代数幾何学シンポジウム記録
2001年度 pp.59-72



# A SIMPLE CHARACTERISATION OF TORIC VARIETIES

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## §1 INTRODUCTION

This paper contains an announcement of some results that will be contained in a paper entitled "A Geometric characterisation of toric varieties". The underlying ideas are quite simple but some of the proofs are a little delicate and technical. In this paper I will state a simplified version of these results and give a sketch of the main ideas of the proof. Complete proofs and statements will of course appear in the paper referred to above.

Toric varieties seem to be ubiquitous in algebraic geometry and indeed other branches of mathematics and physics. This seems intriguing behaviour, for which there seems to be no real explanation. This forms one of the motivating questions for this paper.

**Motivating Question:** Why are toric varieties so ubiquitous?

One possible way to answer this question is to try to find a simple characterisation of toric varieties. If the characterisation is simple enough it might indicate why toric appear so often. Recall the definition of a toric variety:

**1.1 Definition.** Let $X$ be a normal variety and let $D$ be a reduced integral divisor (that is every component of $D$ has coefficient one). We say that the pair $(X, D)$ is **toric** if $U = X \setminus D$ is isomorphic to a torus and the natural action of the torus on itself extends to an action on $X$.

Note that the components of $D$ are precisely the invariant divisors. Note also that toric varieties are rational as they contain an open set isomorphic to a torus. In general it is very hard to give rationality criteria, that is it is extremely hard to decide if a given variety is rational or not. It is an even harder problem to determine if a given surface is isomorphic to $\mathbb{A}^2$ (this is easily seen to be equivalent to the Jacobian conjecture, see [6]). In particular it only seems reasonable to expect a simple criteria when $X$ is proper.

Let us look at some simple examples of proper toric varieties to get some idea of what they look like. The simplest example is $\mathbb{P}^1$. In this case $D$ consists of two points. More generally the pair $(\mathbb{P}^n, D)$ where $D$ consist of $n + 1$ hyperpanes in general position is toric. One can take the product of any two toric varieties and get a toric variety. The simplest example of this is $\mathbb{P}^1 \times \mathbb{P}^1$. In this case $D$ has four components consisting of a pair of fibres for both fibrations. More generally $\mathbb{F}_n$ is toric, where $\mathbb{F}_n$ denotes the unique $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ with a unique section $E_\infty$ of self-intersection $-n$. In this case $D$ consists of $E_0 + E_\infty + F_1 + F_2$ where $E_0$ is any section disjoint from $E_\infty$ and $F_1$ and $F_2$ are two disjoint fibres.

Note that in all of these examples there is a simple formula connecting the number of invariant divisors $d$, the dimension $n$ and the Picard number $\rho$:
The number of invariant divisors is equal to the dimension plus the Picard number.

More generally the same is true if we replace Picard number by the rank of the group of Weil divisors modulo algebraic equivalence. In fact this result is easy to check using the combinatorial description of a toric variety via fans. These considerations motivate the following definition. Recall that a $\mathbb{Q}$-divisor $\Delta$ is said to be a boundary if its coefficients lie between zero and one.

**1.2 Definition.** Let $X$ be an irreducible variety, of dimension $n$, and let $\Delta$ be a boundary. Let $d$ be the sum of the coefficients of $\Delta$. The components of $\Delta$ generate a subgroup of the Weil divisors modulo algebraic equivalence. The **rank** $r$ of $\Delta$ is defined to be the rank of this subgroup. The **absolute rank** $R$ of $X$ is the rank of the group of all Weil divisors modulo algebraic equivalence.

The **complexity** $c$ of the pair $(X, \Delta)$ is $r + n - d$. The **absolute complexity** $C$ of the pair $(X, \Delta)$ is $R + n - d$.

Note that $r \leq R$ so that in particular $c \leq C$. Note also that if $\Delta = D$ is an integral divisor then $d$ simply counts the number of components of $D$. Thus if the pair $(X, D)$ is toric then the absolute complexity is zero. There are two other things to note about toric varieties. The first is that $K_X + D$ is linearly equivalent to zero. Indeed the obvious logarithmic differential defined on the torus extends to a meromorphic differential with simple poles along the invariant divisors. The second is almost a formal consequence of this, that $K_X + D$ is log canonical.

**1.3 Definition.** Let $X$ be a normal variety and let $\Delta$ be a boundary. Suppose that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and let $\pi : Y \longrightarrow X$ be a birational morphism. We may write

$$K_Y + \Gamma = \pi^*(K_X + \Delta).$$

We will call $\Gamma$ the **log pullback** of $\Delta$.

We say that the pair $(X, \Delta)$ is **log canonical** (respectively kawamata log terminal) if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and the log pullback is always a boundary (respectively always a strict boundary, that is every coefficient is less than one).

