$ does not seem to make sense here. (Or could it somehow?)

9.5 Key variety for Type II unprojections

I give a generic form for Type II unprojections, as a preparation for the following examples. Consider the morphism

$$\begin{array}{rcl} \pi\colon \widetilde{D} = \mathbb{C}^{n+1} & \to & D \subset \mathbb{C}^{2n+1} \\ (x_1, \dots, x_n, t) & \mapsto & (x_i, y_i = x_i t, z = t^2) \end{array}$$

that folds the t axis in half, identifying $\pm t$. The image D is a nonnormal toric variety, with coordinate ring the subring $k[D] \subset k[x_1, \dots, x_n, t]$ obtained by outlawing odd pure powers of t. Its equations are

$$\operatorname{rank} N \le 1, \quad \text{where} \quad N = \begin{pmatrix} y_1 & \cdots & y_n & x_1 z & \cdots & x_n z \\ x_1 & \cdots & x_n & y_1 & \cdots & y_n \end{pmatrix}; \tag{9.6}$$

that is, the n^2 equations

$$\begin{cases} x_i y_j - x_j y_i & \text{for } i < j, \\ y_i y_j - x_i x_j z & \text{for } i \le j, \end{cases} \qquad \text{where} \qquad i, j = 1, \dots, n. \tag{9.7}$$

The moral purpose of the equations (9.6) is of course to ensure that $t = y_i/x_i = x_i z/y_i$ is a well defined rational function on D with $t^2 = z$.

I want to treat D as a key variety for a whole series of nonnormal varieties. For $n \ge 2$ it is not Cohen–Macaulay, because its normalisation happens in codimension ≥ 2. Although $\mathcal{O}_D$ is not Cohen–Macaulay as a ring or as a $\mathcal{O}_{\mathbb{C}^{2n+1}}$-module, its normalisation $\pi_*\mathcal{O}_{\tilde{D}}$ is Gorenstein as a ring, and hence as an $\mathcal{O}_{\mathbb{C}^{2n+1}}$-module. Moreover, $\omega_D = \pi_*\omega_{\tilde{D}} \cong \pi_*\mathcal{O}_{\tilde{D}}$, since the dualising sheaf is saturated in codimension 2. From now on, I suppress π_*. Thus ω_D is generated by the single element

$$\overline{s}_0 = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}t$$

over the bigger ring $\mathcal{O}_{\tilde{D}}$ but needs two generators $\overline{s}_0$ and $\overline{s}_1 = t\overline{s}_0$ over $\mathcal{O}_D$ or over $\mathcal{O}_{\mathbb{C}^{2n+1}}$. The Gorenstein $\mathcal{O}_{\mathbb{C}^{2n+1}}$ module $\mathcal{O}_{\tilde{D}} \cong \omega_D$ has a nice presentation:

$$0 \leftarrow \mathcal{O}_{\tilde{D}} \leftarrow \mathcal{O} \oplus \mathcal{O}\cdot t \xleftarrow{N} 2n\mathcal{O} \xleftarrow{P} 2\tbinom{n}{2}\mathcal{O} \leftarrow \cdots$$

where N is the $2 \times 2n$ matrix of (9.6), viewed as relations $(\overline{s}_1, \overline{s}_0)N = 0$, and P is made up of pairs of 4-term syzygies

$$\begin{pmatrix} -y_j & -x_j z \\ y_i & x_i z \\ -x_j & -y_j \\ x_i & y_i \end{pmatrix}.$$

Let $X \subset \mathbb{C}^{2n+1}$ be the general codimension n c.i. containing D, defined by n general linear combinations of the equations (9.7). Then (for $n = 2, 3$, and I conjecture for all n), D unprojects in X by adjoining two elements $s_0, s_1 \in \mathcal{H}om(\mathcal{I}_D, \omega_X)$ with residue $\overline{s}_0, \overline{s}_1 \in \omega_D$.

There should be two proofs, exercises in generalising the respective constructions of [KM] and [PR]. For the former, we have 3 complexes

$$\begin{array}{ccccccccc}
0 & \leftarrow & \mathcal{O}_X & \leftarrow & \mathcal{O} & \xleftarrow{f_i} & n\mathcal{O} & \leftarrow & \cdots \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \leftarrow & \mathcal{O}_D & \leftarrow & \mathcal{O} & \xleftarrow{m_{i,j}} & n^2\mathcal{O} & \leftarrow & \cdots \\
 & & \cap & & \cap & & \downarrow & & \\
0 & \leftarrow & \mathcal{O}_{\tilde{D}} & \leftarrow & \mathcal{O} \oplus \mathcal{O}\cdot t & \xleftarrow{N} & 2n\mathcal{O} & \leftarrow & \cdots
\end{array}$$

resolving $\mathcal{O}_X$, $\mathcal{O}_D$ and $\mathcal{O}_{\tilde{D}}$; of these, the top is the Koszul complex of the c.i. $X : (f_1 = \cdots = f_n = 0)$, and the bottom is the resolution of $\mathcal{O}_{\tilde{D}}$, which has length $n+1$ and Gorenstein symmetry because $\mathcal{O}_{\tilde{D}}$ is a Gorenstein module.

The middle complex is much messier, but we only need it as far as length $n+1$, where it computes $\omega_D = \omega_{\widetilde{D}}$.

The unprojection variables s_0 and s_1 come from putting together the homomorphisms at the end of these complexes much as in [KM]: the linear relations involving s_0, s_1 are

$$ {}^tN \begin{pmatrix} s_0 \\ s_1 \end{pmatrix} = \mathrm{col}(A_1, \dots, A_n, B_1, \dots, B_n), $$

where ${}^tN : (\mathcal{O} \oplus \mathcal{O} \cdot t)^\vee \to (2n\mathcal{O})^\vee$ and the column vector is the nth composite downarrow $\mathcal{O} \to 2n\mathcal{O}$. There is also a single quadratic relation $s_0^2 z - s_1^2 = \cdots$, which is of course the tricky point referred to in Problem 9.2.

In what follows, I restrict to the cases $n = 2, 3$.

Example 9.8 The case $n = 2$ was worked out together with Corti in the context of constructing elliptic involutions of Fano hypersurfaces, and is written up in [CPR], 4.10–4.12 and 7.3. Then $D \subset \mathbb{C}^5$ has codimension 2, and the hypersurface $X \subset \mathbb{C}^5$ containing D has equation a general linear combination

$$ A(x_1y_2 - x_2y_1) + B(y_1^2 - x_1^2 z) + 2C(y_1y_2 - x_1x_2z) + D(y_2^2 - x_2^2 z) $$

of the defining equations (9.7). To unproject D in X, write the adjunction formula for ω_D as usual

$$ 0 \to \omega_X \to \mathcal{H}om(\mathcal{I}_D, \omega_X) \to \omega_D \to 0. $$

Then we need two new generators $s_0, s_1 \in \mathcal{H}om(\mathcal{I}_D, \omega_X)$ to hit the two generators $\overline{s}_0, \overline{s}_1$ of ω_D. The linear relations between these are given by

$$ \begin{pmatrix} y_1 & x_1 & x_1C + x_2D \\ y_2 & x_2 & x_1B - x_2C \\ x_1z & y_1 & x_1A - y_1C - y_2D \\ x_2z & y_2 & x_2A - y_1B + y_2C \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ 1 \end{pmatrix} = 0. \tag{9.9} $$

It so happens in this case that Y has codimension 3, and is the 5×5 Pfaffian

$$ \begin{pmatrix} x_1 & x_2 & y_1 & y_2 \\ & s_0 & -D & -s_1 + C \\ & & s_1 + C & B \\ & & & s_0 z + A \end{pmatrix} = 0. \tag{9.10} $$

See [CPR], 7.3 for further discussion and applications. As explained in 9.5, the matrix in (9.9) can be interpreted as a map between complexes in Kustin and Miller style (although we did not know this at the time).

Example 9.11 When $n = 3$, the variety $X \subset \mathbb{C}^7$ is a codimension 2 c.i. $f = g = 0$ containing the nonnormal variety D.

$$\begin{tikzcd}
0 & \mathcal{O}_X \arrow[l] \arrow[d] & \mathcal{O} \arrow[l] \arrow[d, equal] & 2\mathcal{O} \arrow[l, "{f,g}"'] \arrow[d] & \mathcal{O} \arrow[l, "{\substack{-g \\ f}}"'] \arrow[d] & 0 \arrow[l] \\
0 & \mathcal{O}_D \arrow[l] \arrow[d, phantom, "\cap"] & \mathcal{O} \arrow[l] \arrow[d, phantom, "\cap"] & 9\mathcal{O} \arrow[l, "m_{i,j}"'] \arrow[d] & 20\mathcal{O} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 & \mathcal{O}_{\tilde{D}} \arrow[l] & 2\mathcal{O} \arrow[l] & 6\mathcal{O} \arrow[l, "N"'] & 6\mathcal{O} \arrow[l, "P"'] & 2\mathcal{O} \arrow[l, "{}^tN"'] & 0 \arrow[l]
\end{tikzcd}$$

Here $2\mathcal{O}$ stands for $\mathcal{O} \oplus \mathcal{O} \cdot t$ and its dual, N is the matrix of proportionality relations (9.6) for t, and

$$P = \begin{pmatrix} x_3z & -x_2z & 0 & y_3 & -y_2 \\ & x_1z & -y_3 & 0 & y_1 \\ & & y_2 & -y_1 & 0 \\ & & & x_3 & -x_2 \\ & & & & x_1 \end{pmatrix}$$

its first syzygy matrix.

Conjecture 9.12 Sorry, there is no time to finish writing up this proof properly. I have checked it in Magma [Ma]. The problem is to understand better the quadratic relations between s_0, s_1.

In the case $n = 2$ of Example 9.8, the 5th Pfaffian in (9.10) is quadratic in s_0, s_1:

$$\mathrm{Pf}_{23.45} = s_0^2 z - s_1^2 + As_0 + C^2 + BD.$$

This is not linear, and so not properly accounted for by the construction in terms of maps between complexes. However, the Pfaffian syzygies in (9.10) express each of x_1, x_2, y_1, y_2 times $\mathrm{Pf}_{23.45}$ as a combination of the linear relations (9.9).

The same must hold for all n: there is a single quadratic relation Q that expresses $s_0^2 z - s_1^2$ as an element of $\mathcal{O}_X + \mathcal{O}_X s_0 + \mathcal{O}_X s_1$, and Pfaffians syzygies express Q times $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ as elements of the ideal generated by the linear equations.

Example 9.13 (Altınok) Several of the harder cases in Altınok [A] can be settled using these ideas. Consider a K3 surface X with

$$D^2 = -2 + 7 \times \frac{1}{2} + \frac{4}{5}.$$

This relates closely to the example of a Fano 3-fold of genus -1 and singularities $7 \times \frac{1}{2}$ plus $\frac{1}{5}(1,1,4)$. Simple-minded use of the Hilbert series as in Altınok [A1] gives $\mathbb{P}(2,2,3,4,5,5)$ with coordinates y_1, y_2, z, t, u_1, u_2 as the first guess for the generators, and the multiplied up Hilbert function

$$(1-t^2)^2(1-t^3)(1-t^4)(1-t^5)^2 P(t) = \\ 1 - t^7 - 2t^8 - t^9 - t^{10} + t^{11} + \cdots - t^{21}$$

suggests that, like so many of its colleagues before it, X is a Pfaffian with matrix having weights

$$\begin{pmatrix} 2 & 3 & 3 & 4 \\ & 4 & 4 & 5 \\ & & 5 & 6 \\ & & & 6 \end{pmatrix}.$$

However, this is nonsense: the graded ring of X necessarily has a generator in some degree $\equiv 1 \bmod 5$ to act as a local coordinate at its singularity $\frac{1}{5}(1,4)$. Alternatively, in the alleged Pfaffian model, we can assume that the two entries m_{25} and m_{34} are the degree 5 generators, say $m_{25} = u_1$ and $m_{34} = u_2$, so that any Pfaffian must meet the (u_1, u_2) line at the two coordinate points. At P_{u_1}, the only equations involving u_1 are the 3 Pfaffians containing the terms $u_1 m_{13}$, $u_1 m_{14}$ and $u_1 m_{34}$; but m_{13} and m_{14} both have degree 3, and are thus proportional, because there is only one generator in degree 3. Thus in these degrees, a Pfaffian cannot be quasismooth, and in fact must have at least an elliptic singularity at P_{u_1}.

