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A construction of Calabi-Yau manifolds with non-trivial finite fundamental groups

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1 Introduction

The purpose of this paper is to construct Calabi-Yau manifolds with non-trivial finite fundamental groups. Throughout this paper, a Calabi-Yau manifold is a smooth projective variety X of dimension 3 defined over the complex number field $\mathbf{C}$ such that its canonical line bundle is trivial and $H^1(X, \mathcal{O}_X) = 0$. It is an interesting but difficult problem to find a Calabi-Yau manifold with a non-abelian finite fundamental group. But Beauville constructed a Calabi-Yau manifold whose fundamental group is the quaternionic group $H = \{\pm 1, \pm i, \pm j, \pm k\}$ in the following way;

Example. ([Be]) Let V be the regular representation of H. Then using characters of V, we can find a subvariety $\tilde{X}$ in $\mathbf{P}(V) = \mathbf{P}^7$ defined by four equations of degree 2 such that H acts on $\tilde{X}$ freely. Hence if let $X = \tilde{X}/H$, then X is a Calabi-Yau manifold with $\pi_1(X) = H$.

In this paper, we shall construct Calabi-Yau manifolds with $\pi_1 = H$ in a quite different manner. We use a flat deformation of a normal crossing variety. This idea stems from the work by Friedman [Fr]. Friedman introduced the concept of d-semi-stability for simple normal crossing varieties and showed that every d-semi-stable simple normal crossing K3 surface is smoothable by a flat deformation. In higher dimensional case, the following theorem is shown by Kawamata and Namikawa.

Theorem 2.1. ([Ka-Na] Theorem 4.2) *Let X be a compact Kähler d-semi-stable normal crossing variety of dimension $n \geq 3$ and let $\tilde{X}$ be the normalization of X. Assume the following conditions:*

(a) $\omega_X \cong \mathcal{O}_X$,
(b) $H^{n-1}(X, \mathcal{O}_X) = 0$, *and*
(c) $H^{n-2}(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$.
Then X is smoothable by a flat deformation. □

Let X_t be the smooth variety given by Theorem 2.1. Here we call X_t the smoothing of X. Then there is a natural map $\pi_1(X_t) \to \pi_1(X)$ is surjective (see [Ko] Lemma 5.2.2). Starting with a 3-dimensional normal crossing variety X with $\pi_1(X) = H$, we shall construct a Calabi-Yau manifold X_t by deforming X. In own case, the natural map $\pi_1(X_t) \to \pi_1(X)$ is an isomorphism; hence $\pi_1(X_t) = H$. We shall briefly sketch the construction.

The quaternionic group H acts freely on a 3-dimensional sphere S^3. The quotient space S^3/H called a quaternionic space is given by identifying certain boundaries of the fundamental domain by the action of H on S^3. We will take the triangulation of S^3/H and construct a normal crossing variety X whose dual graph is the triangulation. Then the fundamental group of X is isomorphic to H. However, X is not d-semi-stable. In order to make it d-semi-stable, we must take the blowing-up of X along a suitable curve on the singular locus. If let Y be the blowing-up of X, then we can deform Y to a smooth Calabi-Yau manifold Y_t by Theorem 2.1. We can calculate its Euler number, Betti number and fundamental group. In fact, we have a Calabi-Yau manifold Y_t with

the Euler number $e(Y_t) = 0$,
the Picard number $\rho(Y_t) = 2$, and
the fundamental group $\pi_1(Y_t) = H$.

Moreover we can find a birational map $\varphi : Y_t \dashrightarrow Z$ contracting a del Pezzo surface to a point. Deforming Z, we have a Calabi-Yau manifold Z_s with

the Euler number $e(Z_s) = -16$,
the Picard number $\rho(Z_s) = 1$, and
the fundamental group $\pi_1(Z_s) = H$.

$e(Z_s) = -16$ is equal to the Euler number of Beauville's example. It would be interesting to know if our manifold Z_s is deformation equivalent to Beauville's one.

The fundamental group of a Calabi-Yau manifold in our construction acts on S^3 freely. As such non-abelian finite groups, there are so-called binary polyhedral groups. Hence, starting another binary polyhedral group G instead of H, it is possible to construct a Calabi-Yau manifold with $\pi_1 = G$.