Recall that a divisor is said to be $\mathbb{Q}$-Cartier if some multiple is Cartier and that $X$ is said to be $\mathbb{Q}$-factorial if every $\mathbb{Q}$-Weil divisor is $\mathbb{Q}$-Cartier.

Indeed the point is that to check that the pair $(X, \Delta)$ is log canonical it suffices to check that the log pullback $\Gamma$ of $\Delta$ is a boundary for a single embedded resolution $\pi : Y \longrightarrow X$ of the pair $(X, \Delta)$. If $X$ is a toric variety and $D = D_X$ is the union of the invariant divisors then there is a toric embedded resolution $\pi : Y \longrightarrow X$. Moreover in this case $D_Y$, the union of the invariant divisors, is the log pullback of $D_X$. In fact the equation

$$K_Y + D_Y = \pi^*(K_X + D_X)$$

obviously holds, since it simply asserts that zero is equal to zero.

The following conjecture is an amalgam of a Conjecture of Shokurov and some observations of mine. We work over an arbitrary field, not necessarily algebraically closed. If $\Delta = \sum_i a_i \Delta_i$ then $\llcorner \Delta \lrcorner = \sum_i \llcorner a_i \lrcorner \Delta_i$ and $\ulcorner \Delta \urcorner = \sum_i \ulcorner a_i \urcorner \Delta_i$. Recall that a $\mathbb{Q}$-Cartier divisor is said to be nef if its intersection with every curve is non-negative.

**$1.4_n$ Conjecture.** *Let $X$ be a proper variety of dimension $n$ and let $\Delta$ be a boundary. Suppose that $K_X + \Delta$ is log canonical and $-(K_X + \Delta)$ is nef.*

*Then*

(1) $c \geq 0$.

(2) *If $C < 2$ then $X$ is geometrically rational.*

(3) *If $c < 1$ then there is a divisor $D$ such that the pair $(X, D)$ is toric. Moreover $\llcorner \Delta \lrcorner \subset D$ and $D - S$ is linearly equivalent to a divisor with support in $\Delta$, where $S$ is either empty or an irreducible divisor.*

Geometrically rational means that $X$ is rational over the algebraic closure of the groundfield. This conjecture reflects what I hope is a guiding principle with respect to the complexity:

**Guiding Principle** The smaller the complexity the easier it is to classify the pair $(X, \Delta)$.

The inspiration for this conjecture arises from the theory of complements, which were introduced by Shokurov. A complement is a log canonical pair $(X, \Delta)$ such that some multiple $r$ of $K_X + \Delta$ is linearly equivalent to zero. Complements are extremely useful for classification. In practice one is given a pair $(X, \Delta)$ and the idea is to modify $\Delta$ so that $r$ becomes relatively small. In these terms (1.4) and the guiding principle suggest that the complexity of this problem grows with the complexity of the pair $(X, \Delta)$.

Here are some interesting examples:

**1.5 Examples.**

(i) Let $X = \mathbb{P}^2$ and take $\Delta = aC + L_1 + L_2$, where $L_1$ and $L_2$ are two lines and $C$ is a smooth curve of degree $d$ in general position and $0 < a \leq 1/d$.

In this case the complexity is between zero and one and for $D$ we may take $S+L_1+L_2$ where $S$ is a line in general position. Thus there are some cases where we need the extra divisor $S$.

(ii) Pick three lines $D = L_1 + L_2 + L_3$ in $\mathbb{P}^2$ and consider the following sequence of blows ups $\pi : S \longrightarrow \mathbb{P}^2$. Let $p$ be the intersection of $L_1$ and $L_2$ and first blow up $p$. Let $E_1$ be the exceptional divisor and now blow up the point $E_1 \cap L_1$ (here we abuse notation slightly and refer to a divisor and its strict transform by the same symbol) to get a new exceptional divisor $E_2$. Now blow up the intersection of $E_1$ and $E_2$ to get a third exceptional divisor $E_3$. Finally blow up a point of $E_3$ not on $E_1$ or $E_2$. The whole point of this construction is that $S$ is not toric. On the other hand the complexity of the pair $(\mathbb{P}^2, D)$ is zero and if we let $G$ be the log pullback of $D$ then $G$ contains every exceptional divisor with coefficient one apart from the last (indeed only the last blow up is not toric). Thus the complexity of the pair $(S, G)$ is one.

(iii) Take $X = \mathbb{F}_n$ and let $D = 2E_\infty + \sum F_i$, where $\sum F_i$ consists of $n+2$ fibres, using the notation established above. In this case the complexity is arbitrarily large and negative. Note that if one contracts the negative section, then the image of $D$ is a boundary. The assumption that $K_X + \Delta$ is log canonical is therefore essential.