In this case, the thing that really happens is a codimension 4 embedding $X \hookrightarrow \mathbb{P}(2,2,3,4,5,5,6)$ with coordinates $y_1, y_2, z, t, u_1, u_2, v$. It passes through the point P_{u_1} and has a singularity of type $\frac{1}{5}(4,1)$ there, with t, v as local coordinates. Notice how the nonnormality arises (compare [CPR], 3.4 for local eigencoordinates and their multiplicities on the blown up locus): the local eigencoordinate t at P_{u_1} has weight 4 for the grading of the ring, and has $\mathbb{Z}/5$ weight 4, so vanishes along the blown up curve E with multiplicity $\frac{4}{5}$, which is just right. But the other local eigencoordinate v has graded weight 6 but $\mathbb{Z}/5$ weight 1, and so vanishes along the blown up curve with multiplicity only $\frac{1}{5}$, and so is not in the subring $R_1 \subset R$ of the blowup (compare the discussion in 1.2, (v), and [CPR], Example 4.11). Thus projecting from P_{u_1} eliminates v together with u_1. The image of the projection is the codimension 2 c.i. $Y_{6,10} \subset \mathbb{P}(2,2,3,4,5)$ containing the weighted line $\mathbb{P}(1,4)$ in a nonnormal embedding.

Consider the general embedding

$$\begin{array}{rcll} \mathbb{P}(1,4) & \hookrightarrow & \mathbb{P}(2,2,3,4,5) & \text{given by} \\ (x,t) & \mapsto & (y_1 = x^2, y_2 = 0, z = x^3, t, u = xt), \end{array}$$

or equivalently, the general homogeneous homomorphism from the polynomial ring $k[y_1, y_2, z, t, u]$ to $k[x,t]$. This maps onto every monomial except x, and so models the $\mathbb{P}(1,4)$ extracted from the $\frac{1}{5}(1,4)$ singularity of X. On the other hand, the equations of the image E are

$$y_2 = 0 \quad \text{and} \quad \operatorname{rank}\begin{pmatrix} z & u & y_1^2 & y_1 t \\ y_1 & t & z & u \end{pmatrix} \le 1.$$

This is in a family with the equations (9.6). If $Y_{6,10} \subset \mathbb{P}(2,2,3,4,5)$ is a K3 c.i. containing E, it can be unprojected by adapting the method of 9.5. An alternative strategy is first to construct the Fano 3-fold (see below), then take its section by the single element of degree 1.

The related Fano 3-fold can be projected in a similar way to a Fano 3-fold $W_{6,10} \subset \mathbb{P}(1,2,2,3,4,5)$ containing the nonnormal $\mathbb{P}(1,1,4)$, given by the equations

$$\operatorname{rank}\begin{pmatrix} y_2 & z & u & xy_1 & y_1^2 & y_1 t \\ x & y_1 & t & y_2 & z & u \end{pmatrix} \le 1.$$

This unprojects as in Example 9.11 to a Fano $V \subset \mathbb{P}(1,2,2,3,4,5,5,6)$.

Example 9.14 (Altınok) Anthony Fletcher discovered the codimension 2 c.i. $X_{12,14} \subset \mathbb{P}(2,3,4,5,6,7)$ with $H^0(-K_X) = 0$ at the end of his 1988 Ph.D. thesis [Fl0], as the result of a systematic search. At the end of her thesis 10 years later [A], Selma Altınok discovered 3 more plausible candidates for Fano 3-folds having $H^0(-K_X) = 0$, with codimension at least 4. All three of these are very interesting; we are fairly certain that they exist, and we intend to write them up when time allows. Here I discuss the second of her $H^0(-K_X)=0$ cases. It has singularities $7\times\frac{1}{2}$, $\frac{1}{3}(1,1,2)$, $\frac{1}{8}(1,3,5)$ and genus $g=-2$. That is, $H^0(-K_X)=0$, and

$$(-K_X)^3 = 2g-2+\sum\frac{a(r-a)}{r} = -6+7\times\frac{1}{2}+\frac{2}{3}+\frac{3\cdot 5}{8}=\frac{1}{24}.$$

The Hilbert function calculation (see [A1]) suggests the 8 plausible generators in degrees $2,3,4,5,6,7,8,9$, and, assuming these, the multiplied out Hilbert polynomial is

$$1\quad -2t^{12}-t^{13}-2t^{14}-2t^{15}-t^{16}\quad +2t^{19}+2t^{20}+3t^{21}+\cdots+t^{43}$$

Note however, that although we have specified a singularity of type $\frac{1}{8}(1,3,5)$, the ring has no global element of degree 1 to act as a local eigencoordinate (see [CPR], 3.4). This suggests that we look for a Type II unprojection.

If we write u,v,w for coordinates of $\mathbb{P}(1,3,5)$ and x,v,y,w,z,t for coordinates of $\mathbb{P}(2,3,4,5,6,7)$, the general map $\mathbb{P}(1,3,5)\hookrightarrow\mathbb{P}(2,3,4,5,6,7)$ is the embedding given by

$$x=u^2,\quad v=v,\quad y=uv,\quad w=w,\quad z=uw,\quad t=u^7.$$

That is, we omit u,u^3,u^5 from the polynomial ring $k[u,v,w]$, so that this is a pullback from the D of Example 9.11. Thus, provided that the general $Y_{12,14}$ containing the image $\Pi\cong\mathbb{P}(1,3,5)$ is reasonably nonsingular, the construction of Example 9.11 applies to this to construct X.

Problem 9.15 The nonsingularity calculation is always the nasty part of these constructions. The equations of the image $\Pi=\mathbb{P}(1,3,5)$ are

$$\operatorname{rank}\begin{pmatrix} y & z & t & vx & wx & x^4\\ v & w & x^3 & y & z & t\end{pmatrix}\le 1,$$

that is,

$$\begin{array}{lllll} y^2=v^2x, & yw=vz, & yz=vwx, & yx^3=vt, & yt=vx^4,\\ z^2=w^2x, & zx^3=wt, & zt=wx^4, & t^2=x^7 & \end{array}$$

These are equations of degree $8,9,10,10,11,12,12,13,14$. A general c.i. $Y_{12,14}$ containing $\Pi\cong\mathbb{P}(1,3,5)$ is given by choosing two general linear combinations of these, and it seems likely that $Y_{12,14}$ has only fairly mild singularities on Π.

You should get most of this from Bertini's theorem. You can also try it by computer algebra: plug in random coefficients, write out the equations and its ideal of 2×2 minors, and try to prove that the singularity locus defined by this is contained in Π and consists of fairly mild singularities.

Example 9.16 (Takagi) This is an example of unprojection of Type III. As we saw in Example 4.6, a codimension 2 c.i. $X_{3,4} \subset \mathbb{P}^5(1^4, 2^2)$ containing the plane $\Pi : (x_1 = y_1 = y_2 = 0)$ has defining equations $f = g = 0$, where

$$(f, g) = \begin{pmatrix} x_1 & y_1 & y_2 \end{pmatrix} \begin{pmatrix} a & c \\ b_1 & d_1 \\ b_2 & d_2 \end{pmatrix} = 0,$$

with $\deg a = 3, \deg b_i = \deg c = 2$ and $\deg d_i = 1$. The hyperplane $(x_1 = 0)$ intersects X in the plane Π together with a residual component F defined by

$$F : \left\{ x_1 = 0 \quad \text{and} \quad \operatorname{rank} \begin{pmatrix} -y_2 & b_1 & d_1 \\ y_1 & b_2 & d_2 \end{pmatrix} \leq 1 \right\}. \tag{9.17}$$

I show how to unproject F by introducing two homogeneous forms s_1, s_2 of degree 1 on X with poles along F, related by linear equations only. I calculate s_1, s_2 and the relations between them by copying the method of Kustin and Miller [KM]. This generalisation of [KM] is based on a suggestion of Alessio Corti, who calculated nonsingular Fano 3-folds of genus $g = 6, 7$ by unprojecting a cubic scroll in a Fano of genus $g - 2$.

The resolution of $\mathcal{O}_F$ in the hyperplane $(x_1 = 0)$ is given by the 2×3 matrix in (9.17) and its minors. Set $h = b_1 d_2 - b_2 d_1$ for the 3rd equation. One sees that in the whole space, the resolution of $\mathcal{O}_F$ is

$$\begin{aligned} 0 \leftarrow \mathcal{O}_F \leftarrow \mathcal{O} \xleftarrow{v} \mathcal{O}(-1) \oplus 2\mathcal{O}(-3) \oplus \mathcal{O}(-4) \xleftarrow{M} \\ 2\mathcal{O}(-4) \oplus 3\mathcal{O}(-5) \xleftarrow{U} 2\mathcal{O}(-6) \leftarrow 0, \end{aligned}$$

where $v = (x_1, h, g, -f)$, and M, U are

$$\begin{pmatrix} h & g & -f & ad_1 - b_1 c & ad_2 - b_2 c \\ -x_1 & 0 & 0 & -y_2 & y_1 \\ 0 & -x_1 & 0 & b_1 & b_2 \\ 0 & 0 & -x_1 & d_1 & d_2 \end{pmatrix}, \quad \begin{pmatrix} -y_2 & y_1 \\ b_1 & b_2 \\ d_1 & d_2 \\ x_1 & 0 \\ 0 & x_1 \end{pmatrix}.$$

Now to calculate $\mathcal{E}xt^1(\mathcal{O}_F, \omega_X)$, I write out the homomorphism from the resolution of $\mathcal{O}_X$ to that of $\mathcal{O}_F$:

$$\begin{array}{ccccccccc}
\mathcal{O}_X & \leftarrow & \mathcal{O} & \xleftarrow{g,-f} & \mathcal{O}(-3)\oplus\mathcal{O}(-4) & \xleftarrow{\binom{f}{g}} & \mathcal{O}(-7) & \leftarrow & 0 \\
\downarrow & & \| & & \downarrow & & \downarrow & & \\
\mathcal{O}_F & \leftarrow & \mathcal{O} & \xleftarrow{v} & \begin{matrix}\mathcal{O}(-1)\oplus 2\mathcal{O}(-3)\\ \oplus\mathcal{O}(-4)\end{matrix} & \xleftarrow{M} & \begin{matrix}2\mathcal{O}(-4)\\ \oplus 3\mathcal{O}(-5)\end{matrix} & \xleftarrow{U} & 2\mathcal{O}(-6),
\end{array}$$

where the two downarrows are

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ -a \\ -c \\ y_1 \\ y_2 \end{pmatrix}.$$

Thus I introduce two unprojection variables s_1, s_2 corresponding to the summands of $2\mathcal{O}(-6)$, and write out the linear equations as

$$\begin{pmatrix} -y_2 & y_1 \\ b_1 & b_2 \\ d_1 & d_2 \\ x_1 & 0 \\ 0 & x_1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -a \\ -c \\ y_1 \\ y_2 \end{pmatrix}.$$

Remark 9.18 This type of unprojection adds two new generators s_1, s_2, so could be used to go from codimension 2 to codimension 4 (with resolution 7×12 or smaller). However, it so happens in this case that two of the equations eliminate $y_1 = s_1 x_1$ and $y_2 = s_1 x_1$. Thus the unprojection takes

$$X \dashrightarrow Z_{2,3} : \begin{pmatrix} a + b_1 s_1 + b_2 s_2 = 0 \\ c + d_1 s_1 + d_2 s_2 = 0 \end{pmatrix} \subset \mathbb{P}^5(x_1, \dots, x_4, s_1, s_2). \tag{9.19}$$

As in 2.3, the image is not the *general* $Z_{2,3}$ because s_1, s_2 only appear in the equations (9.19) either explicitly, or via the substitution $y_i \mapsto x_1 x_i$. If for example $b_1 = y_1 + \cdots$ and $b_2 = -y_2 + \cdots$ then X meets the (y_1, y_2) line in two $\frac{1}{2}$ singularities at $y_1 = \pm y_2$, and the first equation of (9.19) is

$$x_1(s_1 - s_2)(s_1 + s_2) = -a. \tag{9.20}$$

Thus $Z_{2,3}$ contains the (s_1, s_2) line L, and in general has two ordinary double points at $s_1 = \pm s_2$.