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2 Deformation theory of normal crossing varieties

The purpose of this section is to describe Theorem 2.1 about the deformation of normal crossing varieties.

Definition. A reduced complex analytic space X of dimension n is a *normal crossing variety* (or *n.c.variety*) if for each point $p \in X$,

$$\mathcal{O}_{X,p} \cong \mathbf{C}\{x_0, \ldots, x_n\}/(x_0 \cdots x_r) \quad (0 \leq r = r(p) \leq n).$$

In addition, if every component X_i of X is smooth, then X is called a *simple normal crossing variety* (or *s.n.c.variety*).

Let X be a normal crossing variety and assume that the smoothing of X exists. Let $\mathcal{X}$ be the smooth total space and $f : \mathcal{X} \to \Delta$ the deformation of X. Then the normal bundle $\mathcal{N}_{X/\mathcal{X}}$ of X is trivial. In general, $\mathcal{N}_{X/\mathcal{X}}$ depends on $\mathcal{X}$, but $\mathcal{N}_{X/\mathcal{X}}|_{\mathrm{Sing}(X)}$ does not depend on $\mathcal{X}$. It is determined by only the structure of X.

Definition. Let X be a n.c.variety of dimension n and $D = \mathrm{Sing}(X)$. Then there is a partial open covering of X with holomorphic functions $\mathcal{U} = \{U_\lambda, z_0^{(\lambda)}, \ldots, z_n^{(\lambda)}\}$ such that the following conditions are satisfied:

(1) $\{U_\lambda\}$ is a partial open covering containing D.

(2) For each λ, there are integers $r = r(\lambda)$ and an isomorphism

$$\varphi_\lambda : U_\lambda \tilde{\to} V_\lambda = \{(x_0, \ldots, x_n) \in \mathbf{C}^{n+1}; x_0 \cdots x_r = 0\}$$

$$\text{such that} \quad z_j^{(\lambda)} = \begin{cases} \varphi_\lambda^*(x_j) & (0 \le j \le r) \\ \text{invertible} & (r+1 \le j \le n). \end{cases}$$

(3) For λ, μ with $U_\lambda \cap U_\mu \neq \emptyset$, there are invertible holomorphic functions $u_j^{(\lambda\mu)}$ $(0 \le j \le n)$ on $U_\lambda \cap U_\mu$ and a permutation $\sigma = \sigma(\lambda, \mu) \in \mathfrak{S}_{n+1}$ satisfying

$$z_{\sigma(j)}^{(\lambda)} = u_j^{(\lambda\mu)} z_j^{(\mu)}$$

Define by $\mathcal{O}_D(-X)$ the line bundle on D induced by the invertible holomorphic functions $\{u_0^{(\lambda\mu)} \cdots u_n^{(\lambda\mu)}|_D\}$ and $\mathcal{O}_D(X) := \mathcal{O}_D(-X)^\vee$, which is called the *infinitesmal normal bundle* on D.

Remark. In the above definition, invertible holomorphic functions $\{u_j^{(\lambda\mu)}\}$ are not uniquely determined. If let

$$u_j'^{(\lambda\mu)} = u_j^{(\lambda\mu)} + a_j^{(\lambda\mu)} z_0^{(\mu)} \cdots \check{z}_j^{(\mu)} \cdots z_n^{(\mu)} \quad (a_j^{(\lambda\mu)} \in H^0(\mathcal{O}_{U_\lambda \cap U_\mu})),$$

$\{u_j'^{(\lambda\mu)}\}$ also satisfies the condition (3a). But restricting these functions to D,

$$u_0^{(\lambda\mu)} \cdots u_d^{(\lambda\mu)}|_D = u_0'^{(\lambda\mu)} \cdots u_n'^{(\lambda\mu)}|_D \quad \text{on } D$$

Hence $\mathcal{O}_D(-X)$ is uniquely determined up to isomorphism.

Remark. For a s.n.c.variety X, Friedman defines $\mathcal{O}_D(-X)$ in his paper as follows [Fr];

Let X_i be a component of X and let I_{X_i} (resp. I_D) be the defining ideal of X_i (resp. D) in X. Then define

$$\mathcal{O}_D(-X) := I_{X_1}/I_{X_1}I_D \otimes_{\mathcal{O}_D} \cdots \otimes_{\mathcal{O}_D} I_{X_m}/I_{X_m}I_D.$$

If X is a s.n.c.variety, Friedman's definition coincides with our definition.

