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MONODROMY PROPERTY OF HYPERGEOMETRIC SERIES IN LOCAL MIRROR SYMMETRY

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ABSTRACT. We study a cohomology valued-hypergeometric series which naturally arises in the description of (local) mirror symmetry. We identify it as a central charge formula for BPS states and study its monodromy property from the viewpoint of Kontsevich's homological mirror symmetry.

1. Introduction – Motivation and Backgrounds

Let us consider a (famous) hypergeometric series of one variable $[1,x] \in \mathbf{P}^1$;

$$w(x) = \sum_{n\geq 0} \frac{(5n)!}{(n!)^5} x^n \tag{1.1}$$

This is a hypergeometric series of type ${}_4F_3(\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5};1,1,1;x)$ which arises in the mirror symmetry of quintic hypersurface $X_5 \subset \mathbf{P}^4$. See the original work by Candelas et al [CdOGP] for the description of the mirror family and the period integral. The hypergeometric series (1.1) represents one of the period integrals of the mirror quintic $X_5^\vee$ and satisfies the following differential equation (Picard-Fuchs) equation:

$$\{\theta_x^4 - 5^5 x(\theta_x - \frac{4}{5})(\theta_x - \frac{3}{5})(\theta_x - \frac{2}{5})(\theta_x - \frac{1}{5})\} w(x) = 0 \,, \tag{1.2}$$

where $\theta_x := x\frac{d}{dx}$. As it is clear in this form, the regular singularity at $x = 0$ has a distinguished property, i.e., the monodromy around this point is *maximally unipotent*. In physics, the point $x = 0$ is called a *large complex structure limit* and plays an important role, e.g., near this point, we evaluate the quantum corrections to the classical (algebraic) geometry of the quintic X_5. Let us focus our attention to the construction of local solutions about $x = 0$ by the classical Frobenius method;

$$w_0(x) := w(x) \,,\; w_1(x) := \tfrac{\partial}{\partial\rho} w(x,\rho)|_{\rho=0} \,,$$
$$w_2(x) := \tfrac{\partial^2}{\partial\rho^2} w(x,\rho)|_{\rho=0} \,,\; w_3(x) := \tfrac{\partial^3}{\partial\rho^3} w(x,\rho)|_{\rho=0} \,,$$

where $w(x,\rho) := \sum_{n\geq 0} \frac{\Gamma(1+5(n+\rho))}{\Gamma(1+(n+\rho))^5} x^{n+\rho}$. With the mirror symmetry of X_5 in $\mathbf{P}^4$ and $X_5^\vee$ in mind, we introduce the following cohomology-valued hypergeometric series;

$$w(x,\frac{J}{2\pi i}) := w(x) + w_1(x)(\frac{J}{2\pi i}) + w_2(x)(\frac{J}{2\pi i})^2 + w_3(x)(\frac{J}{2\pi i})^3 \,, \tag{1.3}$$

where J is the ample, integral generator of $\mathrm{Pic}(X_5) = H^{1,1}(X_5) \cap H^2(X_5,\mathbf{Z})$. In this form, we note that the classical Frobenius method is concisely summarized as the Taylor expansion $w(x,\rho)|_{\frac{J}{2\pi i}}$ with respect to the nilpotent element J. Although this seems just an advantage in bookkeeping, the following observation in [Hos] indicates that we have *more than* that in (1.3):

Observation: *Arrange the Taylor expansion of the cohomology-valued hypergeometric series* $w(x, \frac{J}{2\pi i})$ *as*

$$(1.4)\quad w(x\frac{J}{2\pi i}) = w^{(0)}(x) + w^{(1)}(x)(J - \frac{c_2(X_5)J}{12} - \frac{11}{2})w^{(2)}(x)\frac{J^2}{5} + w^{(3)}(x)(-\frac{J^3}{5}).$$

Then the monodromy matrices of the coefficient hypergeometric series $w^{(0)}(x)$, $w^{(1)}(x)$, $w^{(2)}(x)$, $w^{(3)}(x)$ *are integral and symplectic.*

The integral and symplectic properties of the solutions $w^{(k)}(x)$ $(k = 0,1,2,3)$, of course, originate from those of $H_3(X_5^\vee, \mathbf{Z})$. The point here is that we can recover these properties from $w(x, \frac{J}{2\pi i})$ through a suitable arrangement of a basis of $H^{even}(X, \mathbf{Q})$ near the large complex structure limit.

