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Splitting problems of algebraic vector bundles on projective spaces

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In this article, resuming first several important results on algebraic vector bundles on projective spaces and some splitting problems of those vector bundles, we shall explain an attempt to solve the splitting problems for rank two vector bundles on projective spaces by using determinantal varieties.

1) Vector bundles on projective spaces

Let $\mathbf{P}_k^n$ be an n-dimensional projective space defined over an algebraically closed field k and E an algebraic vector bundle of rank r on $\mathbf{P}_k^n$.

1. 1) n = 1.

We have the following fundamental result for vector bundles on $\mathbf{P}^1$.

Theorem (A.Grothendieck 1957 [G3]) *Every vector bundle on* $\mathbf{P}^1$ *is a direct sum of line bundles.*

$$E = \oplus_{i=1}^r \mathcal{O}(a_i) \quad (a_i \in \mathbf{Z}).$$

As applications, it follows that

a) Embedded deformations.

Let X be a nonsingular algebraic variety (n = $\dim X$) and $X \supset C \simeq P^1$ a curve in X. Then we see that $N = N_{X/C} = \oplus_{i=1}^{n-1}\mathcal{O}(a_i)$ (the normal bundle of C in X) for some integers $\{a_i\}$. *Let* [C] $\in \mathcal{H} = Hilb_X$ be the Hilbert scheme of X. Then it is known that

$$T_{[C],\mathcal{H}} \simeq H^0(C,N) : \text{the embedded deformation space of } C \text{ in } X,$$

$$\dim_{[C]} \mathcal{H} \geq \dim H^0(C,N) - \dim H^1(C,N).$$

Thus $\{a_1, \cdots, a_{n-1}\}$ describes the deformations of C in X. This fact plays important roles in studying the Extremal curves, Contraction maps, Minimal models, $\cdots$ and hence in the Classification theory of algebraic surfaces and 3-folds. (cf. [M4])

b) Jumping lines.

For any line $\ell \subset P^n$, we have

$$E|\ell = \oplus_{i=1}^{r} \mathcal{O}(a_i) \quad (a_1 \geq \cdots \geq a_r)$$

and we denote the set of integers $(a_1, \cdots, a_r)$ by $a_E(\ell)$. There exists an open subset U of the Grassmann variety $Grass(n, 1)$ which is the parameter space of lines in $\mathbf{P}^n$ such that $a_E(\ell)$ are constant for all ℓ *in* U. A line ℓ is called a jumping line if $a_E(\ell)$ is different from that for general lines. (cf [O1])

As for general lines, we have the following

Theorem (H.Grauert, G.Mülich 1975 [G1], H.Spindler 1979 [S3]) *If E is semi-stable, then for a general line ℓ, it is observed that* $0 \leq a_i - a_{i+1} \leq 1$ *for all* i $(1 \leq i \leq r-1)$.

1. 2) n = 2, 3.

There are many vector bundles which are not direct sum of line bundles on $\mathbf{P}^n$ $(n = 2,\ 3)$ contrary to the case $\mathbf{P}^1$.

a) Moduli of stable vector bundles on $\mathbf{P}^2_{\mathbf{C}}$.

Let

$M(0, c)$ = moduli of stable rank 2 vector bundles on $\mathbf{P}^2_{\mathbf{C}}$ with $c_1 = 0$ and $c = c_2$ (≥ 2),

$M(-1, c)$= moduli of stable rank 2 vector bundles on $\mathbf{P}^2_{\mathbf{C}}$ with $c_1 = -1$ and $c = c_2$ (≥ 2).

Then we have the followings.

Theorem (W.Barth 1977 [B1]) *$M(0, c)$ is an irreducible non-singular rational quasi-projective variety of dimension 4c-3.*

Theorem (K.Hulek 1979 [H5]) *$M(-1, c)$ is an irreducible non-singular rational quasi-projective variety of dimension 4c-4.*

There was a small gap in the proof of the rationality and T.Maeda gave a complete proof for the rationality of these varieties in [M1].

On the other hand, M.Maruyama established the following fundamental theorem for the existence of coarse moduli spaces of semi-stable coherent sheaves and also studied the irreducibility and rationality of these varieties.

Theorem (M.Maruyama 1977 [M2], 1978 [M3]) 1) *There exists a coarse moduli space for semi-stable coherent sheaves on non-singular projective varieties except for boundedness.*

2) $M(0,c)$ $(c = 4k)$ is irreducible and rational.

b) Moduli of stable vector bundles on $\mathbf{P}^3_{\mathbf{C}}$.

