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Nobuyuki Kakimi

Freeness of adjoint linear systems on threefolds with $\mathbb{Q}$-factorial terminal singularities or some quotient singularities

Abstract. We define and calculate the weighted multiplicities of non-Gorenstein terminal singularities on threefolds and some quotient singularities. As an application, we improve freeness conditions on threefolds.

0 Introduction

Our results are generalization of the following conjecture by Fujita [F1]:

Conjecture 0.1. *For a smooth projective variety X and an ample divisor L on X, the linear system $|K_X + mL|$ is free if $m \geq dim X + 1$.*

A strong version of Fujita's freeness conjecture is the following:

Conjecture 0.2. *Let X be a normal projective variety of dimension n, $x_0 \in X$ a smooth point, and L an ample Cartier divisor. Assume that $L^n > n^n$, $L^d Z \geq n^d$ for all $Z \subset X$ with $x_0 \in Z$, and $d = \dim Z < n$. Then $|K_X + L|$ is free at x_0.*

We denote the cases where the conjectures are already proved. For smooth complex algebraic surface, Reider [Rdr] proved the strong version of Fujita's freeness conjecture by applying Bogomolov's instability theorem to study adjoint linear series on surfaces. For a projective normal surface, Ein and Lazarsfeld [EL], Matsushita [Mat], Kawachi [KM][Kwc], and Maşek [Ma] extended the result of Reider [Rdr] to singular cases. Langer [La1][La2] obtained the best estimates for a normal surface by applying a rank 2 reflexive sheaf. For Fujita's freeness conjecture, it is quite hard in dimension three proved by Ein and Lazarsfeld [EL]. The lectures of Lazarsfeld [L] provided a

[0]Nobuyuki Kakimi: Department of Mathematical Sciences, University of Tokyo, Meguro, Komaba, Tokyo 153-8914, Japan. e-mail:kakimi@318uo.ms.u-tokyo.ac.jp
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very good introduction. Kawamata[Ka] proved in dimension four case. For the strong version of Fujita's freeness conjecture, Fujita [F2] proved that, if $LC \geq 3$, $L^2S \geq 7$, and $L^3 > 51$, then $|K_X + L|$ is free at x_0. Kawamata [Ka] proved the following:

Theorem 0.3 ([Ka, Theorem 3.1]). *Let X be a normal projective variety of dimension 3, $x_0 \in X$ a smooth point, and L an ample Cartier divisor. Assume that $L^3 > 3^3$ and $L^dZ \geq 3^d$ for all subvariety $Z \subset X$ with $x_0 \in Z$ and $d = \dim Z < 3$. Then $|K_X + L|$ is free at x_0.*

Helmke [He] proved the following:

Theorem 0.4. *Let X be a smooth projective threefold and L be an ample divisor on X. Assume that for some point $x \in X$*
$L^3 > 27$,
$L^2S \geq 9$ for all surfaces S which are smooth at x,
$L^2S \geq 3$ for all surfaces S with a RDP at x,
$LC \geq 3$ for all curves C which are smooth at x.
Then $\mathcal{O}_X(K_X + L)$ is globally generated at x.

For a projective variety X of dimension 3 with some singularities, Oguiso and Peternell [OP] proved that, with only $\mathbb{Q}$-factorial Gorenstein terminal (resp. canonical) singularities and an ample divisor L on X, the linear system $|K_X + mL|$ is free if $m \geq 5$ (resp. $m \geq 7$). Ein, Lazarsfeld and Maşek [ELM], and Matsushita [Mat] extended some of the results of Ein and Lazarsfeld [EL] to projective threefolds with terminal singularities.

We [K1] extended the result of Kawamata [Ka] to normal projective threefolds with terminal Gorenstein singularities or some quotient singularities. Our freeness conditions in terminal Gorenstein singularities or quotient singular points of type $1/r(1,1,1)$ are better than in smooth case. We noticed that our proof [K1, Theorem 3.8] of canonical and not terminal singular point case is wrong. Note that Lee [L1] [L2] also obtained some results on only Gorenstein canonical singularities independently.

Theorem 0.5. *Assume $x_0 \in X$ is a quotient singular point of type $(1/r, a/r, b/r)$ such that an integer $r > 0$, $(r,a) = 1$, and $(r,b) = 1$. Let $L^3 > 3^3/r$, $L^2S \geq 3^2/r$ for all surfaces S which $(S, x_0) \cong \mathbb{C}^2/\mathbb{Z}_r(1, a')$ for $(1, a') = (1, a)$, $(1, b)$, or (a, b), $L^2S \geq 3/r$ for all surfaces S which $(S, x_0) \cong (x^2 + f(y,z) = 0$ or $xy + z^{n+1} = 0 \subset \mathbb{C}^3/\mathbb{Z}_r(1, a", b"))$,* *for* $(1, a'', b'') = (1,a,b),(1,b,a),(a,1,b),(a,b,1),(b,a,1)$, *or* $(b,1,a)$, *and* $LC \geq 3/r$ *for all curves* C *which are smooth at* x_0. *Then* $|K_X + L|$ *is free at* x_0.