(iv) Let $X$ be an elliptic curve and let $\Delta$ be empty. Then the absolute complexity is two and the complexity is one. Clearly $X$ is not geometrically rational.

(v) Let $C$ be a smooth conic in $\mathbb{P}^2$ with no rational points and let $D$ be an element of $|-K_C|$, whose support over the algebraic closure consists of two points. Then the complexity of the pair $(C, D)$ is one but $C$ is not rational. Of course $C$ is geometrically rational but not rational.

Note that in all of the examples above $K_X + \Delta$ is numerically trivial. Indeed (1.4) can be strengthened slightly to include the limiting cases provided we assume that $K_X + \Delta$ is not numerically trivial. Thus if $c = 0$ we expect that this forces $K_X + \Delta$ to be numerically trivial and the conclusions of (1.4.2) and (1.4.3) ought to hold even in the cases $C = 2$ and $c = C = 1$ provided that $K_X + \Delta$ is not numerically tivial.

## §2 Statement of Results

(1.4) has been proved in dimension two. A complete proof appears in [9] and the case of Picard number one and integral boundary is contained in [2] but in fact it is easy to modify the proof given there to the general case. Note that this result in dimension two forms the backbone of the classification of log del Pezzos (normal surfaces with quotient singularities such that $-K_S$ is ample). [7] contains a very special case of (1) and (3) of (1.4) in dimension three. (1) and (3) of (1.4) were also proved by Cheltsov (unpublished), in the case where $X$ is $\mathbb{Q}$-factorial of Picard number one.

Here is what I can prove:

**2.1 Theorem.** *(1.4) holds provided $D = \Delta$ is integral, $X$ is projective and $\mathbb{Q}$-factorial and and the characteristic is zero.*

A similar result holds in characteristic $p$ provided one makes the additional assumption that $h^1(X, \mathcal{O}_X)$ is zero. For those allergic to log canonical singularities here is a weak version of (2.1):

**2.2 Corollary.** *Let $X$ be a smooth projective variety and let $D$ be a normal crossings divisor in $X$. Suppose that the number of components of $D$ is equal to the Picard number of $X$ plus the dimension of $X$.*

*Then the pair $(X, D)$ is toric.*

Note that (2.1) gives a criteria to ensure that a variety is rational. It might be interesting to look for applications of (2.1) in this light. Unfortunately (2.1) applied to hypersurfaces in $\mathbb{P}^n$ does not say anything beyond some easy to prove statements about quadrics and cubics. A more productive place to look might well be moduli spaces. These often come equipped with a divisor $D$ such that $K_X + D$ is log canonical, as the pair $(X, D)$ is often locally a quotient of a smooth variety and a normal crossings divisors. Note that in applications the complexity we generally be more useful than the absolute complexity. Indeed it is relatively straightforward matter to find all the relations between a finite set of divisors and thereby determine $r$. To compute $R$ is potentially far harder since we need to consider every divisor in $X$.

One of the most powerful techniques that has emerged in the last twenty years or so is the MMP (minimal model program) which has been proved to exist up to dimension three in characteristic zero and if conjectured to exist in all dimensions. Here then is an excellent reason to believe (1.4).

**2.3 Theorem.** *Assume that the MMP holds over $k$, a field of characteristic zero, in dimension $n$.*

*Then $(1.4)_n$ holds. In particular (1.4) holds in dimension one and two over an arbitrary field and in dimension three over a field of characteristic zero.*

It is also possible to prove (1.4) in characteristic $p$ provided one replaces the constants 1 and 2 of (1.4) by 1/2 and 3/2.

One interesting feature of the proofs of (2.1) and (2.3) is the fact that the most important point is to find birational morphisms $f : Y \longrightarrow X$ that only extract divisors of coefficient one (or log discrepancy zero). Indeed if the coefficient is one then the complexity only goes down and we are free to replace $X$ with $Y$. Of course the point is to choose $Y$ so that its geometry is more transparent. This goes against the grain of the modern view of higher dimensional geometry where we use the MMP to successively contract divisors on $X$ until we get a Mori fibre space. Unfortunately it is not clear to me why it is easier to prove (1.4) in the case of a Mori fibre space (other than in low dimensions when we use some very deep results about adjunction or the case of Picard number one mentioned which uses a cleaver that does not seem to generalise to other Mori fibre spaces) nor how to conclude from there that the original space satisfies the conclusions of (1.4).

## §3 PROOF OF (2.1)

In this section we indicate how to prove (2.1). In fact we will go backwards in the sense that we will start with the easiest case and gradually improve on this case making it more general until we have in fact proved (2.1). We will also focus mainly on the proof of (1.4.3), whose proof is the most interesting. Finally we will also often prove (2.1) and (2.3) with the complexity replaced by the absolute complexity. This will simplify many of the proofs.

