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On the definitions of the Painlevé equations

Hiroshi Umemura,
Graduate school of mathematics,
Nagoya University

1 Introduction

Today, there are a variety of ways of defining the Painlevé equations. Most of them are unimaginable from the original definition.

(1) Historically the origin of the Painlevé equations goes back to the pursuit of special functions defined by algebraic differential equations of the second order. Around 1900 Painlevé succeeded in classifying algebraic differential equations $y" = f(t, y, y')$ without movable singular points, where F is a rational function of t, y and y' and t is the independent variable so that $y' = dy/dt$ and $y" = d^2y/dt^2$. The property being free from the movable singularites is nowadays called the Painlevé property. After he clasified the equations satisfying the condition, Painlevé then threw away those equations that he could integrate by the so far known functions and thus he arrived at the list of the six Painlevé equations. This is the first definition of the Painlevé equations. It is, however, very lucky that he could discover the Painlevé equations in this manner.

(2) In 1907, R. Fuchs discovered that the sixth Painlevé equation describes a monodromy preserving deformation of a second order ordinary linear equation $y" = p(x)y$. Later R. Garnier generalized this for the other Painlevé equations.

(3) In our former work [2], we showed that we can recover the second Painlevé equation form a raional surface with a rational double point. We can regard this as an algebro-geometric definition of the second Painlevé equation.

(4) Masatoshi Noumi and Yasuhiko Yamada interpreted theory of Painlevé equations form the view point of Kač-Moody Lie algebra. They not only unifromly reviewed the threory of τ function of the Painlevé equations but also generalized the Painlevé equations in the natural frame work.

(5) There is another definition due to J. Drach [1] in 1914. He asserts the equivalence of the following two conditions for a function $\lambda(t)$.

(i) $\lambda(t)$ satisfies the sixth Painlevé equation.

(ii) The dimension of the Galois group of a non-linear differential equation

$$\frac{dy}{dt} = \frac{y(y-1)(t-\lambda)}{t(t-1)(y-\lambda)}$$

is finite.

In the second assertion, the Galois group of general algebraic differential equation is involved. Namely the second assertion depends on his infinite dimensional differential Galois theory, which has been an object of discussion since he proposed it in his thesis in 1898.

In this note, we apply our infinite dimensional Galois theory of differential equations [3] to study the result of J. Drach. It is difficult to imagine the equivalence of the assertions. We prove that (i) implies (ii) for the first Painlevé equation.

Theorem 1 *Let $\lambda(t)$ be a function satisfying the first Painlevé equation $\lambda" = 6\lambda^2 + t$. Then the Galois group $Infgal(L/K) = \widehat{SL_2}$, where*

$$K = \mathbf{C}(t,\ \lambda(t),\ \lambda'(t)),\ L = K(y)$$

such that y is transcendental over K satisfying

$$\frac{dy}{dt} = \frac{1}{2}\frac{1}{y-\lambda(t)}.$$

Then .

$$Infgal(L/K) \simeq \widehat{SL}_{2L^\natural}.$$

Why is the theorem interesting? Because the Galois group, which is a formal group of infinite dimension in general, is very difficult to calculate.We have only two types of examples where we can calculate the Galois group. (1) If L/K is a strongly normal extention in the sense of Kolchin which is his generalization of classical Galois extention so that the Galois group $G := Gal(L/K)$ of the extension is an algebraic group, then $Infgal(L/K) = \hat{G}$ and (2) the Galois group of a Riccati equation coincides with the formal completion of the Galois group of the linearization of the Riccati equation.

Since we can prove only one dirrction of the assertion of Drach, our result is not satisfactory in the sense that it does not give us a new definition of the Painlevé equation. It offers us, however, a highly non-trivial example of a differential field extension of which we can caluculate our Galois group.