Definition. A n.c.variety X is *d-semi-stable* if its infinitesmal normal bundle $\mathcal{O}_D(X)$ is trivial.

Theorem 2.1. ([Ka-Na] Theorem 4.2) *Let X be a compact Kähler d-semi-stable n.c.variety of dimension $n \geq 3$ and let $X^{[0]}$ be the normalization of X. Assume the following conditions:*
(a) $\omega_X \cong \mathcal{O}_X$,
(b) $H^{n-1}(X, \mathcal{O}_X) = 0$, *and*
(c) $H^{n-2}(X^{[0]}, \mathcal{O}_{X^{[0]}}) = 0$.
Then X is smoothable by a flat deformation. □

3 Example of normal crossing varieties

In this section, we construst a n.c.variety whose fundamental group is the quaternionic group H. The quaternionic group acts S^3 freely. So we should just give a triangulation to the quotient S^3/H, and construct a n.c.variey whose dual graph is the triangulation.

Write $S^3 = \{x \in \mathbf{H}; ||x|| = 1\}$ where $\mathbf{H}$ is a set of quaternions. Then the action of $H = \{\pm 1, \pm i, \pm j, \pm k\}$ on S^3 is given by

$$\begin{array}{ccc} S^3 & \longrightarrow & S^3 \\ x & \longmapsto & hx \quad (h \in H). \end{array}$$

Thus the fundamental domain for the quotient space S^3/H is given as a cube. Opposite faces of the cube are identified under a right-helix turn of angle $\frac{\pi}{2}$ as in Figure 1. (see [Mo] Ch.3) So we take a triangulation for S^3/H as in Figure 2. At first, put the points, circles and triangles, on the vertices of the cube. Circles and triangles are identified by right-helix turns respectively. Next, connect a circle to a triangle on a edge and circles on a face. Finally, put the point, a square, on the center of the cube and connect a square to circles and triangles. This gives a triangulation of S^3/H. (Figure 2)

We shall construct a n.c.variety whose dual graph is the above triangulation. Let X_2 and X_3 be the blowing-ups of $\mathbf{P}^3 = \mathrm{Proj}\mathbf{C}[T_0, T_1, T_2, T_3]$ along four points $(1:0:0:0), (0:1:0:0), (0:0:1:0)$ and $(0:0:0:1)$. Let X_1 be the blowing-up of X_2 along the proper transforms of six lines $\{T_i = T_j = 0\}$ $(0 \leq i < j \leq 3)$. Moreover let $D_{ij}^{(k)}$ be the plane in X_i as in

Figure 3. The isomorphism $\varphi_{ij}^{(k)} : D_{ij}^{(k)} \to D_{ji}^{(k)}$ gluinig X_i to X_j are defined as canonical identifications of local coordinates by the correspondence of same numbers in Figure 3.

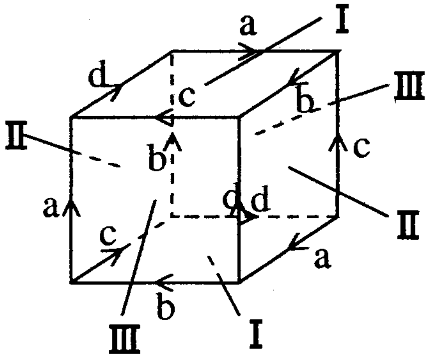


Figure 1.