The aims (and main results) of this note are: 1) to interpret the cohomology-hypergeometric series from the viewpoint of homological mirror symmetry, 2) to present supporting evidences for the interpretation 1) in cases of local mirror symmetry.

As an example of local mirror symmetry, we will consider the crepant resolution $\widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$ of the two dimensional canonical singularity. In studying relevant Gel'fand-Kapranov-Zelevinski(GKZ) hypergeometric series, we connect it to the primitive form by K.Saito in the deformation theory of singularity. This seems to be interesting in its own light, since GKZ hypergeometric series may provide a way to express the 'period integrals' of the primitive form (or oscillating integrals) in the theory of singularity.

2. Central charge formula in terms of $w(x, \frac{J}{2\pi i})$

Here, following [Hos], we will interpret the cohomology valued hypergeometric series in general from Kontsevich's homological mirror symmetry [Ko].

Let X be a Calabi-Yau 3 fold and Y be a mirror of X. On the X side, Kontsevich considers the bounded derived category $D^b(Coh(X))$ of coherent sheaves (D-branes of B type) on X. On the other hand, for the mirror side, he considers the derived Fukaya category $DFuk(Y, \beta)$ with the Kähler form viewed as a symplectic form β. The objects of the latter category consist of (graded) Lagrangian submanifolds with flat $U(1)$ bundle on them (D-brane of type A) and morphisms are given by the Floer homology for Lagrangian submanifolds, and this constitute a triangulated category (, see [FO3] for more precise definition). Kontsevich proposed that these two different category are equivalent (as triangulated category) when X and Y are mirror symmetric, and also this should be a mathematical definition of mirror symmetry. This conjecture itself is of great interest, however let us consider this conjecture at more tractable level (, i.e. at the level of cohomology or K-group as shown in the second line below);

$$(2.1)\qquad \begin{tikzcd} D^b(Coh(X)) \arrow[r, "Mir"] \arrow[d] & DFuk(Y, \beta) \arrow[d] \\ K(X) \text{ or } H^{even}(X, \mathbf{Q}) \arrow[r, "mir"] & H_3(Y, \mathbf{Z}) \end{tikzcd}$$

where the left vertical arrow represents the map from $D^b(Coh(X))$ to the K-group of algebraic vector bundle $K(X)$, and its composition with the Chern character homomorphism $ch(\cdot)$ if we further map to $H^{even}(X, \mathbf{Q}) = \oplus_{p=0}^{3} H^{2p}(X, \mathbf{Q})$. The right vertical arrow is given simply taking the homology classes of the graded Lagrangian cycles. In the second line, the equivalence, Mir, of the two categories becomes simply an isomorphism, mir, between the K-group and $H_3(Y,\mathbf{Z})$. We should note that this is not simply an isomorphism but isomorphism with the symplectic structures, i.e.

$$mir: \ (K(X),\chi(E,F)) \ \tilde{\to} \ (H_3(Y,\mathbf{Z}),\#(L_E\cap L_F)),$$

where $\chi(E,F)=\int_X ch(E^\vee)ch(F)Todd_X$ and $\#(L_E\cap L_F):=\int_Y \mu_{L_E}\cup\mu_{L_F}$ with the Poincaré duals $\mu_{L_E},\mu_{L_F}\in H^3(Y,\mathbf{Z})$ for the mirror homology cycles $L_E:=mir(E), L_F:=mir(F)$. Here we remark that the Euler numbers $\chi(E,F)$ is anti-symmetric due to Serre duality and $K_X=0$, and also non-degenerate. Thus $\chi(E,F)$ introduces a symplectic structure on $K(X)$, which is the mirror of the symplectic structure on $H_3(Y,\mathbf{Z})$.