The inverse map $Z_{2,3} \dashrightarrow X$ is a good example of how the classical idea of linear projection has to be modified to deal with singularities (compare 1.2, (v)). The linear projection from the (s_1, s_2) line L is the map defined by $x_1, \dots, x_4$; in addition to blowing up L, this blows up the two ordinary double points, so is not a primitive extraction in the Mori category. Instead, we make the graded ring by imposing the nth symbolic power of the ideal $\mathcal{I}_L$ on $|-nK_Z|$; this means that we eliminate s_1, s_2, but add $y_1 = x_1s_1$ and $y_2 = x_1s_2$, which vanish twice at the general point of L by (9.20).

10 Gorenstein in codimension 4 – the elusive structure theory

Unprojection has played the role of a *substitute* for a structure theory for Gorenstein rings in codimension 4 (or 5, etc.) throughout the above. We have seen in many examples that it can frequently be used as a reasonably effective way of working with graded rings in low codimension, despite the absence of a general structure theory.

Here I want instead to discuss what we actually know about the structure theory. Most of what I say is almost obvious, but I have not seen it written down in this form. Suppose for simplicity that the ambient space $A = \operatorname{Spec} \mathcal{O}_A$ is a regular local scheme with $\frac{1}{2} \in \mathcal{O}_A$ that is *complete* (or Spec of a polynomial ring localised at the origin, with everything graded in positive degrees). Let $X \subset A$ be a Gorenstein subscheme of codimension $c = 2, 3, 4, 5, 6$, etc., and write

$$\begin{array}{ccccccccc} 0 & \leftarrow & \mathcal{L}_0 & \leftarrow & \mathcal{L}_1 & \leftarrow & \mathcal{L}_2 & \leftarrow & \cdots \\ & & \| & & \| & & \| & & \\ \mathcal{O}_X & \leftarrow & \mathcal{O}_A & \xleftarrow{f_i} & (k+1)\mathcal{O}_A & \xleftarrow{M} & m\mathcal{O}_A & \leftarrow & \cdots \end{array} \tag{10.1}$$

for a free resolution. That is, $(f_1, \dots, f_{k+1})$ is a minimal set of defining equations, and $(f_1, \dots, f_{k+1})M = 0$ the complete set of m first syzygies. For example, if $\operatorname{codim} X = 4$ then $m = 2k$, and we have a $(k+1) \times 2k$ resolution for some $k = 3, 5, 6, 7, \dots$

The point to notice is that, somewhat paradoxically, the matrix M of first syzygies always has more structure and contains more information than the equations f_i themselves. In what follows, please bear in mind the case $c = 3$: then the Buchsbaum–Eisenbud theorem says that k is even, say $k = 2n$, M is a skew $(2n+1) \times (2n+1)$ matrix, and its $2n \times 2n$ minors are the products $\mathrm{Pf}_i \mathrm{Pf}_j$ of the diagonal $2n \times 2n$ Pfaffians $f_i = \mathrm{Pf}_i$, and thus they generate the square of the ideal I_X. A rough qualitative deduction from this is that when the rank of M drops, it drops by $c - 1$ all at one go, and its submaximal minors vanish $c - 1$ times on this locus. This holds generally.

Theorem 10.2 *(i)* rank $M_P = k$ *at every (scheme theoretic) point* $P \in A \setminus X$.

Assume in addition that X *is locally c.i. at every generic point of* X *(this certainly holds if* X *is reduced). Then*

(ii) M_P *has rank* $\le k + 1 - c$ *at every point* $P \in X$ *(where* $c = \operatorname{codim} X$*), with equality where* X *is l.c.i.*

(iii) The ideal sheaf generated by the $k \times k$ *minors of* M *restricted to the l.c.i. locus equals* ${\mathcal{I}_X}^{c-1}$*, the* $(c-1)$*st power of* $\mathcal{I}_X$*. Thus every* $k \times k$ *minor of* M *vanishes* $c - 1$ *times at every generic point of* X*, so that the ideal they generate is contained in the symbolic power* ${I_X}^{[c-1]}$*.*

Problem 10.3 (1) Is the ideal of $k \times k$ minors equal to ${I_X}^{c-1}$ in commonly occurring examples? This always holds in codimension 3. For Gorenstein codimension 4, one checks that the ideal of submaximal minors of M is the cube of I_X in several of the more popular cases. It would be fairly simple to try out a few more cases by computer algebra.

(2) Does the analog of the conclusion (iii) holds without the l.c.i. assumption, for example, when $X \subset A$ is a badly nonreduced cluster (that is, 0-dimensional scheme).

Proof Almost obvious. Localised at $P \notin X$, the ideal sheaf $\mathcal{I}_X = \mathcal{O}_A$, so that each homomorphism in (10.1) splits locally as projection and inclusion of direct summands.

Next, localised at any point $P \in X$ at which X is l.c.i., the ideal sheaf $\mathcal{I}_X$ is generated locally by c equations $x_1, \dots, x_c$, with the remaining $k + 1 - c$ equations expressed as local $\mathcal{O}_A$-linear combinations of the x_i. This means that the matrix of syzygies has a square block of size $k+1-c$ with unit determinant:

$$M \quad \sim \quad \begin{pmatrix} \boxed{\begin{matrix}\text{unit block of}\\ \text{size } k+1-c\end{matrix}} & 0 & 0 \\ 0 & \boxed{\begin{matrix}\text{Koszul matrix of}\\ x_1,\dots,x_c\end{matrix}} & 0 \end{pmatrix}$$

Now the set of $(c-1)\times(c-1)$ minors of a Koszul matrix of a sequence $x_1,\dots,x_c$ is identically equal to $\{0\}$ union the set of monomials of degree $c-1$ in $x_1,\dots,x_c$. For example, if $c=4$, the Koszul matrix is

$$\begin{pmatrix} 0 & x_3 & -x_2 & -x_4 & 0 & 0 \\ -x_3 & 0 & x_1 & 0 & -x_4 & 0 \\ x_2 & -x_1 & 0 & 0 & 0 & -x_4 \\ 0 & 0 & 0 & x_1 & x_2 & x_3 \end{pmatrix};$$

and obviously, every 3×3 minor of this is zero or a cubic monomial, and every cubic monomial appears.

10.4 Codimension 4

In this case, the whole resolution (10.1) is of the form

$$\begin{array}{ccccccccccc} 0 & \leftarrow & L_0 & \xleftarrow{f} & L_1 & \xleftarrow{M} & L_2 & \leftarrow & L_3 & \leftarrow & L_4 & \leftarrow & 0 \\ & & & & & & \| & & \| & & \| & & \\ & & & & & & L_2^\vee & \xleftarrow{M^\vee} & L_1^\vee & \xleftarrow{f^\vee} & L_0^\vee & & \end{array}$$

with M a $(k+1)\times 2k$ matrix. The Buchsbaum–Eisenbud symmetriser trick gives the identifications $L_4 = L_0^\vee$ and $L_3 = L_1^\vee$, and gives a symmetric perfect pairing $L_2 \times L_2 \to L_4 \cong \mathcal{O}_A$ making the identifications commute.

Lemma 10.5 *Under the stated conditions, L_2 together with its perfect pairing is isomorphic to $2k\mathcal{O}_A$ with the standard quadratic form $\left(\begin{smallmatrix}0 & I\\ I & 0\end{smallmatrix}\right)$.*

Proof I can find an isotropic vector in L_2 by successively lifting from the residue field A/m to A/m^n, as in the proof of Hensel's lemma. That is why I assumed A is complete. If $v \in L_2$ is a solution mod m^n, using the fact that the pairing is nondegenerate, I can edit it to $v+v'$ with $v' \in L_2 \otimes (m^n/m^{n+1})$ which is a solution[5] mod m^{n+1}. Then just copy the usual reduction of quadratic forms in linear algebra.

Thus the dual map $L_3 \to L_2$ is given by $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right){}^tM$, and the condition for the composite $L_3 \to L_2 \to L_1$ to be zero (to give a complex) is then simply that

$$M\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right){}^tM = 0.$$

In other words, the rows of M are $k+1$ vectors in L_2 over $\mathcal{O}_A$ that are isotropic with respect to the standard quadratic form. Recall that an isotropic linear subspace of a nondegenerate quadratic form has dimension $\leq k$, so that, any $k+1$ vectors spanning an isotropic subspace must be linearly dependent. If I have a matrix M over an ambient space A representing a family of $k+1$ vectors that span an isotropic subspace, and if M has generic rank k, the linear relation holding between the vectors is generically unique, so that $\operatorname{coker} M$ is a rank 1 sheaf over A.

Theorem 10.6 ("Structure theorem") *Under the above assumptions:*

(iv) For given $k \geq 3$, write

$$V = \left\{ M \;\middle|\; M \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} {}^tM = 0 \right\} \subset \operatorname{Mat}_{\mathbb{C}}(k+1, 2k).$$

Thus V is the universal family of complexes $L_1 \leftarrow L_2 \leftarrow L_3$ with the Gorenstein symmetry described under 10.4.

If $X \subset A$ is a codimension 4 Gorenstein subscheme in a regular local ambient space A over $\mathbb{C}$, with $(k+1) \times 2k$ resolution, then the middle terms of the complex resolving $\mathcal{O}_X$ (plus choices of bases) defines a morphism $\varphi\colon A \to V$ with the properties that $\varphi(P)$ is a matrix of rank k for P in codimension ≤ 3, the complex pulled back to A is exact at $\mathcal{L}_2$, and $(\operatorname{coker} M)^{**}$ is a locally free sheaf of rank 1.

(v) The converse of (iv) holds.

[5] This "proof" needs expanding.

Proof There is almost nothing to prove. Given $X \subset A$, it has a resolution, so a map to V with the stated properties. For the converse, everything is contained in the assumptions, that is, the several conclusions stated in (iv): given a map to V, I get a complex $\mathcal{L}_1 \leftarrow \mathcal{L}_2 \leftarrow \mathcal{L}_3$ with the Buchsbaum–Eisenbud symmetry property, and the rest of the complex $\mathcal{L}_0 \leftarrow \mathcal{L}_1$ comes for free from the inclusion $\operatorname{coker} M \hookrightarrow (\operatorname{coker} M)^{**}$, using the statements in (iv). This inclusion defines an ideal sheaf defining a subscheme X with support in codimension ≥ 4, and the complex is exact of length 4, hence X is a Gorenstein subscheme of codimension 4.

Remark 10.7 The theorem says that a Gorenstein subscheme $X \subset A$ in codimension 4 with $(k+1) \times 2k$ resolution is the degeneracy locus of a family M of $k+1$ vectors spanning an isotropic subspace, with the family satisfying suitable "generality" assumptions. In other words, all you have to do is "something generic in linear algebra". The theorem is thus a formal analog of the Buchsbaum–Eisenbud theorem in codimension 3. The rank, exactness and locally free assumptions can all be expressed in terms of the height (or codimension) of ideals of minors of M.

The theorem is unfortunately completely useless in practice, since it does not prescribe any way of actually filling in the matrix M over an ambient space A. For example, a initial result on Gorenstein codimension 4 is that there are no ideals with 5 as their minimum number of generators. I do not know how to derive even this elementary result from my so-called structure theorem. Part of the difficulty is that we are talking not just about the universal variety V, but about maps M from A to V.