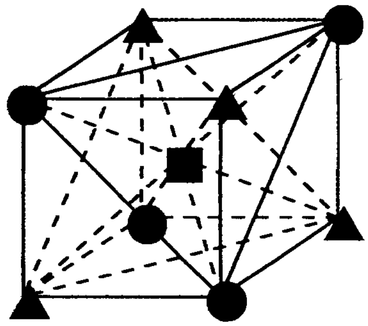
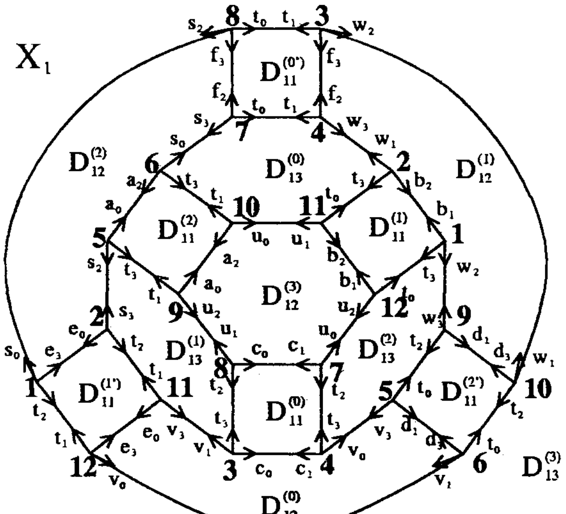
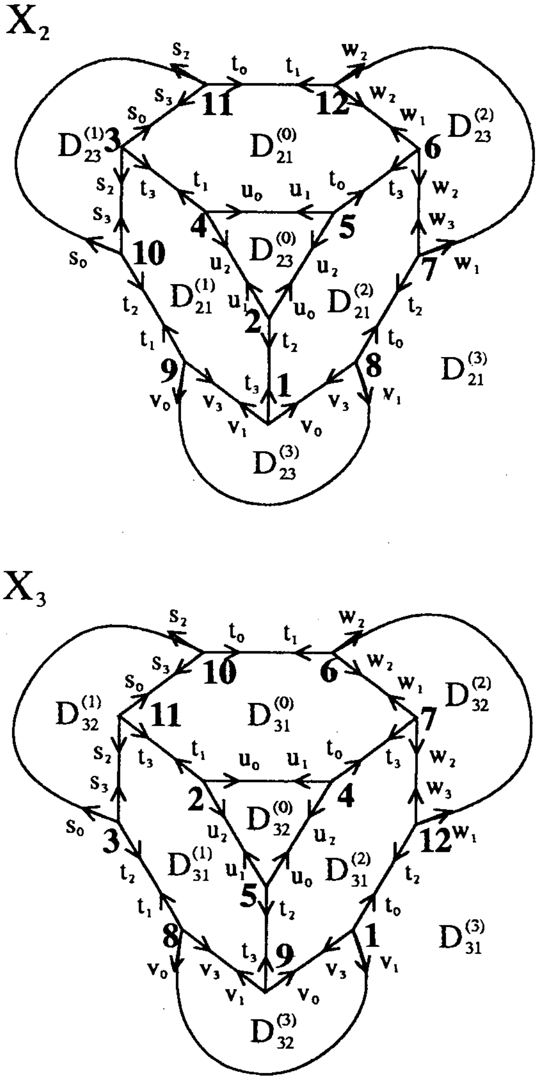


Figure 2.




X2
11
12
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D(2)21
D(3)21
D(3)23
X3
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4
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8
9
1
D(0)31
D(0)32
D(1)32
D(2)32
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D(3)31
D(3)32

Figure 3.

For example, $\varphi_{12}^{(0)} : D_{12}^{(0)} \to D_{21}^{(0)}$ is defined by

$$\begin{aligned}
&\varphi_{12}^{(0)}(c_0) = t_3,\ \varphi_{12}^{(0)}(c_1) = t_1,\ \varphi_{12}^{(0)}(v_0) = u_0,\ \varphi_{12}^{(0)}(v_3) = u_1,\\
&\varphi_{12}^{(0)}(d_1) = t_0,\ \varphi_{12}^{(0)}(d_3) = t_3,\ \varphi_{12}^{(0)}(v_1) = w_1,\ \varphi_{12}^{(0)}(v_0) = w_2,\\
&\varphi_{12}^{(0)}(e_3) = t_1,\ \varphi_{12}^{(0)}(e_0) = t_0,\ \varphi_{12}^{(0)}(v_3) = s_3,\ \varphi_{12}^{(0)}(v_1) = s_0.
\end{aligned}$$

Then by these isomorphisms $\varphi_{ij}^{(k)}$, we can glue X_i together. Let X_1' be the variety given by the gluing of X_1 on $D_{11}^{(k)}$ and $D_{11}^{(k')}$ by $\varphi_{11}^{(k)}$ and let X be the variety given by the gluing of X_i by $\varphi_{ij}^{(k)}$. Note by D the singular locus of X. For these X and D, it follows from van Kampen Theorem that $\pi_1(X) = \pi_1(D) = H$.