2 Review of R. Fuchs' paper

R. Fuch studied a monodromy preserving deformation of a linear differential equation $d^2y/dx^2 = p(x)y$. Namely he considered a system of linear equations

$$(1) \qquad \begin{cases} \frac{\partial^2 y_i}{\partial x^2} &= p y_i, \\ \frac{\partial y_i}{\partial t} &= B y_i - A \frac{\partial y_i}{\partial x}, \end{cases} \qquad \text{for } i = 1,\, 2,$$

where

$$p = \frac{a}{x^2} + \frac{b}{(x-1)^2} + \frac{c}{(x-t)^2} + \frac{e}{(x-\lambda)^2} + \cdots$$

and we assume that λ is not a function of t but it is a function of x, i.e. $\partial\lambda/\partial x = 0$. y_1 and y_2 are linearly independent solutions. The integrability of the system ( ) implies

$$A(x,\, t) = \frac{x(x-1)(t-\lambda)}{t(t-1)(x-\lambda)} \quad \text{and} \quad B(x,\, t) = \frac{1}{2}\frac{\partial A}{\partial x}$$

and $\lambda(t)$ satisfies the sixth Painlevé equation P_{VI}.

Where comes the non-linear differential equation

$$\frac{dy}{dt} = \frac{y(y-1)(y-\lambda)}{t(t-1)(t-\lambda)}$$

from?

Lemma 1 *We may assume that the Wronskian*

$$W_r = \begin{vmatrix} y_1 & y_2 \\ \frac{\partial y_1}{\partial x} & \frac{\partial y_2}{\partial x} \end{vmatrix} = 1.$$

Proof. It is an exercise to check $\partial W_r/\partial t = \partial W_r/\partial x = 0$.

From now on we write T for t, W for x so that we consider the system

$$(2) \qquad \begin{cases} \frac{\partial^2 y_i}{\partial W^2} &= p y_i, \\ \frac{\partial y_i}{\partial T} &= B(W,\, T) y_i - A(W, T) \frac{\partial y_i}{\partial W}, \end{cases} \qquad \text{for } i = 1,\, 2,$$

Lemma 2 *If we set $y = y_1/y_2$, then we have*

$$\begin{cases} \frac{\partial y}{\partial W} &= \frac{1}{y_1^2}, \\ \frac{\partial y}{\partial T} &= -A \frac{1}{y_1^2}. \end{cases}$$

We are working in the differebtial field

$$\mathbf{C}(W,T)\langle\lambda(T)\rangle = \mathbf{C}(W,T,\lambda(T),\lambda'(T),\cdots)(y_1,y_2,\frac{\partial y_1}{\partial T},\frac{\partial y_2}{\partial T}) \tag{3}$$

with derivations $\{\partial/\partial W,\ \partial/\partial T\}$. The differential field extension

$$\mathbf{C}(W,T)\langle\lambda(T)\rangle(y_1,y_2,\frac{\partial y_1}{\partial T},\frac{\partial y_2}{\partial T})/\mathbf{C}(W,T)\langle\lambda(T)\rangle$$

is defined by the adjunction of the solutions y_i, y_2 of the system (1) of linear equations.

Now we introduce differential operators

$$\begin{cases} D_t &= \frac{\partial}{\partial T}+\frac{\partial}{\partial W}, \\ D_w &= y_1^2\frac{\partial}{\partial W}. \end{cases}$$

so that the field (3) is a differntial field with derivations $\{D_t,\ D_w\}$. If we regard the the field (3) as a differential field with derivations $\{D_t,\ D_w\}$, then it involvs non-linear differential equations.

Lemma 3 $D_tW = A(W,\ T)$.

Proof. This follows from the definition of the operator D_t.

Lemma 4

$$\frac{\partial y}{\partial T}+A\frac{\partial y}{\partial W}=0 \qquad \textit{so that } D_ty=0.$$

Proof. This is a consequaence of Lemma 2.

Lemma 4 shows that y is a first integral of $dY/dT = A(Y,\ T)$.

It follows from Lemma 2 $D_w(W) = y_1^2$ and hence y_1 is algebraic of degree (at most 2) over $\mathbf{C}(t)\langle\lambda\rangle\langle y\rangle\langle W\rangle$. Here $\langle\,\rangle$ should be interpreted in the differential field (3) with derivations $\{D_t,\ D_w\}$. Since $y_2 = yy_1$,

$$(\mathbf{C}(W,\ T)\langle\lambda\rangle(y_1,y_2,\frac{\partial y_1}{\partial W},\frac{\partial y_2}{\partial W}:\mathbf{C}(t)\langle\lambda\rangle\langle y\rangle\langle W\rangle)=2.$$