Example 10.8 (Dicks' format) Dicks [D] proposes a universal "rolling factors" format for Gorenstein varieties with 9×16 resolution (see also Reid [R2], Section 5). For this, consider some regular ambient space A in which

$$\sum_{i=1}^{4} \alpha_i z_i \equiv \sum_{i=1}^{4} \beta_i y_i \tag{10.9}$$

holds "as an identity". He writes down the system of 9 equations

$$\operatorname{rank}\begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \end{pmatrix} \quad \text{and} \quad \begin{aligned} &\textstyle\sum_{i=1}^4 \alpha_i y_i = 0 \\ &\textstyle\sum_{i=1}^4 \alpha_i z_i \equiv \sum_{i=1}^4 \beta_i y_i = 0 \\ &\textstyle\sum_{i=1}^4 \beta_i z_i = 0 \end{aligned}$$

The last 3 equations are in *rolling factors format*, meaning that we go from the first to the second and from the second to the third by substituting $y_i \mapsto z_i$ in one factor. Note that Dicks' identity (10.9) is precisely the condition for the 9 rows m_i of the syzygy matrix

$$\begin{pmatrix}
y_2 & -z_2 & -y_3 & z_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_0 & \alpha_0 & -\beta_1 & -\alpha_1 \\
-y_1 & z_1 & \cdot & \cdot & y_3 & -z_3 & \cdot & \cdot & \cdot & \cdot & \beta_0 & \alpha_0 & \cdot & \cdot & -\beta_2 & -\alpha_2 \\
\cdot & \cdot & y_1 & -z_1 & -y_2 & z_2 & \cdot & \cdot & \beta_0 & \alpha_0 & \cdot & \cdot & \cdot & \cdot & -\beta_3 & -\alpha_3 \\
y_0 & -z_0 & \cdot & \cdot & \cdot & \cdot & -y_3 & z_3 & \cdot & \cdot & \beta_1 & \alpha_1 & -\beta_2 & -\alpha_2 & \cdot & \cdot \\
\cdot & \cdot & -y_0 & z_0 & \cdot & \cdot & y_2 & -z_2 & \beta_1 & \alpha_1 & \cdot & \cdot & -\beta_3 & -\alpha_3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & y_0 & -z_0 & -y_1 & z_1 & \beta_2 & \alpha_2 & -\beta_3 & -\alpha_3 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & z_3 & \cdot & z_2 & \cdot & z_1 & \cdot & z_0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & z_3 & -y_3 & z_2 & -y_2 & z_1 & -y_1 & z_0 & -y_0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -y_3 & \cdot & -y_2 & \cdot & -y_1 & \cdot & -y_0 & \cdot
\end{pmatrix}$$

to span an isotropic subspace of the standard quadratic form $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. In fact, all the scalar products $m_i \cdot m_j$ cancel out by trivial skewsymmetry, except that $v_1 \cdot v_6$, $v_2 \cdot v_5$, $v_3 \cdot v_4$ work out to be plus or minus $\sum \alpha_i z_i - \sum \beta_i y_i$.

To achieve the condition (10.9), one way is to take a 4×4 symmetric matrix (a_{ij}) and set $\alpha_i = \sum a_{ij} y_j$ and $\beta_i = \sum a_{ij} z_j$. This system of equations includes the case of a hypersurface $X_{d,d+2} \subset \mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ in the Segre embedding: indeed, take coordinates t_1, t_2 in $\mathbb{P}^1$ and $x_0, \dots, x_3$ coordinates in $\mathbb{P}^3$. The equation of $X_{d,d+2}$ is given by a bihomogeneous form

$$h \in S^d(t_1, t_2) \otimes S^{d+2}(x_0, \dots, x_3).$$

I can write h (in many ways) as a quadratic form $h = \sum a_{ij} x_i x_j$ in the x_i, with coefficients a_{ij} that are bihomogeneous of degree (d, d) in t_1, t_2 and $x_0, \dots, x_3$. Then the substitution $t_1 x_i \mapsto y_i$, $t_2 x_i \mapsto z_i$ expresses

$$f_0 = t_1^2 h = \sum a_{ij} y_i y_j, \quad f_1 = t_1 t_2 h = \sum a_{ij} y_i z_j, \quad f_2 = t_2^2 h = \sum a_{ij} z_i z_j.$$

Note that specialising $\sum a_{ij} x_i x_j \mapsto x_0 x_2 - x_1 x_3$ exhibits these equations as a flat deformation of the affine cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

The "identity" (10.9) in whatever ring the $\alpha_i, \beta_i, y_i, z_i$ have their values illustrates the problem implicit in Theorem 10.6 of how to map a regular ambient space A into the singular variety V that is the universal space of complexes. When y_i, z_i are not independent, there are of course many other ways of satisfying (10.9), and this format includes other anticanonical divisors in scrolls. Compare Stevens [S].

11 $\mathbb{G}_m$ covers of Mori flips of Type A

11.1 Introduction

This section is a brief outline of a joint paper in preparation with Gavin Brown [BR], that develops the ideas of my old preprint and notes "What is a flip?" [wF] and Brown's thesis [B], [B1]. The idea is to study a Mori flip $X \searrow Y \swarrow X^+$ in terms of the $\mathbb{Z}$-graded ring $R(Y, K_Y)$ arising as the canonical and anticanonical algebra of the two sides of the flip. The $\mathbb{Z}$-graded ring $R = R(Y, K_Y)$, or the corresponding affine variety $\operatorname{Spec} R$ together with the $\mathbb{G}_m$ action coming from the $\mathbb{Z}$-grading is called the $\mathbb{G}_m$ *cover* of the flip (I write $\mathbb{G}_m = \mathbb{C}^*$ for the multiplicative group).

If we assume that the general elephant $S \in |-K_Y|$ of the flip is a Du Val singularity, the $\mathbb{G}_m$ cover of S is an affine Gorenstein 3-fold that we can take as a known object. For a Type A flip it is a toric variety. The problem translates into how to deform this to a 4-fold whose $\mathbb{G}_m$ quotient has only terminal singularities. We carry out this deformation by introducing a class of "double-headed toric varieties" V_{ABLM} (the terminology is currently under construction – we apologise to the reader for any discomfort and hope to resume normal service shortly). These are affine 6-folds, determined by combinatorics and serial unprojection, that have a 4-parameter family of $\mathbb{G}_m$ actions, and that play the role of "key varieties" containing the $\mathbb{G}_m$ covers of Mori flips of Type A. (Here I only give examples, but we conjecture that all Mori flips of Type A can be covered in this way.) The main point of this section is to introduce the varieties V_{ABLM} as examples of serial unprojection, and I really only discuss the basic setup and our main results as far as required to make this point.

Although our work could in principle be presented as a logically self-contained treatment of flips (modulo assumptions or conjectures on the existence of flips, and the nature of their general elephant), it is more reasonable to see it as an attempt to get to grips with and reinterpret a brilliant calculation of Mori [M] dating back to the early 1980s, describing flips of the Mori category whose general elephant is an A_n singularity. Mori has explained this calculation to me on several occasions since 1986, and it provides motivation, logical foundation and a frequently invoked sanity check for our work.

Definition 11.1.1 I adopt the following narrow definition of 3-fold flips, that is sufficient for present purposes: a *flipping contraction* is a projective morphism $f\colon X \to Y$, where X is a quasiprojective 3-fold with $\mathbb{Q}$-factorial terminal singularities, f contracts a single irreducible curve $C \subset X$ to a point $P \in Y$ and is an isomorphism on the complement, and $-K_X$ is relatively ample for f. A *Mori flip* is a diagram

$$\begin{tikzcd} X \arrow[dr] & & X^+ \arrow[dl] \\ & Y & \end{tikzcd} \tag{11.1.1}$$

where $X \to Y$ is a flipping contraction and $X^+ \to Y$ a projective morphism from a quasiprojective 3-fold with $\mathbb{Q}$-factorial terminal singularities extracting a single curve C^+ with K_{X^+} relatively ample on X^+. We assume that $Y = \operatorname{Spec} R_0$ is affine, or even a local analytic neighbourhood of $P \in Y$.

For any nonzero element $t \in H^0(Y, \mathcal{O}_Y(-K_Y))$, the divisor $\operatorname{div} t = S \subset Y$ is a Gorenstein surface, called an *elephant* of Y. It is a theorem of Mori and Kollár and Mori that for a general choice of t the elephant S is a Du Val singularity. It follows from this that t defines a diagram of subvarieties

$$\begin{tikzcd} S^- \arrow[dr] & & S^+ \arrow[dl] & & X \arrow[dr] & & X^+ \arrow[dl] \\ & S & & \subset & & Y & \end{tikzcd} \tag{11.1.2}$$

with $S^- \subset X$ and $S^+ \subset X^+$ also the divisor of t, and $S^- \to S$ and $S^+ \to S$ crepant partial resolutions of the Du Val singularity of S.

The flip (11.1.1) is of *Type A* if its general elephant $S \subset Y$ is a Du Val singularity of Type A.

For simplicity in setting up the graded ring, assume first that K_X is a generator of the class group of X (the more general case is discussed in

Remark 11.2.2). Define

$$R_n = H^0(Y, \mathcal{O}_Y(nK_Y)) = \begin{cases} H^0(X, nK_X) & \text{if } n \le 0, \\ H^0(X^+, nK_{X^+}) & \text{if } n \ge 0, \end{cases} \tag{11.1.3}$$

(the multiplication is most easily defined at the level of $\mathcal{O}_Y(nK_Y)$) and set

$$R = \bigoplus_{n\in\mathbb{Z}} R_n, \quad R_- = \bigoplus_{n\le 0} R_n, \quad \text{and} \quad R_+ = \bigoplus_{n\ge 0} R_n.$$

Then R is a $\mathbb{Z}$-graded Gorenstein ring. Both R_- and R_+ are finitely generated rings, and according to [wF], the flip diagram (11.1.1) is the $\mathbb{Z}$-graded Proj R, meaning that $X = \operatorname{Proj}_Y R_-$, $Y = \operatorname{Spec} R_0$ and $X^+ = \operatorname{Proj}_Y R_+$. The grading of R defines an action of $\mathbb{G}_m$ on the corresponding affine variety Spec R, and the three varieties Y, X and X^+ are different GIT interpretations of the quotient $(\operatorname{Spec} R)/\mathbb{G}_m$ under different notions of "stability". The case when R is a hypersurface treated by Gavin Brown [B], [B1] already leads to examples of flips with an interesting diversity of behaviour.

11.2 Mori's codimension 2 example

Consider the codimension 2 c.i. $V \subset \mathbb{C}^6$ defined by

$$x_1y_0 = x_0^e u^\alpha + t^{\mu e} \quad \text{and} \quad x_0y_1 = u^\beta + x_1^d t^{\lambda d}, \tag{11.2.1}$$

where x_0, x_1, y_0, y_1, t, u are coordinates on $\mathbb{C}^6$ and $d, e, \alpha, \beta, \lambda, \mu$ are given positive integers, with λ, μ coprime. Write

$$R = \mathbb{C}[V] = \mathbb{C}[x_0, x_1, y_0, y_1, t, u]/(\text{equations (11.2.1)}).$$

for the coordinate ring of V. Specify a monomial $\mathbb{G}_m$ action on $\mathbb{C}^6$ and on V by setting

$$\begin{array}{rcl} u & \mapsto & u \\ t & \mapsto & g^{-1}t \end{array} \quad \text{and} \quad \begin{array}{rcl} x_1 & \mapsto & g^\lambda x_1 \\ y_1 & \mapsto & g^\mu y_1 \\ x_0 & \mapsto & g^{-\mu} x_0 \\ y_0 & \mapsto & g^{-\lambda-\mu e} y_0 \end{array} \quad \text{for} \quad g \in \mathbb{G}_m.$$

This makes R into a $\mathbb{Z}$-graded ring, assigning weights

$$\begin{aligned} &\operatorname{wt} u = 0, \quad \operatorname{wt} t = -1, \\ &\operatorname{wt} x_1 = \lambda, \quad \operatorname{wt} y_1 = \mu, \quad \operatorname{wt} x_0 = -\mu, \quad \operatorname{wt} y_0 = -\lambda - \mu e. \end{aligned} \tag{11.2.2}$$

In fact, since the two equations (11.2.1) are assumed to be homogeneous, the given weights of u and t determine everything up to a finite torsion ambiguity (this is discussed further in Remark 11.2.2).