Theorem 3.1. *X is a projective n.c.variety.*

Proof. We can construct an divisor L on X such that $L|_{X_i}$ is an ample divisor for all i. □

4 Trivialization of infinitesmal normal bundle

In section 3, we constructed a n.c.variety X with $\pi_1(X) = H$. But X is not d-semi-stable. To apply Theorem 2.1 to X, we will blow-up X along the divisor C on $D = \mathrm{Sing}(X)$ associated to $\mathcal{O}_D(X)$. At first, we must construct the divisor C.

Define the hypersurface R in $\mathbf{P}^3$ by $R = \{\sum_{i<j} T_iT_j = 0\} \subset \mathbf{P}^3$. Let R_i be the proper transform of R in X_i and let

$$D_i = \bigcup_{j,k} D_{ij}^{(k)} \subset X_i, \quad \text{the anti canonical divisor on } X_i$$

$$C_i = R_i|_{D_i} \text{ and } C_{ij}^{(k)} = C_i|_{D_{ij}^{(k)}}.$$

Then C_i are patched each other by $\varphi_{ij}^{(k)}$, so define by it a Cartier divisor C on D.

Proposition 4.1 $\mathcal{O}_D(C) \cong \mathcal{O}_D(X)$.

Proof. To show this, we may observe invertible holomorphic functions defining $\mathcal{O}_D(X)$. □

Next, by blowing-up X along C, we construct a d-semi-stable n.c.variety. To preserve the projectivity, we must blow-up X according to the order of indices of components of X. This operation is as follows. (locally, in Figure 4)

step1 Blow up X_1 along C_1.

step2 Blow up X_2 along $C_{23}^{(k)}$.

step3 Blow up X_1 along the proper transform of $D_{12}^{(k)}$ to resolve ordinary double points.

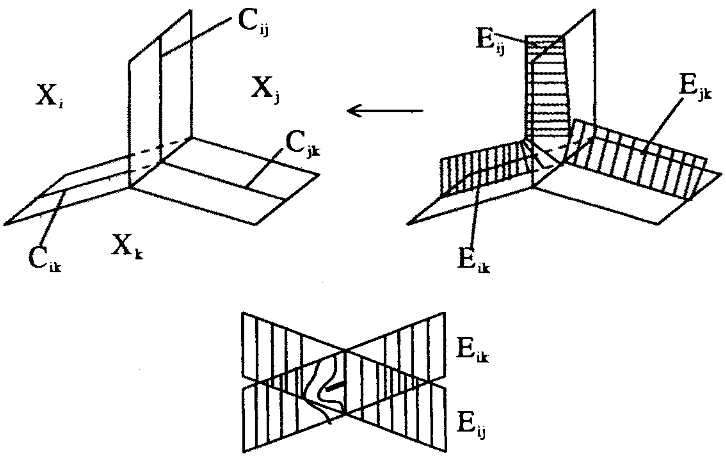

Figure 4.

Let Y_i be the blowing-up of X_i and let $E_{ij}^{(k)}$ be the exceptional divisor over $C_{ij}^{(k)}$. Replace D (resp. D_i, D_{ij}) by the proper transform of D (resp. D_i, D_{ij}). Let $Y = Y_1 \cup Y_2 \cup Y_3$, then Y is a d-semi-stable n.c.variety and there is a birational map $\pi : Y \to X$.

Theorem 4.2. *Y is a projective d-semi-stable n.c.variety satisfying all of the assumptions in Theorem 2.1.* □

5 Smoothing

By Theorem 2.1, Y is smoothable by a flat deformation. Let $f : \mathcal{Y} \to \Delta$ be this deformation, $Y = f^{-1}(0)$ and $Y_t = f^{-1}(f)$ $(t \neq 0)$ the general fiber of f. Then Y_t is a Calabi-Yau manifold. We can culculate topological invariants of Y_t such as the Euler number, the Betti number and the fundamental group.