3 Infinite dimensional differential Galois theory

We start from a differentail field extension $L = \mathbf{C}(t)\langle\lambda\rangle\langle W\rangle$ over $K = \mathbf{C}(T)\langle\lambda\rangle$ with derivation D_t. They are subfields of

$$\mathbf{C}(W,\ T)\langle\lambda\rangle(y_1,y_2,\frac{\partial y_1}{\partial W},\frac{\partial y_2}{\partial W}).$$

Recall that we have

$$D_t(T) = 1, \quad D_tW = \frac{W(W-1(t-\lambda)}{T(T-1)(W-\lambda)}$$

and W is transcentental over the field K.

Let us now review our differential Galois theory of infinite dimension using a particular example. We start from the differential field extension $L = K(W)/K$ with derivation D_t. We define its Galois group. We consider the universal Taylor morphism $i : L \to L^\natural[[\tau]]$. Namely we set for an element $a \in L^\natural[[\tau]]$

$$i(a) = \sum_{n=0}^{\infty} \frac{1}{n!} D_t^n(a)\tau^n.$$

Here $L^\natural$ is the abstract field structure of the differential field L. Namely we forget in the differential field L the drivation D_t. The map i introduced above is a morphism of rings compatible with the derivations D_t and $\partial/\partial\tau$.

Consider now on $L^\natural$, the derivation $\partial/\partial W$, which we denote by $(\partial/\partial W)^\natural$ to avoide confusions. So we have in the power series ring $L^\natural[[\tau]]$ two mutually commutative derivations $\partial/\partial\tau$ and $(\partial/\partial W)^\natural$. The latter operates as a drivation of coeficients of a power series.

The quotient field of $L^\natural[[\tau]]$ is the field $L^\natural[[\tau]][\tau^{-1}]$ of Laurent series that is the differential field with derivations $\partial/\partial\tau$ and $(\partial/\partial W)^\natural$. In this differential field $L^\natural[[\tau]][\tau^{-1}]$, let $\mathcal{L}$ be the differential sufield generated by $i(L)$ and $L^\natural$ and we define $\mathcal{K}$ as the differential subfield generated by $i(K)$ and $L^\natural$.

Remark. Since the $L^\natural$-vector space $Der(L^\natural/K^\natural)$ of $K^\natural$-derivations of $L^\natural$ is 1-dimensional and so it is spanned by any non zero element of the $L^\natural$-vector space $Der(L^\natural/K^\natural)$. Hence we have

$$Der(L^\natural/K^\natural) = L^\natural(\partial/\partial W)^\natural$$

Therefore the construction of $\mathcal{L}$ and $\mathcal{K}$ is independent of the choice of a generator of the $L^\natural$-vector space $Der(L^\natural/K^\natural)$.

Now considering again the Taylor expansion of the coefficients of a Laurent series, we have a differential algebra morphism $L^\natural[[\tau]][\tau^{-1}] \to L^{\natural\natural}[[\xi]][[\tau]][\tau^{-1}]$, where ξ is the variable appearing when we expand the coefficients of our Laurent series.

$$L^\natural \to L^{\natural\natural}[[\xi]] \quad a \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} \left(\left(\frac{\partial}{\partial W}\right)^\natural\right)^n (a)\xi^n.$$

So now $\mathcal{L}$ and $\mathcal{K}$ are differential subfields of $L^{\natural\natural}[[\xi]][[\tau]][\tau^{-1}]$ with derivations $\{\partial/\partial\xi, \partial/\partial\tau\}$.

Now we consider the functor of infinitesimal deformations of $\mathcal{L}/\mathcal{K}$ in

$$L^{\natural\natural}[[\xi]][[\tau]][\tau^{-1}]$$

that is a principal homogeneous space of a formal group $Infgal(L/K)$ of infinite dimension in general. This is the definition of our Galois group. To be more precise, we consider the category $Alg(L^{\natural\natural})$ of commutative $L^{\natural\natural}$-alebras. We define the functor $F : Alg(L^{\natural\natural}) \to (Sets)$ by setting for $A \in Alg(L^{\natural\natural})$

$$F(A) := \{\varphi \in \mathcal{L} \to A[[\xi]][[\tau]][\tau^{-1}] \quad | \varphi \text{ is a differential algebra morphism satisfying the following two conditions below}\}$$

(i) φ induces the identity map on $\mathcal{K}$.