We can now calculate the $\mathbb{Z}$-graded $\operatorname{Proj} R$ and obtain a flip diagram (11.1.1): R_- is clearly generated over R_0 by the generators of negative weight x_0, y_0 and t, and a pure power of t appears in the first equation of (11.2.1), so that $X = \operatorname{Proj}_Y R_-$ is covered by two affine pieces:

$$X_{x_0\neq 0} = \left(\begin{array}{rcl} x_1 y_0 & = & u^\alpha + t^{\mu e} \\ y_1 & = & \text{function} \end{array} \right) \Big/ \frac{1}{\mu}(\lambda, -\lambda, 0, -1),$$

$$X_{y_0\neq 0} = \left(\begin{array}{rcl} x_1 & = & x_0^e u^\alpha + t^{\mu e} \\ x_0 y_1 & = & u^\beta + (x_0^e u^\alpha + t^{\mu e})^d t^{\lambda d} \end{array} \right) \Big/ \frac{1}{\lambda + \mu e}(-\mu, \mu, 0, -1).$$

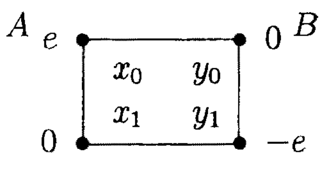
The fractional notation $\frac{1}{\mu}(\lambda, -\lambda, 0, -1)$ means the cyclic group $\mathbb{Z}/\mu$ acting on x_1, y_0, u, t by the characters $\varepsilon^\lambda, \varepsilon^{-\lambda}, 1, \varepsilon^{-1}$, where $\varepsilon = \exp \frac{2\pi i}{\mu}$. This is the standard way of choosing an affine cover and inhomogeneous coordinates on each affine piece of a $\mathbb{G}_m$ quotient, just as for w.p.s.s (compare Fletcher [Fl], 5.3). Both of these are standard terminal singularities of Type A, so that $\operatorname{Proj} R$ is a Mori flip of Type A.

Remark 11.2.1 One observes in this and other examples that $u^\alpha, u^\beta, t^\lambda, t^\mu$ appear as indivisible tokens. It thus simplifies the notation to replace them by independent variables A, B, L, M, giving rise to the codimension 2 c.i. $V_{ABLM} \subset \mathbb{C}^8(x_0, x_1, y_0, y_1, A, B, L, M)$ defined by

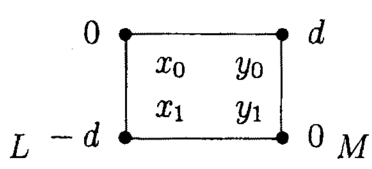
$$x_1 y_0 = x_0^e A + M \quad \text{and} \quad x_0 y_1 = B + x_1^d L. \tag{11.2.3}$$

Thus V_{ABLM} is a 6-fold; in fact, $V_{ABLM} \cong \mathbb{C}^6$, because we can just solve for M and B. We can view V_{ABLM} as a "key variety", and (11.2.1) as obtained by pulling back the key variety by a morphism $A = u^\alpha$, $B = u^\beta$, $L = t^{\lambda e}$, $M = t^{\mu d}$.

In the general notation for the two-headed toric 6-folds V_{ABLM} introduced below, the equations (11.2.3) are encoded in the pair of rectangles



and



Remark 11.2.2 In setting up the graded ring over a flip in (11.1.3), mainly for simplicity of notation, I assumed that K_X is a generator of the class group of X. More generally, K_X may be divisible in $\operatorname{Cl} X$, or $\operatorname{Cl} X$ may have torsion, so that K_X is only a $\mathbb{Q}$-generator. As mentioned in Definition 1.1, in this case we just use the slightly bigger ring

$$R(Y) = \bigoplus_{D \in \operatorname{Cl} X} H^0(Y, \mathcal{O}_Y(D)),$$

which is graded by $\operatorname{Cl} X \cong \mathbb{Z} \oplus \text{torsion}$.

The more general case is conveniently illustrated in Mori's Example 11.2. Choose positive integers $d, e, \alpha, \beta, \lambda, \mu$ with λ, μ coprime, and a common factor $\delta \mid \operatorname{hcf}(d, e)$. Define a codimension 2 c.i. $V = \operatorname{Spec} R \subset \mathbb{C}^6$, replacing (11.2.1) by

$$x_1 y_0 = x_0^e u^\alpha + t^{\mu e/\delta} \quad \text{and} \quad x_0 y_1 = u^\beta + x_1^d t^{\lambda d/\delta}.$$

Write μ_n for the cyclic group of nth roots of 1. The more general monomial action of $\mathbb{G}_m \times \mu_{de}$ on $\mathbb{C}^6$ and on V is given by[6]

$$\begin{array}{rcl} u & \mapsto & u \\ t & \mapsto & g^{-\delta} t \end{array} \quad \text{and} \quad \begin{array}{rcl} x_1 & \mapsto & g^\lambda \varepsilon_1 x_1 \\ y_1 & \mapsto & g^\mu \varepsilon_2 y_1 \\ x_0 & \mapsto & g^{-\mu} {\varepsilon_2}^{-1} x_0 \\ y_0 & \mapsto & g^{-\lambda - \mu e} {\varepsilon_1}^{-1} y_0 \end{array}$$

for $g \in \mathbb{G}_m$ and $\varepsilon_1 \in \mu_d, \varepsilon_2 \in \mu_e$. This just takes into account the finite ambiguity in the weights mentioned in (11.2.2).

The key variety V_{ABLM} itself is not affected by the generalisation, only the pullback and the choice of the group action. For this reason I suppress the generalisation in most of what follows (it is easily restored).

11.3 One long rectangle

As explained in (11.1.2), the general elephant of a flip $X \searrow Y \swarrow X^+$ of Type A is a diagram of subvarieties $S^- \searrow S \swarrow S^+$, with S a Du Val

[6] This treatment is too hurried. I have notes and a letter from Mori somewhere doing it properly. I should choose $n = \operatorname{lcm}(d, e)$, and $\varepsilon_1 \in \mu_n$, $\varepsilon_2 \in \mu_n$ in a coherent way in order that the action of $\mathbb{G}_m \times \mu_n$ has isolated fixed points and corresponds to the dual of the class group of Y.

singularity of Type A_n, and the two sides $S^- \searrow S \swarrow S^+$ crepant partial resolutions extracting at most one curve. (The general elephant $S^- \subset X$ need not contain the flipping curve, in which case $S^- \to S$ is an isomorphism. Compare Remark 11.5.1.)

The $\mathbb{G}_m$ cover of the elephant $S^- \searrow S \swarrow S^+$ is a Gorenstein affine toric 3-fold $V_u = \operatorname{Spec} k[\sigma \cap M]$ whose cone of monomials $\sigma \subset M_{\mathbb{R}}$ is the quadrilateral cone of Figure 11.3.1. Here the x_i, y_j and u are monomial

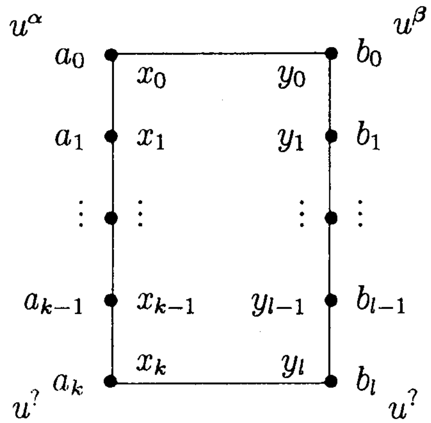


Figure 11.3.1: A Gorenstein cone σ in $M = \mathbb{Z}^3$. The origin is behind the page; the monomials x_i, y_j and the internal generator u are not coplanar.

generators, with u the unique internal generator. (The dualising sheaf of an affine toric variety V is isomorphic to the ideal of internal monomials, so V is Gorenstein if and only this ideal is principal.) In Figure 11.3.1, the *tags* a_i, b_j down the sides represent *tag equations*

$$\begin{aligned} x_{i-1}x_{i+1} &= x_i^{a_i} \quad \text{with } a_i \geq 2 \text{ for } i = 1, \dots, k-1, \\ y_{j-1}y_{j+1} &= y_j^{b_j} \quad \text{with } b_j \geq 2 \text{ for } j = 1, \dots, l-1. \end{aligned} \tag{11.3.1}$$

The top two corners are *annotated* by powers of u, which modify their tag equations to

$$x_1 y_0 = x_0^{a_0} u^\alpha \quad \text{and} \quad x_0 y_1 = y_0^{b_0} u^\beta \quad \text{with } \alpha, \beta \geq 0. \tag{11.3.2}$$

The monomials down either side of the rectangle (11.3.1) form the Newton polygon of a surface cyclic quotient singularity, and are thus governed by standard rules in terms of the Hirzebruch continued fraction $[a_1, \dots, a_{k-1}]$ and $[b_1, \dots, b_{l-1}]$. The restriction $a_i, b_j \geq 2$ comes from this. All this follows easily because any two consecutive monomials v_i, v_{i+1} around the perimeter of (11.3.1) together with the internal generator u form a $\mathbb{Z}$-basis of the lattice of monomials M.

On the other hand, the tags at the world's four corners a_0, a_k, b_l, b_0 cannot all be ≥ 2. In fact, each tag equation corresponds to a change of basis

$$\begin{pmatrix} v_i \\ v_{i+1} \\ u \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & a_i & ? \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_{i-1} \\ v_i \\ u \end{pmatrix}$$

in M, where ? is the annotation $u^?$ at the corners. Circumnavigating (11.3.1) by successive changes of bases gives

$$\prod \begin{pmatrix} 0 & 1 & 0 \\ -1 & a_i & ? \\ 0 & 0 & 1 \end{pmatrix} = \mathrm{id}_3, \quad \text{in particular} \quad \prod \begin{pmatrix} 0 & 1 \\ -1 & a_i \end{pmatrix} = \mathrm{id}_2. \qquad (11.3.3)$$

The tag equations so far (11.3.1–11.3.2) determine each x_i and y_j as a Laurent monomial in x_0, y_0, u. Writing out these Laurent monomials allows us to calculate all the equations between the generators $x_0, \dots, x_k, y_0, \dots, y_l, u$, and in particular the powers of u annotating the tag equations at the bottom corners. Write

$$k[V_u] = k[\sigma \cap M] = k[x_0, \dots, x_k, y_0, \dots, y_l, u]$$

for the ring generated by these monomials, the affine coordinate ring of the Gorenstein toric 3-fold V_u.