Proposition 5.1. ([Pe]) *Let $f : \mathcal{Y} \to \Delta$ be a flat deformation of a n.c.variety. Let $Y = f^{-1}(0)$ be a smoothable n.c.variety and let $Y_t = f^{-1}(t)$ be a smoothing of Y. Then*

$$e(Y_t) = e(Y) - e(\mathrm{Sing}(Y))$$

Proof. Topologically, Y_t is given as a so-called real blowing-up of Y along $\mathrm{Sing}(Y)$. □

Proposition 5.2. *Let Y be a n.c.variety with a flat deformation $f : \mathcal{Y} \to \Delta$ and a smoothing Y_t. Assume that $H^1(Y, \mathcal{O}_Y) = 0$ and $\omega_Y \cong \mathcal{O}_Y$. Then*

$$b_2(Y_t) = b_2(Y) + h^0(Y^{[0]}, \mathcal{O}_{Y^{[0]}}) - h^0(Y, \mathcal{O}_Y). \ \square$$

Theorem 5.3. *Let Y and Y_t be the above. Then*

$$\pi_1(Y_t) \cong \pi_1(Y) \cong H.$$

Proof. In general, there is a natural surjective map $s : \pi_1(Y_t) \to \pi_1(Y)$. ([Ko] Lemma 5.2.2) Now, Ker(S) is generated by cycles in S^1 which is a fiber over Sing(Y). By observing the relations among the cycles, we can show that Ker$(S) = \{1\}$. □

Corollary 5.4. *Y_t is a Calabi-Yau manifold with*

$$e(Y_t) = 0, b_2(Y_t) = 2\ b_3(Y_t) = 6, \text{ and}$$
$$\pi_1(Y_t) = H. \ \square$$

6 Birational contraction map

In the previous sections, we constructed a Calabi-Yau manifold Y_t with $\pi_1(Y_t) = H$ and the Picard number $\rho(Y_t) = 2$. In this section, we find a birational contraction map of Y_t to a Calabi-Yau threefold with $\rho = 1$.

Let R_1 be the proper transform of $\{\sum_{i<j} T_iT_j = 0\} \subset \mathbf{P}^3$ in X_1 as in section 4. Let S be the proper transform of R_1 in Y. Then S is a del Pezzo surface of degree 4. There is an obstruction in $H^1(S, \mathcal{N}_{S/Y})$ to extending S to a subvariety in Y_t. ([Mu]) Since $S \cap \mathrm{Sing}(Y) = \emptyset$ by the construction of Y,

$$H^1(S, \mathcal{N}_{S/Y}) = H^1(S, \omega_S) = H^1(S, \mathcal{O}_S) = 0$$

by the adjunction formula. So S extends to a del Pezzo surface S_t in Y_t.

Proposition 6.1. *There is a birational map $\varphi : Y_t \to Z$ contracting S_t to a point $p \in Z$.*

Proof. This follows from contraction theorem and intersection theory. □

Since S_t is del Pezzo surface of degree 4, the singularity (Z, p) is an isolated complete intersection singularity defined by two equations f and g in $\mathbf{C}^5$. Let f_0 and g_0 be the initial parts of f and g. f_0 and g_0 are homogenious of degree 2, so we may assume $f_0 = x_1^2 + \cdots + x_5^2$. It follows from the next theorem by Namikawa that Z smooth by a flat deformation.

Theorem 6.2. ([Na] Theorem 5) *Let Z be a Calabi-Yau threefold with isolated rational Gorenstein singuralities, that is, Z is a projective variety of dimension 3 with isolated rational Gorenstein singularities such that $\omega_Z \cong \mathcal{O}_Z$ and $H^1(Z, \mathcal{O}_Z) = 0$. Assume that*
(a) Z is $\mathbf{Q}$-factorial,
(b) every singularity on Z is locally smoothable, and
(c) Kuranishi space of every singularity on Z is smooth.
Then Z is smoothable by a flat deformation. □

It is easy to show that Z satisfies all of the assumptions in Theorem 6.2. So Z is smoothable. Let Z_s be a smoothing of Z. Then Z is a Calabi-Yau manifold. Moreover, we can calculate topological invarints of Z_s.

Theorem 6.3. *Z_s is a Calabi-Yau manifold with*

$$e(Z_s) = -16, \ b_2(Z_s) = 1 \text{ and } b_3(Z_s) = 20. \ \square$$

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