(ii) Let $N(A)$ be the ideal of the algebra A consisting of all the nilpotent elements of A. So we have a canonical morphism

$$r : A[[\xi]][[\tau]][\tau^{-1}] \to A/N(A)[[\xi]][[\tau]][\tau^{-1}]$$

of reducing the coefficients of Laurent series modulo the ideal $N(A)$. Let $j : \mathcal{L} \to A[[\xi]][[\tau]][\tau^{-1}]$ be the composite of the inclusions

$$\mathcal{L} \subset L^{\natural\natural}[[\xi]][[\tau]][\tau^{-1}] \subset A[[\xi]][[\tau]][\tau^{-1}].$$

Using this notation, the condition that we require is $r \circ \varphi = r \circ j$.

Intuitively φ is an infinitesimal deformation of the inclusion map j. Let

$$\mathcal{W}(\xi, \tau) \in L^{\natural\natural}[[\xi]][[\tau]][\tau^{-1}]$$

be the image of $W \in L$ by the canonical map

$$L \to L^{\natural\natural}[[\xi]][[\tau]][\tau^{-1}].$$

Let $\varphi \in F(A)$. The there exists a power series

$$\psi(\xi) = a_0 + a_1\xi + a_2\xi^2 + \cdots \in A[[\xi]]$$

such that

$$\varphi(W) = \mathcal{W}(\psi(\xi), \tau)$$

and such that $\psi(\xi)$ is congruent to ξ modulo $N(A)$. More precisely

$$a_0,\ a_1 - 1,\ a_2,\ a_3, \cdots \in N(A).$$

The infinitesimal defromation φ is determined by the pwoer series $\psi(\xi)$ because $\{D_t, (\partial/\partial W)^\natural\}$-differential field $\mathcal{L}$ over $\mathcal{K}$ is generated by $\mathcal{W}(\xi, \tau)$. The set

$$G(A) = \{\psi(\xi) = a_0 + a_1\xi +_2 \xi^2 + \cdots \in A[[\xi]] \,|\, a_0,\, a_1 - 1,\, a_2,\, a_3,\, \cdots \in N(A).\}$$

of formal power series congruent to the identity ξ modulo $N(A)$ forms a group by the composite of power series. The group functor G plays the role of the Lie pseudo-group of all the coordinate transformations of 1-variable. We can show that

$$H(A) = \{\psi \,|\, \mathcal{W}(\psi(\xi), \tau) \text{ defines an element of } F(A)\}$$

forms a subgroup of $G(A)$. The subgroup functor H is defined by a set of algebraaic diferential equations. So in the classical language H is a Lie pseudo-group and $H(A)$ operates on $F(A)$ functorially. The group functor H is the Galois group of L/K.

We want to show $Infgal(L/K) \simeq \widehat{SL}_{2L^\natural}$. It follows form Lie's classification of Lie algebras operating on a manifold of dimension 1. We have to show tr.d.$[\mathcal{L} : \mathcal{K}] = 3$. We have to connect L/K with the differential field (3) of §2. Ignoring the technical points, we have to show

Question. *The field of constants of the differential field*

$$\mathbf{C}(T,\, \lambda,\, \lambda',\, W,\, y_1,\, \partial y_1/\partial W)$$

with derivation D_t coincides with $\mathbf{C}$?

We can not answer the Question but we can answer an analogue of the Question for the first Painlevé equation.

Theorem 1. *Let us consider a differential field extension $L = \mathbf{C}(T,\, \lambda,\, \lambda',\, W)$ over $K = \mathbf{C}(T,\, \lambda,\, \lambda')$ with derivation D_t such that $D_tT = 1$,*

$$D_tW = \frac{1}{2}\frac{1}{W - \lambda(t)}$$

and such that λ satisfies the first Painlevé equation $D_t^2(\lambda) = 6\lambda + t$. We assume that W is transcendental over K. Then

$$Infgal(L/K) \simeq \widehat{SL}_{2L^\natural}.$$

The proof of the theorem is as much invilved as the proof of the irreducility theorem.

References

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