In the diagram (2.1), we assumed a complex structure is fixed for $D^b(Coh(X))$ side, which is mapped to the symplectic (Kähler) form β in the right hand side. On the other hand, we may change the (complexified) Kähler class of X which is mapped to the complex structure moduli of Y under the mirror map. Changing the (complexified) Kähler structure amount to changing the polarization and thus results in varying the stability condition on the sheaves on X. This change of the stability (Π-stability) condition has been studied in [Do] and its mathematical aspects are elaborated in [Br]. Here, without going into the detailed definition of Π-stability, we propose a closed formula for the *central charge* which is indispensable for the definition of Π-stability.

Definition 2.1. (Central charge formula.) Assume $K(X)$ is torsion free, and let $E_1,\cdots,E_r$ be a $\mathbf{Z}$ basis of $K(X)$. Let $\Omega(Y_x)$ be a holomorphic 3-form of the mirror family $\{Y_x\}_{x\in B}$ of X. Under the mirror symmetry (2.1), we define the following $\mathcal{Z}_x$ as an element in $K(X)\otimes\mathbf{C}\{x\}$,

$$\mathcal{Z}_x := \sum_{i,j}\int_{mir(E_i)} \Omega(Y_x)\ \chi^{ij}\ E_j \tag{2.2}$$

with $(\chi^{ij}):=(\chi(E_i,E_j))^{-1}$. Then the central charge of $F\in K(X)$ is defined by

$$Z_t(F)=\int_X ch(F)\ ch(\mathcal{Z}_x^\vee)\ Todd_X\ , \tag{2.3}$$

where $t=t(x)$ is the (complexified) Kähler moduli.

In the above definition, it should be noted that $\mathcal{Z}_x$ does not depend on the choice of a basis $E_1,\cdots,E_r$. Also the central charge $Z_t(F)$ contains full 'quantum corrections' and coincides with that appeared in the literature[Do] (, where only asymptotic forms are given).

Now we may connect our hypergeometric series $w(x,\frac{J}{2\pi i})$ to the central charge above. Before doing this we remark that, in the mirror symmetry of hypersurfaces by Batyrev[Ba1], the hypergeometric series (1.1) is naturally generalized to Gel'fand-Kapranov-Zelevinski(GKZ) hypergeometric series of multi variables $x_1,\cdots,x_r$[GKZ]. Using the GKZ hypergeometric series, and also suitable integral, (semi-)ample generators $J_1,\cdots,J_r$ of $H^{1,1}(X)\cap H^2(X,\mathbf{Z})$, we have the cohomology-valued hypergeometric series $w(x,\frac{J}{2\pi i})$ as a generalization of (1.3), see [Hos] for more details.

Conjecture 2.2. *The cohomology-valued hypergeometric series (1.3)(, precisely its generalization defined in Section 2 of* [Hos],*) gives the central charge;*

$$(2.4) \quad w(x_1, \cdots, x_r; \frac{J_1}{2\pi i}, \cdots, \frac{J_r}{2\pi i}) = \sum_{i,j} \int_{mir(E_i)} \Omega(Y_x)\, \chi^{ij}\, ch(E_j)\ (= ch(\mathcal{Z}_x))\,.$$

Using this, and also the mirror map $t = t(x)$, we can write the central charge $Z_t(F)$ of $F \in K(X)$ as

$$(2.5) \qquad Z_t(F) = \int_X ch(F)\, w(x; \frac{-J}{2\pi i})\, Todd_X \ .$$

Here we note that the hypergeometric series has a finite radius of convergence and shows a monodromy property when it is analytically continued around its (regular) singularities. As noticed by Kontsevich, this monodromy property should be mirrored to some linear (symplectic) transformations on $ch(E_i)$ which come from Fourier-Mukai transforms on $D^b(Coh(X))$. If we postulate that the cohomology-valued hypergeometric series has an invariant meaning under these monodromy actions, our cohomology-valued hypergeometric series $w(x, \frac{J}{2\pi i})$ provides a connection between these two different 'monodromy' transforms in both sides. The conjectural formula (2.4) has been tested in case X is an elliptic curve, (lattice polarized) K3 surfaces, and several Calabi-Yau hypersurfaces[Hos].