The following Lemma is not logically necessary to the argument but it does gives an indication of what is involved in the proof of (1.4).

**3.1 Lemma.** *(2.1) holds if $X$ is a curve.*

*Proof.* As $C < 2$ and $n + R = 2$ it follows that $d > 0$. Thus $-K_X$ is ample and $X$ is certainly geometrically rational. If $c < 1$ and $n + r \geq 1$ it follows that $\Delta$ is non-empty so that $n + r = 2$. In this case $d > 1$ and so at least one point of $\Delta$ must be geometrically irreducible, that is $X$ has a rational point. But then $X$ is rational and (2.1) certainly holds. □

**3.2 Lemma.** *(2.1) holds if $X$ is a product of copies of $\mathbb{P}^1$.*

*Proof.* It clearly suffices to prove (1.4.1) and (1.4.3). To simplify matters we will work with the absolute complexity rather than the ordinary complexity. Suppose that $C < 1$. Then by considerations of multi-degree it is clear that the projection of every component of $D$ onto one of the factors must be a point and that there are two such components for every projection. In this case the pair $(X, D)$ is clearly toric and the result is clear. □

**3.3 Lemma.** *(2.1) holds if there is a birational morphism $\pi : X \longrightarrow Y$ where $Y$ is a product of copies of $\mathbb{P}^1$.*

*Proof.* As before, it clearly suffices to prove (1.4.1) and (1.4.3).

Let $G$ be the pushforward of $D$. Recall that a divisor is said to be pseudo-effective if it is a limit of effective divisors. As $-(K_X + D)$ is nef it is the imit of ample divisors and so certainly it is pseudo-effective. But the pushforward of a pseudo-effective divisor is certainly pseudo-effective and so $-(K_Y + G)$ is pseudo-effective. But as $Y$ is a product of curves it follows that $-(K_Y + G)$ is nef.

On the other hand a moments thought will convince the reader that the complexity can only decrease under pushforward of divisors. It follows that the complexity of the pair $(Y, G)$ is at most zero, and if we have equality then the complexity of the pair $(X, D)$ is also zero. But by (3.2) the pair $(Y, G)$ must be toric and the complexity of the pair $(Y, G)$ is equal to zero.

Thus the complexity of the pair $(X, D)$ is equal to the complexity of the pair $(Y, G)$. If we work with the absolute complexity it is clear that this is only possible if every divisor contracted by $\pi$ has coefficient one in $D$. It is not too hard to prove that the same is true for the complexity.

By (3.4) every divisor contracted by $\pi$ corresponds to a toric valuation of $Y$. Extracting those toric valuations of $Y$ we reduce to the case that $\pi$ does not contract any divisors (however $Y$ need no longer be a product of copies of $\mathbb{P}^1$, just a toric variety and $\pi$ need not be a morphism anymore, just a birational map). In this case $X$ and $Y$ are isomorphic in codimension one. As $Y$ is toric, the MMP holds for $Y$ and so we may factor this morphism into a sequence of flips. Each flip is toric, so by induction on the number of flips the pair $(X, D)$ is toric as required. □

**3.4 Definition-Lemma.** *Suppose that $X$ is a toric variety. We will call a valuation $\nu$* **toric**, *if there is a birational toric morphism $Y \longrightarrow X$ such that $\nu$ corresponds to an exceptional divisor.*

*Let $\Delta$ be a boundary which is supported on the invariant divisors. Then every valuation $\nu$ of log discrepancy less than one with respect to $K_X + \Delta$ is toric.*

*Proof.* We may as well assume that $\Delta$ is the union of the invariant divisors with coefficient one. Let $\nu$ be a valuation of log discrepancy less than one. Suppose that $\pi : Y \longrightarrow X$ is a toric morphism. Let $\Gamma$ be the log pullback of $\Delta$. Then $\Gamma$ is also the union of the invariant divisors and $\nu$ has log discrepancy less than one with respect to $K_X + \Gamma$. Thus we are free to replace the pair $(X, \Delta)$ by the pair $(Y, \Gamma)$. Hence passing to a toric resolution of the pair $(X, \Delta)$, we may assume that $X$ is smooth and that $\Delta$ has normal crossings.

Now $\nu$ determines a tower of blow ups, each blow up with centre the centre of $\nu$, such that the centre of $\nu$ is eventually a divisor (see for example (2.45) of [5]). If the centre of $\nu$ is a divisor we are done. By induction on the number of blow ups, therefore we may as well assume that there is one blow up $\pi : Y \longrightarrow X$ such that $\nu$ becomes a divisor. Working locally about the centre of $\nu$, we may assume that $X$ is affine and that the centre of $\nu$ is a point. Therefore we may assume that $X = \mathbb{A}^n$ and that $\Delta$ is the union of some of the co-ordinate hyperplanes and that the centre of $\nu$ is the origin.