The following shows that the conditions in Theorem 0.5 is best possible:

Example 0.6. *Let* $X = \mathbb{P}(1,1,1,r)$ *and* $x_0 = (0:0:0:1)$. *Then* x_0 *is a quotient singular point of type* $(1/r, 1/r, 1/r)$ *and* $K_X = \mathcal{O}(-r-3)$. *If* $K_X + L$ *is Cartier at* x_0 *and* L *is effective, we have* $L = \mathcal{O}(rk+3)(k \in \mathbb{Z}, rk+3 \geq 0)$. *If* $L = \mathcal{O}(3)$, $S = \mathbb{P}(1,1,r)$, *and* $C = \mathbb{P}(1,r)$, *then* $|K_X + L|$ *is not free at* x_0 *and we have* $L^3 = 27/r$, $L^2S = 9/r$, *and* $LC = 3/r$.

We have the following that $K_X + L$ is not free at a quotient terminal singular point for $L^3 > 27/r$ but $LC < 3/r$:

Example 0.7. *Let* $X = \mathbb{P}(1, a, r-a, r)$ *for* $r > 2a$ *and* $x_0 = (0:0:0:1)$. *Then* x_0 *is a quotient singular point of type* $(1/r, a/r, (r-a)/r)$ *and* $K_X = \mathcal{O}(-2r-1)$. *If* $K_X + L$ *is Cartier at* x_0 *and* L *is effective, we have* $L = \mathcal{O}(rk+1)(k \in \mathbb{Z}, rk+1 \geq 0)$. *If* $L = \mathcal{O}(r+1)$ *and* $C = \mathbb{P}(r-a, r)$, *then* $|K_X + L|$ *is not free at* x_0 *and* $L^3 = (r+1)^3/ra(r-a) > 27/r$ *but* $LC = (r+1)/r(r-a) < 3/r$.

We obtain estimates for not quotient $\mathbb{Q}$-factorial terminal singularities.

Theorem 0.8. *Assume* $x_0 \in X$ *is a nonhypersurface and not quotient* $\mathbb{Q}$-*factorial terminal singular point of* $\mathrm{ind}_{x_0} X = r \geq 1$. *Let* $L^3 > 2^3 \cdot 2/r$, $L^2S \geq 2^2 \cdot 2/r$ *for all surfaces* S *with* $x_0 \in S$, *and* $LC \geq 2/r$ *for all curves* C *which are smooth at* x_0. *Then* $|K_X + L|$ *is free at* x_0.

Helmke[He] proved the following of dimension n:

Theorem 0.9. *Let* X *be a smooth projective variety of dimension* n *and* L *an ample divisor on* X. *Let* $x \in X$ *and assume that*
$L^n > n^n$, $L^{n-1}H \geq n^{n-1}$ *for all hypersurfaces* H *containing* x, $L^dZ \geq m_x(Z) \cdot n^d$ *for all* $Z \subset X$ *with* $d = \dim Z \leq n-2$ *and multiplicity* $m_x(Z) \leq \binom{n-1}{d-1}$ *at* x. *Then* $\mathcal{O}_X(K_X + L)$ *is globally generated at* x.

1 Definition and Calculation of the weighted multiplicities

We define the new following notions which we derive from the multiplicity of a point on a normal variety X and the multiplicity of an effective $\mathbb{Q}$-Cartier divisor D on X at a point:

Definition 1.1. Let X be a normal variety of dimension n, x_0 a point of X, $\mu : Y \to X$ a weighted blow up at x_0 with exceptional divisors E, $W \subset X$ the subvariety of dimension p such that W is normal at x_0, $\bar{W}$ the strict transform of W, and $\bar{D}_{\bar{W}}$ on $\bar{W}$ the strict transform of an effective $\mathbb{Q}$-Cartier divisor D_W on W. The *weighted multiplicity* of W at x_0 for μ (w-mult$_{\mu:x_0}W$) is defined by the equation

$$\dim \frac{O_{W,x_0}}{\mu_* O_{\bar{W}}(-hE|_{\bar{W}})} = \text{w-mult}_{\mu:x_0}W \cdot \frac{h^p}{p!} + \text{lower term in } h.$$

The *weighted order* of D_W on W at x_0 for μ (w-ord$_{\mu:x_0}D_W$) is defined by the equation

$$\mu^*(D_W) = \bar{D}_{\bar{W}} + \text{w-ord}_{\mu:x_0}D_W \cdot E|_{\bar{W}}.$$

Definition 1.2. Assume $x_0 \in X$ is a n-dimensional quotient singular point of type $(1/r, a_1/r, \cdots, a_{n-1}/r)$. Then we denote $(X, x_0) \cong \mathbb{C}^n/\mathbb{Z}_r(1, a_1, \cdots a_{n-1})$.