Convexity considerations and combinatorics of concatenated continued fractions (compare Craw and Reid [CR], Section 2) reduce us to just a few cases. The main one is the rectangle of Figure 11.3.2 with tags forming *complementary continued fractions*

$$[a_0, a_1, \dots, a_{k-2}] = \frac{n}{q} \quad \text{and} \quad [b_1, b_2, \dots, b_{l-1}] = \frac{n}{n-q} \qquad (11.3.4)$$

for some n and q. This is a well known way of fixing up identities such as the second of (11.3.3); see for example [CR], Figure 4 or [Rie], §3, pp. 220–3. However, as Jan Stevens taught us, it is also a way of *deconstructing* the

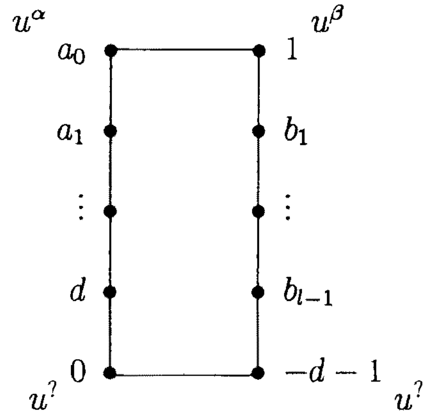


Figure 11.3.2: Main case: the corner tags have signs $a_0 > 0$, $b_0 = 1$, $a_k = 0$ and $b_l < 0$, and the side tags form complementary continued fractions.

continued fraction, successively eliminating the tag 1 and decrementing its two neighbours. For example, take $n = 11$ and $q = 7$; then $\frac{11}{7} = [2,3,2,2]$ and $\frac{11}{4} = [3,4]$. Concatenating the two continued fractions with a 1 gives $[2,3,2,\underline{2,1,4},3]$, that deconstructs by replacing $2,1,4$ by $1,3$:

$$\to [2,3,\underline{2,1,3},3] \to [2,\underline{3,1,2},3] \to [2,\underline{2,1,3}] \to [\underline{2,1,2}] \to [1,1] = 0.$$

Compare [CR], §2. My main point is:

> *the same calculation gives successive Gorenstein projections of the toric coordinate ring $k[V_u]$ down to a codimension 2 complete intersection.*

In fact, if y_0 has the tag $b_0 = 1$ then also x_0, y_1 and u base M. This is basically the same reason that allowed us to assume that $a_i, b_j \geq 2$ down the sides (if $a_i = 1$, I can eliminate x_i as a generator), but here eliminating y_0 cuts down the cone σ, so makes a birational change to V_u.

To see the effect of this change in more detail, note that because y_0 has the tag $b_0 = 1$, it occurs linearly in 3 tag equations:

$$x_1 y_0 = x_0 u^\alpha, \quad y_0 y_2 = y_1^{b_1} \quad \text{and} \quad x_0 y_1 = y_0 u^\beta$$

(and also in long equations $y_0 x_i$ for $i \geq 2$ and $y_0 y_j$ for $j \geq 3$, but we do not need these). To eliminate y_0, I multiply the first two equations by u^β and

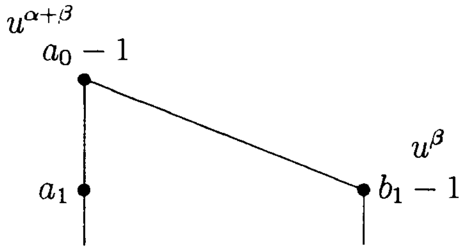


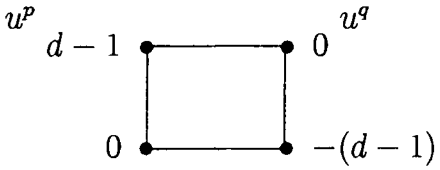
Figure 11.3.3: Projecting from y_0

substitute $y_0u^\beta = x_0y_1$ in each, then cancel a power of x_0 or y_1 respectively, to get the new tag equations

$$x_1y_1 = x_0^{a_0-1}u^{\alpha+\beta} \quad \text{and} \quad x_0y_2 = y_1^{b_1-1}u^\beta.$$

In words, chop off the top right-hand corner of σ, giving the new Gorenstein quadrilateral cone σ' with top corners given as in Figure 11.3.3. Note that the two monomials adjacent to the recently executed y_0 have their tags decremented by 1, but inherit a factor of u^β in their annotation.

Under the current assumption that $[a_0, a_1, \dots, a_{k-2}]$ and $[b_1, b_2, \dots, b_{l-1}]$ are complementary continued fractions, one of $a_0 - 1$ and $b_1 - 1$ is again equal to 1, allowing me to eliminate x_0 or y_1 by another Gorenstein projection, and so on. The serial projection ends with a rectangle



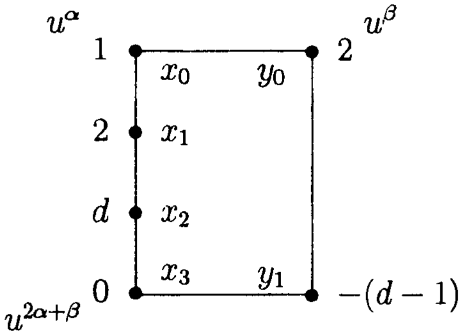
where the exponents p, q of u are linear combinations of α and β determined by cumulatively multiplying the annotations. This rectangle represents the codimension 2 complete intersection

$$x_{k-1}y_l = u^q, \quad x_ky_{l-1} = x_{k-1}^{d-1}u^p.$$

11.4 A pair of long rectangles

Example 11.4.1 I illustrate in a simple codimension 4 case how to deform a Gorenstein toric variety by projection and unprojection to obtain the $\mathbb{G}_m$

cover of a Mori flip. The quadrilateral cone of monomials and its tag equations are as follows:



$$\Longrightarrow \quad \begin{aligned} x_1y_0 &= x_0u^\alpha & \qquad x_0y_1 &= y_0^2u^\beta \\ x_0x_2 &= x_1^2 & & \\ x_1x_3 &= x_2^d & & \\ x_2y_1 &= u^{2\alpha+\beta} & \qquad x_3y_0 &= x_2^{d-1}u^\alpha \end{aligned}$$

for some $\alpha > 0$ and $\beta \geq 0$.

Remark 11.4.2 The bottom right equation $x_3y_0 = x_2^{d-1}u^\alpha$ is nonstandard: because the tag $b_1 = -(d-1)$ is negative, the regular tag equation would be

$$x_3y_0 = y_1^{-(d-1)}u^?, \tag{11.4.1}$$

which is not a polynomial equation. I therefore replace y_1^{-1} by $x_2u^?$ using the bottom left equation, getting the *modified tag equation* $x_3y_0 = x_2^{d-1}u^?$. This replaces the negative exponent of the corner monomial y_1 with *a positive exponent of the monomial* x_2 *opposite the corner.* We certainly pay for this substitution when we do syzygies, although I do not know how to express this sentiment mathematically.

Note also that (11.4.1), rewritten as $x_3y_0y_1^{d-1} = u^?$, is a general feature of the monomial rectangle (11.3.1) that may at first sight seem somewhat unexpected: the internal monomial u chooses to live right in one corner, namely in the convex hull of y_l, x_k, y_{l-1}.

I deform this ring by guessing the two equations at the top:

$$x_1y_0 = x_0u^\alpha + t^{2\lambda+\mu} \quad \text{and} \quad x_0y_1 = y_0^2u^\beta + x_1t^\lambda \tag{11.4.2}$$

where t is an elephantine deformation parameter ($t = 0$ will be the anticanonical section for the $\mathbb{Z}$-grading). There is nothing very special about the exponents of t: they are a priori arbitrary coprime integers satisfying some inequalities and possibly divisibility conditions; the point of writing them in this form is to make the bottom two equations and the $\mathbb{Z}$-grading pretty.

Now (11.4.2) is a codimension 2 c.i. that contains the codimension 3 c.i. (x_0, y_0, t^λ). It thus unprojects by Theorem 5.2. The unprojection variable x_2 satisfies 3 new equations given by a game of Pfaffians similar to that of Example 4.1:

$$\begin{pmatrix} y_1 & -x_1 & -y_0u^\beta & x_2 \\ & y_0 & -t^\lambda & -u^\alpha \\ & & x_0 & -t^{\lambda+\mu} \\ & & & x_1 \end{pmatrix} \implies \left\{ \begin{array}{rcl} x_0x_2 & = & x_1^2 + y_0u^\beta t^{\lambda+\mu} \\ x_2y_0 & = & x_1u^\alpha + y_1t^{\lambda+\mu} \\ x_1y_1 & = & y_0u^{\alpha+\beta} + x_2t^\lambda \end{array} \right.$$

These 5 equations define a Gorenstein codimension 3 variety that contains the codimension 4 c.i. $(x_0, x_1, y_0, t^\lambda)$. This again unprojects by Theorem 5.2, adjoining x_3. In fact, it is a Jerry unprojection (see Example 6.8): rows and columns Nos. 3 and 4 of the Pfaffian matrix (all entries except for the 125 triangle) are in the ideal $(x_0, x_1, y_0, t^\lambda)$.

As in Example 6.8, there is an easy trick to derive x_3 as a rational function, namely elimination of x_0 (a projection). Notice that x_0 appears linearly in 3 of our 5 Pfaffians; the two not involving x_0 are the above equations

$$x_2y_0 = x_1u^\alpha + y_1t^{\lambda+\mu} \quad \text{and} \quad x_1y_1 = y_0u^{\alpha+\beta} + x_2t^\lambda.$$

These define a codimension 2 c.i. that contains two separate codimension 3 c.i.s, namely the ideal of denominators (x_2, y_1, u^α) of x_0 and the new one (x_1, y_0, t^λ) that is the ideal of denominators of x_3. I indulge myself in just one final round of Pfaffians:

$$\begin{pmatrix} y_1 & -x_2 & -u^{\alpha+\beta} & x_3 \\ & y_0 & -t^\lambda & -u^\alpha \\ & & x_1 & -y_1t^\mu \\ & & & x_2 \end{pmatrix} \implies \left\{ \begin{array}{rcl} x_1x_3 & = & x_2^2 + y_1u^{\alpha+\beta}t^\mu \\ x_3y_0 & = & x_2u^\alpha + y_1^2t^\mu \\ x_2y_1 & = & u^{2\alpha+\beta} + x_3t^\lambda \end{array} \right.$$

The first of these equations proves that if we make a deformation of the ring in Example 11.4.1, starting from the top two equations (11.4.2) and adopting the above style of unprojection, then necessarily $d = 2$. As in Example 6.8, we do not really have a good way of deriving the "long equation" for x_0x_3. As far as we know, it is not contained in a Pfaffian in any useful way. Messing around with explicit syzygies eventually gives

$$x_0x_3 = x_1x_2 + y_0y_1u^\beta t^\mu + u^{\alpha+\beta}t^{\lambda+\mu}. \tag{11.4.3}$$

There is a unique way of putting a $\mathbb{Z}$-grading on this set of equations with $\operatorname{wt} u = 0$, $\operatorname{wt} t = -1$, provided that $3 \mid \mu$. Namely,

$$\begin{aligned} &\operatorname{wt} x_3 = \lambda > 0, \quad \operatorname{wt} y_1 = \mu/3 > 0, \quad \text{and} \\ &\operatorname{wt} x_2 = -\mu/3, \quad \operatorname{wt} x_1 = -\lambda - 2\mu/3, \\ &\qquad\qquad \operatorname{wt} x_0 = -2\lambda - \mu, \quad \operatorname{wt} y_0 = -\lambda - \mu/3. \end{aligned}$$

The $\mathbb{G}_m$ quotient is the flip diagram $X \searrow Y \swarrow X^+$, where

$$X = \operatorname{Proj} R_- = \begin{pmatrix} x_1 y_0 & = & x_0 u^\alpha + t^{2\lambda+\mu} \\ x_0 y_1 & = & y_0^2 u^\beta + x_1 t^\lambda \end{pmatrix} \Big/ \mathbb{G}_m$$

is covered by the two affine pieces $x_0 = 1$ and $y_0 = 1$ and

$$X^+ = \operatorname{Proj} R_+ = \begin{pmatrix} x_3 y_0 & = & x_2 u^\alpha + y_1^2 t^\mu \\ x_2 y_1 & = & u^{2\alpha+\beta} + x_3 t^\lambda \end{pmatrix} \Big/ \mathbb{G}_m$$

is covered by the two affine pieces $x_3 = 1$ and $y_1 = 1$.