The condition that the blow up is not toric, is equivalent to requiring that the support of $\Delta$ has at most $n-1$ components. In this case, by direct calculation, the log discrepancy of $\nu$ is at least one, a contradiction. □

So our strategy is now clear. Modify $X$ in such a way that there is a birational morphism to a product of copies of $\mathbb{P}^1$. Clearly we must first find a morphism to $\mathbb{P}^1$, or in other words a pencil. The obvious place to look for pencils is to find two effective divisors $D_0$ and $D_1$ with support in $D$ such that $D_0$ is linearly equivalent $D_1$. Unfortunately we are not told that there are many linear equivalences amongst the components of $D$, we are just given algebraic relations. So we need a result that reduces algebraic equivalence to linear equivalence. In other words we need the fact that the Albanese is trivial.

**3.5 Lemma.** *Let $X$ be a projective variety of dimension $n$ over a field of characteristic zero. Suppose that $-(K_X+\Delta)$ is nef and $K_X+\Delta$ is log canonical. Suppose that $C<2$ and that (2.1) holds in dimension $n-1$.*

*Then the dimension of the Albanese variety of $X$ is zero.*

*Proof.* Suppose not. Then the Albanese map $g : X \dashrightarrow A$ is non-trivial. The idea is to exhibit a rational curve in $X$ whose image in $A$ is not a point and thereby derive a contradiction.

Let $f : X' \longrightarrow Z$ be the graph of $g$. Consider the induced birational morphism $\pi : X' \longrightarrow X$. Let $E$ be any $\pi$-exceptional divisor. Then the image of some fibre of $E$ over $\pi(E)$ inside $f(E)$ is not a point. Suppose that the log discrepancy of $E$ is greater than zero. Then possibly rechoosing $E$, we may assume that every fibre of $E$ over $\pi(E)$ is uniruled by (3.6) and so $f(E)$ is uniruled, a contradiction as $A$ is an abelian variety.

It follows that every divisor extracted by $g$ has log discrepancy zero. Then replacing the pair $(X,\Delta)$ by $(X',\Delta')$ where $\Delta'$ is the log pullback of $\Delta$ we may assume that $g$ is a morphism.

Let $Y$ be the fibre of $g$ over the generic point of the image of $X$. Let $\Gamma$ be the restriction of $\Delta$ to $Y$. Then the pair $(Y,\Gamma)$ is log canonical and so by induction the complexity of $(Y,\Gamma)$ is at least zero. It follows that there is a fibre $F$ of $g$ such that every component of $F$ is contained in the support of $\Delta$.

Let $\Delta=\Delta_h+\Delta_v$ be the decomposition of $\Delta$ into horizontal and vertical components. Pick any curve $C$ that dominates $A$. Then $C\cdot\Delta_v\geq C\cdot F>0$. If (1) holds then it follows by the cone Theorem that there is an extremal ray $R$ that is $F$ positive. $R$ is then generated by a rational curve $C$ which is not contained in a fibre as $F\cdot C>0$. But then $A$ contains a rational curve which is impossible. □

**3.6 Lemma.** *Let $(X, \Delta)$ be a log canonical pair. Let $\pi : Y \longrightarrow X$ be a projective birational morphism and suppose that $\pi$ extracts at least one divisor of coefficient less than one.*

*Then there is at least one component $F$ of the exceptional locus $E$ of coefficient less than one such that the fibres of $F$ over $\pi(F)$ are uniruled.*

*Proof.* We may write

$$K_Y + \Gamma + E = \pi^*(K_X + \Delta) + R$$

where $\Gamma$ is the strict transform of $\Delta$, $E$ is the sum of all the exceptional divisors taken with coefficient one and $R$ is effective and exceptional. By assumption $R$ is non-empty. First observe that there is a component $F$ of the exceptional locus that is covered by curves $C$ that can be chosen to avoid any closed subset of codimension three or more such that $R \cdot C < 0$. Indeed the result is local about the base and so we may as well assume that $Y$ is affine; cutting by hyperplanes we reduce to the case of dimension two. But then this is a well known result due to Artin, see for example (2.19) of [4].