In case $n = 3$, it turns out that

Theorem (W.Barth, R.Hartshorne 1978 [H2]) 1) *Every irreducible component of $M(0,c)$ has dimension $\geq 8c-3$. If odd $c \geq 5$, then there exist irreducible components of dimension 8c-3.*

2) *For $c \geq 3$, $M(0,c)$ is disconnected.*

c) Instanton bundles.

A stable rank two bundle E on $\mathbf{P}^3_{\mathbf{C}}$ is called a k-instanton bundle (k being a positive integer) if it satisfies the following conditions :

$$1)\quad c_1(E) = 0,\ c_2(E) = k, \qquad 2)\quad H^1(E(-2)) = 0.$$

Let $M(k)$ be the moduli of k-instanton bundles. Then the real points $M(k)_{\mathbf{R}}$ of $M(k)$ is a real analytic manifold of (real) dim $= 8k-3$. When $k = 1, 2$, $M(k)$ is connected.

Theorem (M.Atiyah, R.Ward 1977 [A2]) *There is a natural one to one correspondence between*

a) self dual solutions of the $SU(2)$ Yang-Mills equations on S^4 (4-dimensional sphere) up to gauge equivalence.

b) isomorphism classes of rank two algebraic vector bundles E on $\mathbf{P}^3_{\mathbf{C}}$ satisfying the conditions :

1) E has a symplectic structure.

2) The restriction of E to every real line of $\mathbf{P}^3_{\mathbf{C}}$ is (algebraically) trivial.

d) Topological bundles on $\mathbf{P}^n_{\mathbf{C}}$ ($n = 2, 3$).

<u>Theorem</u> (R.Schwartzenberger 1961 [S2], M.Atiyah, E.Rees 1976 [A1])
Every topological bundle on $\mathbf{P}^n_{\mathbf{C}}$ $(n = 2, 3)$ *has at least one holomorphic structure.*

1. 3) n = 4.

The only essentialy known example that is an indecomposable rank two vector bundle on $\mathbf{P}^4_{\mathbf{C}}$ is the Horrocks-Mumford bundle.

<u>Theorem</u> (G.Horrocks, D.Mumford 1963 [H4]) *There exists a rank two stable vector bundle* E *on* $\mathbf{P}^4_{\mathbf{C}}$ *with* $c_1(E) = 5,\ c_2 = 10$ *and 15,000 symmetries.*

On the other hand, H.Grauert and M.Schneider claimed the following important theorem concernig splitting of rank two vector bundles on $\mathbf{P}^4_{\mathbf{C}}$. However there was a gap in their proof.

<u>Theorem ?</u> (H.Grauert, M.Schneider 1977 [G2]) *A rank two vector bundle on* $\mathbf{P}^n_{\mathbf{C}}$ $(n \geq 4)$ *which is unstable is a direct sum of line bundles.*

If the above were true, then we have

<u>Corollary ?</u> *There exist rank two topological vector bundles on* $\mathbf{P}^n_{\mathbf{C}}$ $(n \geq 4)$ *which are not algebraic.*

1. 4) n = 5

Any indecomposable rank two vector bundles on $\mathbf{P}^5_{\mathbf{C}}$ are not constructed in characteristic zero case yet.

Although we have in positive characteristic cases

<u>Theorem</u> (H.Tango 1976 [T1]) *There exists a rank two stable vector bundle* E *on* $\mathbf{P}^5_k$ *in characteristic 2.*

and in addition,

<u>Theorem</u> (G.Horrocks 1978 [H3]) *There exist irreducible rank three vector bundles on* $\mathbf{P}^5_k$ *in characteristic different 2. In characteristic 2, they split as the sum of an indecomposable rank two bundle with a line bundle.*

2) Open problems

Under these background, R.Hartshorne posed the following famous conjectures concerning algebraic vector bundles on $\mathbf{P}^n_{\mathbf{C}}$ $(n \geq 5)$.

2. 1) Conjectures (R.Hartshorne 1979 [H2])

1) Does there exist an indecomposable vector bundle of rank two on $P^5_{\mathbf{C}}$?

2) Is every rank two vector bundle on $\mathbf{P}^n_{\mathbf{C}}$ $(n \geq 7)$ a direct sum of line bundles ?

and further since there are no examples of non-singular closed subvarieties of low codimensions in projective spaces that are not complete intersections,

3) If X is a nonsingular closed subvariety of $\mathbf{P}^n_{\mathbf{C}}$ and if $\dim X > \frac{2n}{3}$, then is X a complete intersection ?