This is a Mori flip of Type A, with $t = 0$ the general elephant, and $u = 0$ the general hyperplane section. For example, the $y_0 = 1$ affine piece of the left-hand side X is the hyperquotient singularity

$$\begin{pmatrix} x_1 & = & x_0 u^\alpha + t^{2\lambda+\mu} \\ x_0 y_1 & = & u^\beta + (x_0 u^\alpha + t^{2\lambda+\mu}) t^\lambda \end{pmatrix} \Big/ \frac{1}{\operatorname{wt} y_0} (\operatorname{wt} x_0, \operatorname{wt} y_1, 0, 1).$$

If we look up the weights, we see that this is

$$(x_0 y_1 = u^\beta + \cdots + t^{3r}) \Big/ \frac{1}{r}(-a, a, 0, 1), \quad \text{with } r = \lambda - \frac{\mu}{3},\ a = \frac{\mu}{3},$$

which is a standard Type A terminal singularity.

11.5 Conclusions from this example

The rectangle of Example 11.4.1 defines an affine Gorenstein toric 3-fold V_u; I have shown how to deform it to a 4-fold $V_{u,t}$ with a $\mathbb{G}_m$ action such that $V_{u,t}$ has isolated singularities modulo the action and isolated fixed points. Requiring isolated fixed points means that the corner equations contain pure powers of t (possibly after a substitution). The deformation style adopted keeps track of the powers of t by introducing the right-hand rectangle of

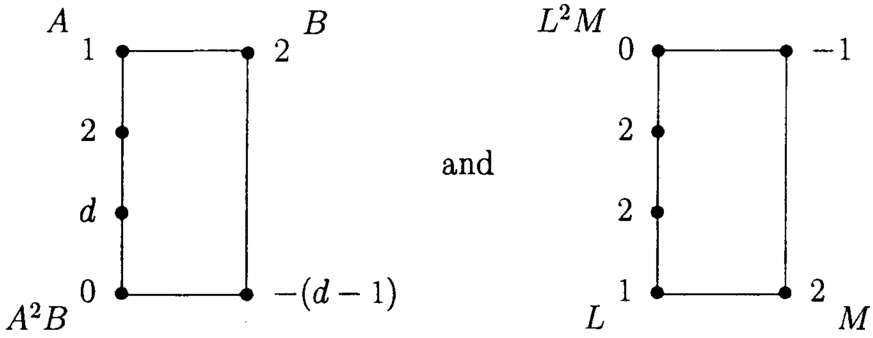


Figure 11.5.1: The pair of long rectangles for Example 11.4.1

Figure 11.5.1, having different top and bottom corner tags and annotations, but identical torso. This imposes $d = 2$ on the original rectangle.

As in Remark 11.2.1, the final expression only contains u and t within the tokens u^α, u^β, t^λ, t^μ. Replacing

$$u^\alpha \mapsto A, \quad u^\beta \mapsto B, \quad t^\lambda \mapsto L, \quad t^\mu \mapsto M$$

in the equations (say, taking the top two equations to $x_1y_0 = x_0A + L^2M$ and $x_0y_1 = y_0^2B + x_1L$) gives an affine Gorenstein 6-fold V_{ABLM} with a regular sequence $A, B, L, M \in k[V_{A,B,L,M}]$ such that the codimension 2 sections $V_{AB} : (L = M = 0)$ and $V_{LM} : (A = B = 0)$ are the toric 4-folds with respective cones of monomials documented by the long rectangles of Figure 11.5.1. There is a 4-dimensional torus $\mathbb{G}_m^4$ with a monomial action on V_{ABLM}, namely, the subgroup of the diagonal group $\mathbb{G}_m^8$ acting on $\mathbb{C}^8$ with coordinates $x_0, x_1, y_0, y_1, A, B, L, M$ that leaves semi-invariant the top two equations $x_1y_0 = x_0A + L^2M$ and $x_0y_1 = y_0^2B + x_1L$.

Remark 11.5.1 At the start of Example 11.4.1, the assumption on β was only that $\beta \geq 0$. If $\beta > 0$ then $\mathbb{P}^1(x_0 : y_0)$ is contained in the elephant and the point $P_{y_0} = (0 : 1) \in V_u$ is a terminal singularity of index $(-\operatorname{wt} y_0)$. If $\beta = 0$, the top right equation contains a pure power of y_0, so that $P_{y_0} \notin V_u$. In this case, $S^- \to S$ of (11.1.2) is an isomorphism. This is Mori's distinction between cases kA_2 and kA_1, having respectively two and one singularities of index > 1 on S. In this case the angle at y_0 in the left-hand long rectangle straightens out, so that the elephant is represented by a "long triangle".

11.6 Pairs of long rectangles and serial unprojection via pentagrams

The combinatorial classification of pairs of long rectangles is solved in Brown and Reid [BR]. We obtain a number of families labelled by Roman numerals: the case of Figure 11.5.1 is currently called III(1, 0). Each pair gives rise to a *two-headed toric 6-fold* V_{ABLM} by the serial unprojection method outlined below, with the properties sketched in 11.5. We still have some work to do to identify our treatment with Mori's calculation [M], but the pairs II(d, e, k) of Figure 11.6.1 seem to be most closely related to it. The figure illustrates

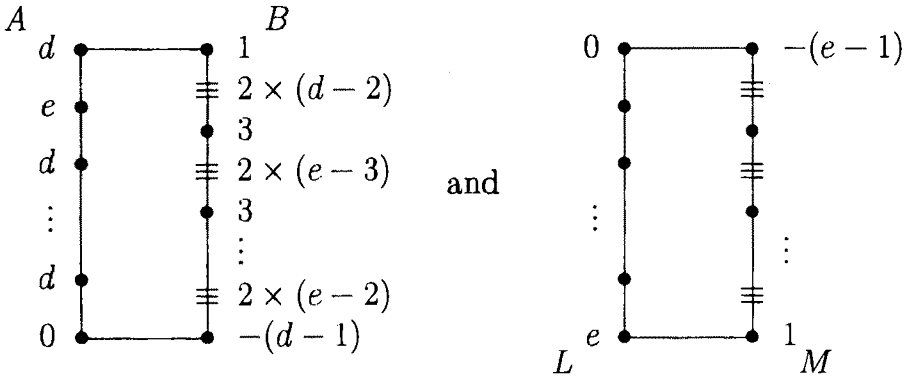


Figure 11.6.1: The pair of long rectangles II(d, e, k).

the case k even and $d, e \geq 4$. The two rectangles have the same torso tags (excluding tops and tails): k terms $e, d, e, d, \ldots, d$ down the left and k blocks $2, \ldots, 2, 3$ of $d-2$ and $e-2$ terms each down the right, giving rise to complementary continued fractions

$$[e, d, \ldots] = \frac{n}{q} \quad \text{and} \quad [2, \ldots, 2, 3, 2, \ldots] = \frac{n}{n-q}$$

as in (11.3.4).

Remark 11.6.1 (1) The two rectangles correspond to the elephant $t = 0$ and the general section $u = 0$ of a flip. Mori has proved that if the elephant is of Type A then the section is a cyclic quotient singularity, this is where V_{ABLM} gets its two toric heads from.

(2) We would be interested to know if the V_{ABLM} have already been studied elsewhere. We hope that they have other descriptions; it is conceivable that they are quasihomogeneous varieties for some slightly bigger group than $\mathbb{G}_m^4$, for example, something with a unipotent radical such as two copies of $\left(\begin{smallmatrix}\mathbb{G}_m & \mathbb{G}_a\\ 0 & \mathbb{G}_m\end{smallmatrix}\right)$. All those Pfaffian equations of V_{ABLM} that become binomial (toric) on cutting to V_{AB} and V_{LM} might to have something to do with extending toric varieties to quasihomogeneous varieties modelled on $\mathrm{GL}(2)$.

(3) It should be reasonably straightforward to extend much of the apparatus of toric geometry to deal with the V_{ABLM}. Invariant divisors, coherent cohomology, Betti cohomology and Hodge theory, derived categories, Gromov–Witten invariants, mirror partners ... Get on with it, this isn't a research grant application!

(4) It seems likely that the two-headed toric varieties V_{ABLM} can also act as key varieties in other contexts. For example, we expect that they can be given nice positive gradings, and so act as key varieties for projective varieties coming from serial unprojection constructions, as illustrated in Examples 7.1–7.6. This could extend the range of our artillery for attacking surfaces and 3-folds, bringing other interesting targets within range.

Example 11.6.2 I illustrate serial unprojection with a little workout in the case $k = 2$, and get a final fix (*ultimo Pfaffiano!*). The bullets down the left side of the rectangle in Figure 11.6.2 are monomials x_0, x_1, x_2, x_3. To avoid going into double suffixes, I write $y_0, \dots, y_d$ and then $z_0, \dots, z_e$ for the

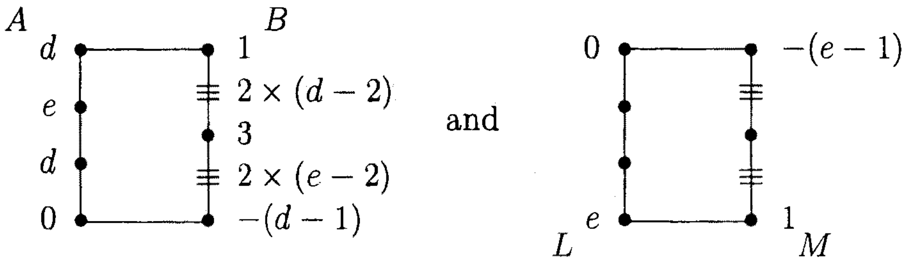


Figure 11.6.2: The pair of long rectangles $\mathrm{II}(d, e, k)$.

monomials down the right side, with an overlap of 3:

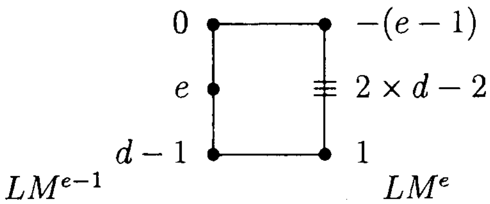
$$y_{d-2} = z_0, \quad y_{d-1} = z_1 \text{ is the monomial with tag 3, and } \quad y_d = z_2.$$

I start work on the right-hand rectangle (the cone of monomials for V_{LM}), with the initial objective of discovering the annotations at its top corners. The bottom right tag is a 1, so that I can project from z_e as in Figure 11.3.3 (but bottom-up this time), then successively from $z_{e-1}, \dots, z_2$. By the rules given around Figure 11.3.3, the new tag at x_3 is decremented by 1 at each projection, and the annotation at x_3 inherits a factor of M, so that after $e-1$ steps the tag is 1 and the annotation is LM^{e-1}, giving successive tag equations

$$x_1x_3 = x_2^d, \quad x_2z_1 = x_3LM^{e-1}, \quad x_3z_0 = z_1^2M.$$

The last projection of z_2 decrements the tag at z_1 from 3 to 2, and gives z_1 an annotation of M.

Now x_3 can be projected: I chop it off, and its annotation LM^{e-1} is multiplied into that of x_2 and $z_1 = y_{d-1}$. The score at half-time is:



Next project from $y_{d-1}, \dots, y_2$. At each point the annotation LM^e of y_i is passed on to that of y_{i-1}, and is multiplied into the tag of x_2. Since we project $d-2$ times, the tag of x_2 decrements down to 1, and its annotation multiplies up to

$$LM^{e-1} \times (LM^e)^{d-2} = L^{d-1}M^{de-e-1}.$$

Finally project x_2, so that its annotation is passed on to x_1 and multiplied into that of y_1 to give L^dM^{de-1}. This leaves us with a rectangle representing the two equations

$$x_1y_0 = L^dM^{de-1} \quad \text{and} \quad x_0y_1 = x_1^{e-1}L^{d-1}M^{de-e-1}$$

at the top corners of the right-hand rectangle.