As $C$ can be chosen to avoid any codimension three subset it follows that we may find a resolution $\psi : Z \longrightarrow Y$ in a neighbourhood of $C$ such that if we write

$$K_Z + \Gamma' + E' + R' = \psi^*(K_Y + \Gamma + E)$$

where $\Gamma'$ and $E'$ denote strict transforms then $R'$ is effective and exceptional. Indeed this is reduces to a problem about surfaces and in this case the minimal desingularisation suffices. But then by adjunction $K'_F \cdot C < 0$. As $C$ is contained in the smooth locus of $\pi$ we are done by (7.6) of [3]. □

*3.7 Remark.* Presumably much more is true, presumably the conclusion of (3.6) holds for every divisor of coefficient less than one. For example if one has the MMP then one can selectively contract any divisor of log discrepancy between one and zero.

**3.8 Example.** There is an interesting example to show why we need (3.6). Let $S$ be the cone over an elliptic curve and let $\pi : T \longrightarrow S$ be the minimal desingularisation. Then $\pi$ extracts a copy $E$ of the elliptic curve and it follows easily by adjunction that

$$K_T + E = \pi^* K_S.$$

Thus $S$ is log canonical but not log terminal. Clearly $E$ is not uniruled. Moreover $T$ is naturally a ruled surface over $E$ so that the Albanese of $T$ is given by the natural morphism $T \longrightarrow E$. In this case the Albanese map for $S$ is not a morphism.

Using (3.5) we may conclude that if two divisors $A$ and $B$ are algebraically equivalent then there is some $r$ such that $rA$ is linearly equivalent to $rB$. This property goes under the handy catch phrase **algebraic equivalence implies linear equivalence.** Note that this is the only place where we use the condition that $X$ is projective and that the characteristic is zero.

Suppose that $C < 2$ and algebraic equivalence implies linear equivalence. Then we can find two divisors $D_0$ and $D_1$ with support in $D$ such that $rD_0$ is linearly equivalent to $rD_1$. Let $Y \subset X \times \mathbb{P}^1$ be the total space of the corresponding pencil. Thus there is a birational morphism $\pi : Y \longrightarrow X$ and a morphism $f' : Y \longrightarrow \mathbb{P}^1$. Let $f : Y \longrightarrow C$ be the Stein factorisation so that $f$ is a contraction morphism. Let $G$ be the log pullback of $D$. We want to replace $X$ with $Y$. Arguing as in the proof of (3.3) this is surely okay provided the complexity of the pair $(Y, G)$ is the same as the complexity of the pair $(X, D)$. In other words we have to prove that every divisor extracted by $\pi$ has coefficient one.

**3.9 Lemma.** *Every divisor extracted by $\pi$ has coefficient one and $C$ is isomorphic to $\mathbb{P}^1$.*

*Proof.* Let $E$ be an exceptional divisor for $\pi$. Let $V$ be the image of $E$ in $X$. Then the fibres of $E$ over $V$ are copies of $C$. It follows that $V$ has codimension two. Cutting by hyperplanes we may asssume that $X$ is a surface $S$ and $E$ is a copy of $C$. $G_0$ and $G_1$ (the inverse image of $D_i$) are two components of $G$ that meet $E$ and are disjoint. Let us try to apply adjunction to $E$,

$$(K_S + G)|_E = K_E + P_0 + P_1 + R.$$

Here $P_i$ is the intersection of $E$ with $G_i$ and $R$ is whatever is left over. The crucial point is that $R$ is effective (indeed this is a crucial and basic fact about adjunction of log divisors see for example Chapter 16 of [4]) so that in fact $R = 0$, $C$ is isomorphic to $\mathbb{P}^1$ and $(K_S + E) \cdot E = 0$. But then

$$K_S + G + E = \pi^*(K_S + D)$$

since both sides are in fact zero. □

Note that we can squeeze one more fact from the proof of (3.9). Since $R$ is empty in fact $G_0$ and $G_1$ are the only fibres of $f$ that are contained in the support of $G$. So replacing the pair $(X, D)$ by $(Y, G)$ we may assume that there is a contraction morphism $f : X \longrightarrow \mathbb{P}^1$. Moreover if $F$ denotes the generic fibre of $f$ and $G$ the restriction of $D$ to $F$ then it is not hard to check that the complexity of the pair $(F, G)$ is no more than the complexity of the pair $(X, D)$. Now we use linear equivalence on $F$ to induce linear equivalences on $X$. Proceeding by an obvious induction in this way we reduce to the case where $X$ admits a contraction morphism to the product of $n$ copies of $\mathbb{P}^1$. By (3.3) this completes the proof of (2.1).

It is instructive to consider a few examples to see how this argument works in practice.