Needless to say, the book : Geometric Invariant Theory (for short, GIT) is one of the most important reference book of invarint theory and moduli theory and we can find the following interesting comment therein.

G I T (2nd Enlarged Edition 1982 [M5]) The question of the existence of non-trivial rank two vector bundle on $\mathbf{P}^n_{\mathbf{C}}$, $n \geq 5$, is the most interesting problem in projective geometry that I know of.

2. 2) Several results

Though many mathematicians have tried to solve the conjectures for almost thirty years, we do not have obtained any complete answers yet.

However it is proved as for splitting of vector bundles on projective spaces that

Theorem (W.Barth,Van de Ven 1974 [B2], E.Sato 1977 [S1]) *Any infinitely extendable vector bundle on* $\mathbf{P}^n_k$ *is a direct sum of line bundles.*

On the other hand, it is shown as for complete intersectionness of nonsingular closed subvarieties of low codimensions in projective spaces that

Theorem (Z.Ran 1983 [R1]) *Let X be a locally complete intersection subvariety of codimension 2 in P^{m+2} over an algebraically closed field of arbitrary characteristic. Let N denote the normal bundle of X and d its degree. Assume that $\wedge^2 N \simeq \mathcal{O}(\nu), \nu \in \mathbf{Z}$, and moreover that either (1) $\nu \geq \frac{d}{m} + m$, or (2) $d \leq m$. Then X is a complete intersection.*

Recently the following splitting criterion is obtained using determinantal variety method.

<u>Theorem</u> (H.Sumihiro 1999 [S4], H.Sumihiro, S.Tagami 2001 [S5]) *Let E be a rank two vector bundle on $\mathbf{P}^n_k$ $(n \geq 4)$ in any characteristic and P a 4- or 5-dimensional projective linear subspace of $\mathbf{P}^n_k$ and let $\overline{E} = E|P$ be the restriction of E to P. Then E splits into line bundles if and only if $H^1(P, End(\overline{E})) = 0$.*

3) Determinantal variety method

Now let us explain the determinantal variety method to deal with the splitting problems of vector bundles on projective spaces. Please see [S4] for the precise definition of determinantal varieties and algebro-geometric properties that determinantal varieties enjoy.

3. 1) Problems (in any characteristic)

We are aiming at to prove the following.

<u>Problem 1</u> : If E is a rank two bundle on $\mathbf{P}^4_k$ with $c_1^2 - 4c_2 \geq 0$ where c_i $(i = 1, 2)$ is the i-th Chern number of E , then does E split into line bundles ?

<u>Problem 2</u> : If E is an unstable rank two bundle on $\mathbf{P}^5_k$, then does E split into line bundles ?

<u>Problem 3</u> : If E is a rank two bundle on $\mathbf{P}^6_k$, then does E split into line bundles ?

<u>Remark</u> : Be careful for characteristic 2 !

3. 2) Reduction to positive characteristic (in zero characteristic case).

Assume that E is a vector bundle situated in Problems 1), 2) or 3). Let $\mathcal{E}$ be a rank two vector bundle on $\mathbf{P}^n_\Lambda = \mathbf{P}^n \times Spec(\Lambda)$, where Λ is a finitely generated $\mathbf{Z}$ algebra contained in $\mathbf{C}$ such that $E \simeq \mathcal{E}|\mathbf{P}^n_\eta \otimes \mathbf{C}$ (η being the generic point of $Spec(\Lambda)$). Then for a closed point x in $Spec(\Lambda)$, $\mathcal{E}_x = \mathcal{E}|\mathbf{P}^n_x$ is a rank two vector bundle on $\mathbf{P}^n_x$ in positive characteristic. If we could show that $\mathcal{E}_x$ does split, then it follows from the above splitting criterion that E splits into line bundles.

3. 3) We shall treat with Problem 1) in the sequel. Assume that E is a very ample rank two bundle on $\mathbf{P}^4_k$ with $c_1^2 - 4c_2 \geq 0$ in positive characteristic p.

(3.3.1) Determinantal varieties.

Take sufficiently general three sections $\{s_1, s_2, s_3\}$ of E which satisfy the following conditions :

(1) $Y = D_1 \cap D_2 \cap D_3$ is a smooth closed subscheme of pure codimension 3 in $P(E)$, where D_i $(i = 1,2,3)$ is the tautological divisor defined by s_i.