Merging the right hand side of these with those at the top corners of the left-hand rectangle gives the top equations for the 6-fold V_{ABLM}:

$$x_1y_0 = x_0^dA + L^dM^{de-1} \quad \text{and} \quad x_0y_1 = y_0B + x_1^{e-1}L^{d-1}M^{de-e-1}. \qquad (11.6.1)$$

Now this is where the fun really starts. Consider the series of 5×5 Pfaffians

$$\begin{pmatrix} x_2 & -B & -x_1^{e-1} & y_i \\ & x_1 & -LM^e & -x_0^{d-i}AB^{i-1} \\ & & x_0 & -x_2^{i-1}L^{d-i}M^{(d-i)e-1} \\ & & & y_{i-1} \end{pmatrix}$$

$$\Longrightarrow \quad \left\{ \begin{array}{rcl} x_1y_{i-1} & = & x_0^{d-i+1}AB^{i-1} + x_2^{i-1}L^{d-i+1}M^{(d-i+1)e-1} \\ x_0y_i & = & y_{i-1}B + x_1^{e-1}x_2^{i-1}L^{d-i}M^{(d-i)e-1} \\ x_2y_{i-1} & = & x_0^{d-i}x_1^{e-1}AB^{i-1} + y_iLM^e \\ x_1y_i & = & x_0^{d-i}AB^i + x_2^iL^{d-i}M^{(d-i)e-1} \\ x_0x_2 & = & x_1^e + BLM^e \end{array} \right.$$

for $i = 1, \dots, d-1$. When $i = 1$, the first two equations are just (11.6.1), and the final three, linear in x_2, are the equations defining the unprojection of the codimension 3 c.i. $(x_0, y_0, L^{d-1}M^{de-e-1})$ in the codimension 2 c.i. (11.6.1).

For $i \geq 2$, the matrix is easily read off the *pentagram* of Figure 11.6.3, (a). We start from the two known equations for x_0x_2 and x_1y_{i-1}. The 5 points of the pentagram are $x_2, x_1, x_0, y_{i-1}, y_i$. We write these cyclically around the superdiagonal and top right of the matrix, so that adjacent vertexes do not multiply in Pfaffians, but vertexes that are two apart do so, as in the pentagram: the new unprojection variable y_i goes in the top right, from whence it will multiply x_0, x_1 and the middle entry m_{24}, but not x_2 or y_{i-1}. The remaining matrix entries are uniquely determined by requiring that $\mathrm{Pf}_{12.34}$ is the known equation for x_0x_2 and $\mathrm{Pf}_{23.45}$ the known equation for x_1y_{i-1} with the middle entry $m_{24} = LM^e$ the hcf of the terms BLM^e and $x_2^{i-1}L^{d-i+1}M^{(d-i+1)e-1}$ in those two equations (taking a factor smaller than the hcf would lead to a nonnormal variety). The 3 new Pfaffians determine the new unprojection variable y_i as a rational function, and one proves via Theorem 5.2 that it defines a Gorenstein variety with coordinate ring generated by $x_0, x_1, x_2, y_0, \dots, y_i, A, B, L, M$.

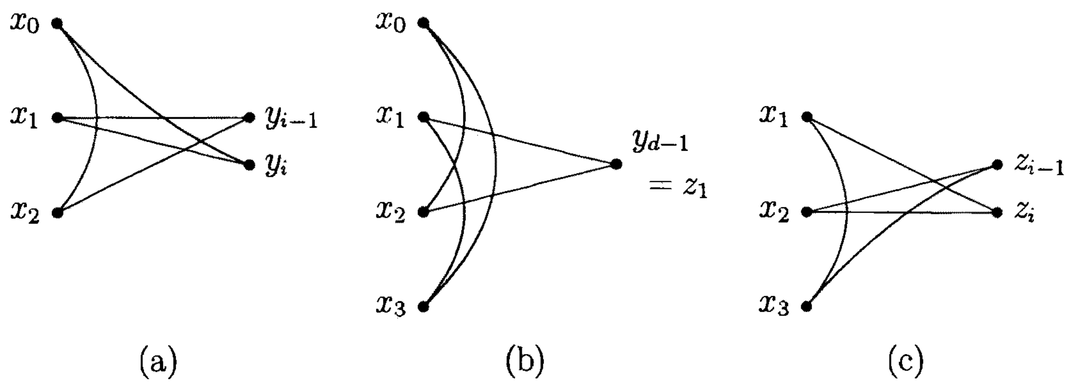


Figure 11.6.3: Pentagrams for $\mathrm{II}(2, d, e)$

For $1 \le i \le d-2$, suppose by induction that these equations hold for i; then projecting from y_{i-1} deletes the first three equations, leaving the last two as a codimension 2 c.i. containing (x_0, x_1, LM^e). Thus we can introduce a new unprojection variable y_{i+1}, with three new relations contained in the same set of equations with $i \mapsto i+1$.

The rest is similar. At the end of the first half, the first series of Pfaffians culminates at $i = d-1$ with the equation

$$x_1 y_{d-1} = x_0 AB^{d-1} + x_2^{d-1} LM^{e-1}.$$

At half-time, we use this together with the equation for $x_0 x_2$ as input to a 5×5 Pfaffian matrix corresponding to the pentagram (b), switching to $z_1 = y_{d-1}$ for the second half:

$$\begin{pmatrix} x_2 & -BM & -x_1^{e-1} & x_3 \\ & x_1 & -LM^{e-1} & -AB^{d-1} \\ & & x_0 & -x_2^{d-1} \\ & & & z_1 \end{pmatrix}$$

$$\implies \quad \begin{aligned} x_0 x_3 &= x_1^{e-1} x_2^{d-1} + z_1 BM \\ x_1 x_3 &= x_2^d + AB^d M \\ x_2 z_1 &= x_1^{e-1} AB^{d-1} + x_3 LM^{e-1} \end{aligned}$$

Notice that the middle term of the matrix $m_{24} = LM^{e-1}$ has slipped down to the hcf of two terms in the input equations $\mathrm{Pf}_{12.34}$ and $\mathrm{Pf}_{23.45}$.

The two last equations form the input to the series of Pfaffians corresponding to the pentagram Figure 11.6.3, (c) that play during the second half:

$$\begin{pmatrix} x_3 & -AB^d & -x_2^{d-1} & z_{i+1} \\ & x_2 & -M & -x_1^{e-i-1}A^iB^{d-i} \\ & & x_1 & -x_3^iLM^{e-i-1} \\ & & & z_i \end{pmatrix}$$

This works by induction as before, and I leave it at that.

References

[A] S. Altınok, Graded rings corresponding to polarised K3 surfaces and $\mathbb{Q}$-Fano 3-folds, Univ. of Warwick Ph.D. thesis, Sep. 1998, 93+ vii pp.

[A1] S. Altınok, Hilbert series and applications to graded rings, submitted

[B1] G. Brown, Flips in low codimension – classification and quantitative theory, Univ. of Warwick Ph.D. thesis, Jun 1995, 100 + iv pp.

[B] G. Brown, Flips arising as quotients of hypersurfaces, Math. Proc. Camb. Phil. Soc. **127** (1999) 13–31

[BR] G. Brown and M. Reid, Mori flips of Type A (provisional title), in preparation

[Ca] F. Catanese, Commutative algebra methods and equations of regular surfaces, in Algebraic geometry (Bucharest, 1982), LNM 1056, pp. 68–111

[Ca1] F. Catanese, Canonical rings and "special" surfaces of general type, in Algebraic geometry (Bowdoin, 1985) Proc. Sympos. Pure Math., **46**, Part 1, pp. 175–194,

[Ca2] F. Catanese, Homological algebra and algebraic surfaces, in Algebraic geometry (Santa Cruz, 1995), Proc. Sympos. Pure Math., **62**, Part 1, 3–56,

[Ci] C. Ciliberto, Canonical surfaces with $p_g = p_a = 4$ and $K^2 = 5, \dots, 10$, Duke Math. J. **48** (1981), 121–157

[Co] A. Corti, Factoring birational maps of 3-folds after Sarkisov, J. Alg. Geom. **4** (1995), 223–254

[Co2] A. Corti, Singularities of linear systems and 3-fold birational geometry, in Explicit birational geometry of 3-folds, A. Corti and M. Reid (eds.), CUP 2000, 259–312

[CM] A. Corti and M. Mella, Birational geometry of terminal quartic 3-folds. I, in preparation. http://can.dpmms.cam.ac.uk/~corti/cm.ps.gz

[CPR] A. Corti, A. Pukhlikov and M. Reid, Birationally rigid Fano hypersurfaces, in Explicit birational geometry of 3-folds, A. Corti and M. Reid (eds.), CUP 2000, 175–258

[CR] A. Craw and M. Reid, How to calculate A-Hilb$\mathbb{C}^3$, 29 pp., math/9909085

[D] D. Dicks, Surfaces with $p_g = 3$, $K^2 - 4$ and extension–deformation theory, Univ. of Warwick Ph.D. thesis, 1988

[Fl0] A. Iano-Fletcher, Plurigenera of 3-folds and weighted hypersurfaces, Univ. of Warwick Ph.D. Thesis, 1988

[Fl] A. Iano-Fletcher, Working with weighted complete intersections, in Explicit birational geometry of 3-folds, A. Corti and M. Reid (eds.), CUP 2000, 101–173

[GW] S. Goto and K-i Watanabe, On graded rings. I, J. Math. Soc. Japan **30** (1978) 179–213

[Ka] KAWAKITA Masayuki, Divisorial contractions in dimension three which contract divisors to smooth points, math.AG/0005207, 16 pp.

[Ka2] KAWAKITA Masayuki, Divisorial contractions in dimension three which contract divisors to compound A_1 points, math.AG/0010207, 23 pp.

[KM] A. Kustin and M. Miller, Constructing big Gorenstein ideals from small ones, J. Algebra **85** (1983) 303–322

[M] MORI Shigefumi, On semistable extremal neighborhoods, to appear, pp. 26

[Ma] Magma (John Cannon's computer algebra system): W. Bosma, J. Cannon and C. Playoust, The Magma algebra system I: The user language, J. Symb. Comp. **24** (1997) 235–265. See also www.maths.usyd.edu.au:8000/u/magma

[Mu] MUKAI Shigeru, Curves and symmetric spaces. I, Amer. J. Math. **117** (1995) 1627–1644

[P] Stavros Papadakis, Unprojection, Gorenstein rings and applications to algebraic geometry, Univ. of Warwick Ph.D. thesis (in preparation)

[PR] Stavros Papadakis and Miles Reid, Kustin–Miller unprojection without complexes, J. algebraic geometry (to appear), preprint math.AG/0011094, 15 pp.

[R] M. Reid, Surfaces with $p_g = 0$, $K^2 = 1$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **25** (1978) 75–92

[R1] M. Reid, Surfaces with $p_g = 3$, $K^2 = 4$ according to E. Horikawa and D. Dicks, in Proceedings of Algebraic geometry mini-symposium (Tokyo, Dec 1989), 1–22

[R2] M. Reid, Graded rings over K3 surfaces, in abstracts of Matsumura memorial conference (Nagoya, Aug 1996), 11 pp.

[R3] M. Reid, McKay correspondence, in Proc. of algebraic geometry symposium (Kinosaki, Nov 1996), T. Katsura (Ed.), 14–41, Duke file server alg-geom 9702016, 30 pp., rejected by Geometry and Topology

[wF] M. Reid, What is a flip? Lecture at Univ. of Utah, Dec. 1992, plus subsequent notes

[Homework] M. Reid, Worksheet on graded rings, in www.maths.warwick.ac.uk/~miles/Homework

[Rie] O. Riemenschneider, Deformationen von Quotientensingularitäten (nach zyklischen Gruppen), Math. Ann. **209** (1974), 211–248

[S] J. Stevens, Rolling factors deformations and extensions of canonical curves, Chalmers Univ. preprint 2000:44, 37 pp.

[T] TAKAGI Hiromichi, On the classification of $\mathbb{Q}$-Fano 3-folds of Gorenstein index 2. I, II, RIMS preprint 1305, Nov 2000, 66 pp.

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