**3.10 Example.** Let $X = \mathbb{P}^2$ and take $D$ to be three lines. We look for a linear equivalence. Take two lines $D_0$ and $D_1$. The base locus of this pencil is $D_0 \cap D_1$ so that the morphism from the total space $Y \longrightarrow X$ blow ups this point. Let $E$ be the exceptional divisor. Then the generic fibre $F$ of $f : Y \longrightarrow \mathbb{P}^1$ is a copy of $\mathbb{P}^1$ and the third line meets $F$ in a single point. So locally in a neighbourhood of the generic fibre $E \sim D_3$. We try to lift this to the whole of $X$. This is easy, for example $E + D_0$ is linearly equivalent to $D_3$. Since the base locus is a point then we get $\mathbb{P}^2$ blown up at two points and a birational morphism to $\mathbb{P}^1 \times \mathbb{P}^1$. In fact in this way we recover the classical birational map between a quadric and $\mathbb{P}^2$.

Now suppose that we start with a different pencil. For example $D_1 + D_2$ is linearly equivalent to $2D_3$. In this case the base locus is not a reduced scheme and in fact $Y$, the total space of the pencil is not smooth. In fact the fibre corresponding to $D_3$ contains two nodes (or $A_1$ singularities). So even if we only want to prove (2.2) in the proof of (2.1) given here, we need to consider varieties that are not smooth.

Suppose that we look at a similar example but now in $\mathbb{P}^3$. Thus now we have four planes $D = D_0 + D_1 + D_2 + D_3$. Then $D_0 + D_1 \sim D_2 + D_3$. Locally about the common point of intersection of $D_0$, $D_1$ and $D_2$ the equation for $D_0 + D_1$ is $xy = 0$ and $D_2$ is $z = 0$ so that the total space of the pencil is given locally as $xy + zt = 0$. Thus $Y$ need not even be $\mathbb{Q}$-factorial. However the generic fibre $F$ of $f : Y \longrightarrow \mathbb{P}^1$ is $\mathbb{Q}$-factorial and of course this is all we need for the induction to go through.

## §4 Proof of (2.3)

In this section we indicate how to prove (2.3). The first point to note is that passing to a log terminal model we may assume that $X$ is projective and $\mathbb{Q}$-factorial. The idea, of course, is to manipulate $\Delta$ until all its coefficients are one and then to apply (2.1). In principle this ought to be easy. We have already proved that algebraic equivalence implies linear equivalence and so there are plenty of linear equivalences between reduced divisors $D_0$ and $D_1$ where $D_0$ and $D_1$ are contained in the support of $\Delta$. Repeatedly replacing $\Delta$ by $\Delta_t = \Delta + t(D_1 - D_0)$ for an approapariate positive value of $t$, we ought to be able to reduce to the case where $\Delta$ is integral. Here of course we order $D_0$ and $D_1$ so that the sum of the coefficients of $D_1$ is greater than the sum of the coefficients of $D_0$. With this choice of $D_0$ and $D_1$ the complexity of the pair $(X, \Delta_t)$ is no more than the complexity of the pair $(X, \Delta)$.

However there are three potential problems with this idea. First of all we must make sure that $K_X + \Delta_t$ remains log canonical. So we increase $t$ until $K_X + \Delta_t$ is maximally log canonical and then extract any divisors of coefficient one and repeat the same process on the new space. Naively it would seems that the best one can do using this method is to reduce to the case where there are at most $r$ components of $\Delta$ of coefficient not equal to one. However this is not enough to apply (2.1). We need $\llcorner \Delta \lrcorner$ to have a lot of components (in fact we need at least $r+1$ components to ensure that we get a pencil of divisors with support in $\llcorner \Delta \lrcorner$). More to the point if we want to do better it is not at all clear that we can come up with a process that will terminate in a finite number of steps. Both of the problems above can be resolved using the idea of connectedness. Conjecturally if $\Delta$ is any effective divisor such that $-(K_X + \Delta)$ is nef then either the locus where the pair $(X, \Delta)$ is not kawamata log terminal is connected or the situation is very special indeed.

We recall some useful definitions due to Kawamata [1].

**4.1 Definition.** Let $X$ be a normal variety and let $\Delta$ be an effective divisor. A subvariety $V$ of $X$ is called a (respectively strict) log canonical center if it is the image of a divisor of log discrepancy at most zero (respectively less than zero). A (respectively strict) log canonical place is a valuation corresponding to a divisor of log discrepancy at most zero (respectively less than zero). The **log canonical locus** $\mathrm{LCS}(X, \Delta)$ (resp. $\mathrm{LCS}^+(X, \Delta)$) of the pair $(X, \Delta)$ is the union of the (respectively strict) log canonical centers.

It is easy to give examples that show we need to consider the possibility that the locus of log canonical singularities is not connected.

**4.2 Example.** Let $X = \mathbb{P}^1$ and $\Delta$ consist of two points $p + q$. Then $\mathrm{LCS}(X, \Delta) = \{p, q\}$ which is obviously not connected. Now take this example and cross it with an elliptic curve $C$. Thus we get two disjoint copies of $C$ living inside $C \times \mathbb{P}^1$. Now pick a point of either copy and blow up each of these points. The strict transforms of these curves are contractible and contracting them we get a projective surface $S$ such that $K_S$ is numerically trivial. The images of these two curves is the locus of log canonical singularities.