(2) $W(s_1) \cap W(s_2) \cap W(s_3) = \emptyset$, where $W(s_i)$ is the zero locus of s_i $(i = 1,2,3)$.

Let X be the closed subscheme of $\mathbf{P}^4_k$ with the following defining equations :

$$X : s_i \wedge s_j = 0 \quad (1 \leq i \leq j \leq 3).$$

Then X is isomorphic to Y through the structure morphism $P(E) \to \mathbf{P}^4_k$. Hence X is a smooth surface contained in $\mathbf{P}^4_k$ which is called the determinantal variety of E.

(3.3.2) Divisors on X.

We shall put the following distinguished divisors on X.

D : the restriction of the tautological divisor of $P(E)$ to $Y \simeq X$.

H : the restriction of the hyperplane of $\mathbf{P}^4_k$ to X.

a) Let $F = c_1 H - D$. Since the complete linear system $|F|$ is free from base points and of dimension 2, it defines a morphism $\varphi : X \to P^2_k$ which is finite and separable.

b) Let $a = \max\{n \in \mathbf{Z} \mid H^0(P^4_k, E(-n)) \neq 0\}$ and let

$$Z = D - aH, \quad Z^* = D - (c_1 - a)H.$$

Since E is unstable, i.e., $2a \geq c_1$, $Z^* = Z + (2a - c_1)H$ is an effective divisor and

$$Z \cdot Z^* = -(a^2 - c_1 a + c_2)H^2 = -(a^2 - c_1 a + c_2)(c_1^2 - c_2).$$

Hence it follows that E splits into line bundles if and only if Z^* is a nef divisor because $c_2(E(-a)) = a^2 - c_1 a + c_2 \geq 0$.

As for nefness of effective divisors on nonsingular projective surfaces, we have the following asymptotic criterion.

Lemma *An effective divisor W on a nonsingular projective surface X is nef if and only if* $\dim H^1(X, \mathcal{O}(-rW)) \leq O(r^1)$ *for* $r \gg 0$.

It is wellknown that Kodaira-Nakano vanishing theorem does not hold in positive characteristic. Let us pose the following problem because if it were true, it might be usefull in studies of algebraic geometry in positive characteristc. For example, the above lemma is easily obtained from it.

Problem(Kodaira-Nakano vanishing in positive characteristic)

Let X be a nonsingular projective variety defined over an algebraically closed field k of positive characteristic and L an ample line bundle on X. Then there exists a constant c which depends only on X and p and q such that

$$\dim H^q(X, L \otimes \Omega_X^p) \leq c \quad \text{for } p+q \geq \dim X + 1.$$

(3.3.3) Let $\sigma : X \to X$ be the (absolute) Frobenius morphism with the exponent $q = p^n$ and $F : X \to X'$ the relative Frobenius morphism with respect to $\varphi : X \to \mathbf{P}_k^2$.

There exists the exact sequence :

$$(*) \qquad 0 \to \mathcal{O}_{X'} \to F_*(\mathcal{O}_X) \to F_*(\mathcal{O}_X)/\mathcal{O}_{X'} \to 0,$$

where $F_*(\mathcal{O}_X)/\mathcal{O}_{X'}$ has support in the ramification divisor of φ.

Let $\psi : X' \to X$ be the projection morphism. Then it is calculated that

$$\dim H^1(X', \psi^*(\mathcal{O}_X(-Z^*)) = \dim H^1(X, \mathcal{O}_X(-Z^*)),$$

$$\dim H^1(X', F_*(\mathcal{O}_X) \otimes \psi^*(\mathcal{O}_X(-Z^*)) = \dim H^1(X, \mathcal{O}_X(-qZ^*)).$$

Therefore it is observed that $\dim H^1(X, \mathcal{O}_X(-qZ^*)) \leq O(q^1)$ using the exact sequence $(*)$ if we can prove that $\dim H^1(X, F_*(\mathcal{O}_X)/\mathcal{O}_{X'} \otimes \psi^*(\mathcal{O}_X(-Z^*)) \leq O(q^1)$. By use of the above lemma, it follows that Z^* is nef and so E is a direct sum of line bundles.

How can we describe the structure of the torsion sheaf $F_*(\mathcal{O}_X)/\mathcal{O}_{X'}$ under the condition $c_1^2 - 4c_2 \geq 0$?

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