We calculate the weighted multiplicities of some quotient singularities and non-Gorenstein terminal singularities on threefolds.

Theorem 1.3. *Let $(X, x_0) \cong \mathbb{C}^n/\mathbb{Z}_r(1, a_1, \cdots a_{n-1})$ such that an integer $r > 0$, $(r, a_1) = 1$, and integers a_j $(0 \le a_j < r)$ for $1 \le j \le n-1$. Let $l := \min\{i \mid a_j i \equiv i (\ mod\ r)\ (1 \le j \le n-1) \text{ for } 0 < i \le r\}$. Let $\mu : Y \to X$ be the weighted blow up of X at x_0 such that $\mathrm{wt}(x_0, x_1, \cdots, x_{n-1}) = (l/r, l/r, \cdots, l/r)$ with the exceptional divisor E of μ. Then we have*

$$\text{w-mult}_{\mu:x_0}X = r^{n-1}/l^n.$$

Theorem 1.4. *Let (X, x_0) be a 3 folds nonhypersurface and not quotient terminal singular point of $\mathrm{ind}_{x_0}X = r > 1$ over $\mathbb{C}$ and $\mu : Y \to X$ the weighted blow up with the weights $\mathrm{wt}(x, y, z, u) = (1, 1, 1, 1)$ with the exceptional divisor E of μ such that $K_Y = \mu^* K_X + E$. Then*

$$\text{w-mult}_{\mu:x_0}X = 2/r.$$

By applying the weighted multiplicities, we improve freeness conditions on threefolds.

2 General methods for freeness of adjoint linear systems

We recall notation of [Ka] (cf [KMM]).

Definition 2.1. Let X be a normal variety and $D = \Sigma_i d_i D_i$ an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. If $\mu : Y \to X$ is an embedded resolution of the pair (X, D), then we can write

$$K_Y + F = \mu^*(K_X + D)$$

with $F = \mu_*^{-1} D + \Sigma_j e_j E_j$ for the exceptional divisors E_j.

The pair (X, D) is said to have only *log canonical singularities* (LC) (resp.*kawamata log terminal singularities*(KLT)) if $d_i \leq 1$(resp. < 1) for all i and $e_j \leq 1$(resp. < 1) for all j.

A subvariety W of X is said to be a *center of log canonical singularities* for the pair (X, D), if there is a birational morphism from a normal variety $\mu : Y \to X$ and a prime divisor E on Y with the coefficient $e \geq 1$ such that $\mu(E) = W$. The set of all the centers of log canonical singularities is denoted by $CLC(X, D)$. For a point $x_0 \in X$, we define $CLC(X, x_0, D) = \{W \in CLC(X, D); x_0 \in W\}$.

We can construct divisors which have high weighted order at a given point from the following:

Lemma 2.2. *Let X be a normal projective variety of dimension n, L an ample $\mathbb{Q}$-Cartier divisor, $x_0 \in X$ a point, and t,t_0 a rational number such that $t > t_0 > 0$. We assume that $\mu : Y \to X$ is the weighted blow up of X at x_0 with the exceptional divisor E of μ. Let $W \subset X$ be a subvariety of dimension p such that W is normal at x_0. Then there exists an effective $\mathbb{Q}$-Cartier divisor D_W such that $D_W \sim_{\mathbb{Q}} tL|_W$ and*

$$\text{w-ord}_{\mu:x_0} D_W \geq (t_0 + \epsilon) \sqrt[p]{\frac{L^p W}{\text{w-mult}_{\mu:x_0} W}}$$

which is a rational number for $0 \leq \epsilon \ll \sqrt[p]{\text{w-mult}_{\mu:x_0} W / L^p W}$.

Proof. We change the multiplicity of subvariety at the point with the weighted multiplicity of subvariety at the point for μ in [K1 2.1 (cf [Ka 2.1])]. The proof is the same as [K1 2.1 (cf [Ka 2.1])]. □

The following proposition is the key of the proofs of our results of freeness:

Proposition 2.3 ([K1, 2.2] cf [Ka, 2.3]). *Let X be a normal projective variety of dimension n, $x_0 \in X$ a KLT point, and L an ample $\mathbb{Q}$-Cartier divisor such that $K_X + L$ is Cartier at x_0. Assume that there exists an effective $\mathbb{Q}$-Cartier divisor D which satisfies the following conditions:*
(1) *$D \sim_{\mathbb{Q}} tL$ for a rational number $t < 1$,*
(2) *(X, D) is LC at x_0,*
(3) *$\{x_0\} \in CLC(X, D)$.*
Then $|K_X + L|$ is free at x_0.

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