However this is the worse that can happen. In fact we have the following conjecture:

**4.3 Conjecture.** *Let $\pi : X \longrightarrow Y$ be a contraction morphism between irreducible varieties, where $X$ is normal.*

*Let $\Delta$ be an effective divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Suppose that $-(K_X + \Delta)$ is $\pi$-nef. Let $F$ be any geometric fibre of $\pi$. We work in a neighbourhood of $F$. Then every connected component of* $\mathrm{LCS}^+(X, \Delta)$ *meets every irreducible component of* $\mathrm{LCS}(X, \Delta)$.

*If further* LLC$(X, \Delta) \cap F$ *is not connected then* $K_X + \Delta$ *is log canonical, there are exactly two log canonical places* $\nu_1$ *and* $\nu_2$ *with respect to* $K_X + \Delta$ *and there is a covering family of rational curves* $C_s$ *such that for all* $s$, $C_s$ *intersects the centre of* $\nu_1$ *and* $\nu_2$ *and* $(K_X + \Delta) \cdot C_s = 0$.

Connectedness was first observed by Shokurov in his proof of 3-fold log flips [8], where he proved it for surfaces. In fact I can prove (4.3) assuming the MMP in dimension $n$. However the proof does not add much conceptually to Shokurov's original proof and so I will not reproduce it here. Essentially the idea is to undo the construction of (4.2) by running the MMP to unveil the covering family of curves we are looking for.

Using (4.3) it is easy to prove that we can reduce to the case where all but one component of $\Delta$ has coefficient one. For example if $X$ is a surface then the first time we construct a log canonical centre then we simply extract it and ignore that component of $\Delta$. The next time we construct a log canonical centre then either this centre intersects the original curve and we have constructed a log canonical centre of dimension zero (the intersection of these two components) and we can construct no more strict log canonical centres or the component is disjoint from the original component and it is still true that we cannot construct anymore strict log canonical centres. From there it is easy to prove that the process I sketched above terminates when all but one component of $\Delta$ has coefficient one.

The final problem is to convert the last component of coefficient less than one to a component of coefficient one. As we already have $n + r - 1$ components of coefficient one, a careful examination of the proof given in the last section will convince the reader that we can reduce to the case where we have a contraction morphism $X \longrightarrow Y$ where $Y$ is the product of $n - 1$ copies of $\mathbb{P}^1$. One component of $\llcorner \Delta \lrcorner$ is then a section of this fibration and the component with coefficient less than one then dominates $Y$. Now if $c < 1/2$ then the coefficient of this component is greater than $1/2$ and so it too must be a section and it is easy to argue that we can increase its coefficient to one.

Otherwise we need to apply the theory of complements. The theory of complements guarantees the existence of a reduced divisor $S$ such that $K_X + \llcorner \Delta \lrcorner + S = K_X + D$ is linearly equivalent to zero (this only works in in characteristic zero as we need to apply Kawamata-Viehweg vanishing, see Chapter 19 of [4]). In fact this will be the stage where we pick up the mysterious irreducible component $S$. Since the complexity of the pair $(X, D)$ is at most zero, $S$ can have at most one component. On the other hand if $c > 0$ then $K_X + \llcorner \Delta \lrcorner$ is not numerically trivial so $S$ cannot be empty.

Replacing $\Delta$ by $D$ as we have reduced to the case where $\Delta = \llcorner \Delta \lrcorner$ we can apply (2.1) and this finishes the proof.

Finally note one thing about the proof of (1.4). Since the MMP is known to hold only up to dimension three, it would be useful to eliminate its use. There are only two places where we need the MMP. First every time that $K_X + \Delta_t$ becomes maximally log canonical we need to extract a divisor of coefficient one. As the situation we are working with is so special we ought to be able to get around this point. The second place we need the MMP is to prove (4.3). Now (4.3) is known to hold if $-(K_X + \Delta)$ is nef and big. For this reason we ought to be able to prove (1.4) if there is an ample (or at least nef and big) divisor which is linearly equivalent to a sum of $K_X$ and the components of $\Delta$ (indeed there would then be a small deformation $\Delta'$ of $\Delta$ such that $-(K_X + \Delta')$ is big and nef that would not change $\mathrm{LLC}^+(X, \Delta)$). If $X$ is projective, and $0 \leq c \leq C < 1$ then this is automatic, since then every divisor is a linear combination of the components of $\Delta$. Thus it would seem that the most significant part of (1.4) is the fact that the complexity is always non-negative.

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