As studied in [Mu] for the cases of K3 surfaces and abelian varieties, and in [Or] for general, the Fourier-Mukai transform is an equivalence of the category $D^b(Coh(X))$ which takes the form

$$\Phi^{\mathcal{P}}(\cdot) = \mathbf{R}_{p_{2*}}(p_1^*(\cdot) \overset{\mathbf{L}}{\otimes} \mathcal{P})$$

where $\mathcal{P}$ is an object in $D(X \times X)$, called the *kernel*, and p_1 and p_2 are, respectively, the natural projections to the first and the second factor from $X \times X$. Due to a result in [Or], we may always assume the above form, i.e., there exists a suitable kernel $\mathcal{P}$, for any equivalence $\Phi : D^b(Coh(X)) \simeq D^b(Coh(X))$ as triangulated category. It is rather easy to see that the monodromy transforms around the large complex structure limit are given by tensoring invertible sheaves, which may be expressed by the kernels;

$$\mathcal{P}: \ \cdots \to 0 \to \mathcal{O}_\Delta \times p_2^*(O_X(D)) \to 0 \cdots \ ,$$

with $D \in Pic(X)$ and Δ representing the diagonal in $X \times X$. Kontsevich predicted that a monodromy transform associated to a vanishing cycle, a Picard-Lefschetz transform, has its mirror FM transform with its kernel,

$$\mathcal{P}: \ \cdots \to 0 \to \mathcal{O}_{X \times X} \to \mathcal{O}_X \to 0 \cdots \ .$$

Seidel and Thomas [ST] (and Horja [Hor]) generalized the above kernel associating it to so-called spherical objects $\mathcal{E} \in D^b(Coh(X))$ with defining property: $Ext^i(\mathcal{E}, \mathcal{E}) = 0\ (i \neq 0, n), \mathbf{C}\ (i = 0, n)$ where $n = dim X$. For each spherical object, we have a kernel given by the mapping cone;

$$\mathcal{P} = Cone(\mathcal{E}^\vee \otimes^{\mathbf{L}} \mathcal{E} \to \mathcal{O}_\Delta) \ .$$

The equivalence $\Phi^{\mathcal{P}}$ is called Seidel-Thomas twist. We will see these equivalences in the corresponding monodromy property of certain hypergeometric series.

3. Local mirror symmetry — $X = \widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$

In this section and the subsequent sections, we will test our Conjecture 2.2 for the case of mirror symmetry of non-compact toric Calabi-Yau manifolds (local mirror symmetry). Batyrev's mirror symmetry still makes sense for such non-compact toric Calabi-Yau manifolds although the attractive proposal by Strominger-Yau-Zaslow(SYZ)[SYZ], which is closely related to the homological mirror symmetry (2.1), becomes less clear. Mirror symmetry of non-compact toric manifolds are also formulated in physical terms[HIV].

(3-1) **Mirror symmetry and hyperkähler rotation.** Let us consider the minimal resolution of a two dimensional simple singularity; $X = \widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$. This is an example of two dimensional, non-compact, toric Calabi-Yau manifold. Two dimensional Calabi-Yau manifolds are hyperkähler, and it is known that the mirror symmetry of them is well-understood by the hyperkähler rotation, see e.g. [GW][Huy]. Our minimal resolution X has a natural hyperkähler structure, and therefore its mirror is X itself with a different complex structure after a suitable rotation. To describe the mirror symmetry, let us first write the quotient $\mathbf{C}/\mathbf{Z}_{\mu+1}$ by a hypersurface $UV = W^{\mu+1}$ in $\mathbf{C}^3$. Bowing up the singularities at the origin μ times results in the minimal resolution X, and thereby we introduce exceptional curves $C_i \cong \mathbf{P}^1$ $(i = 1, \cdots, \mu)$. On the other hand, we may deform the defining equation $UV = W^{\mu+1}$ to $UV = a_0 + a_1 W + \cdots + a_{\mu+1} W^{\mu+1}$ with introducing finite sizes to the vanishing cycles $L_i \cong S^2$ $(\mu = 1, \cdots, \mu)$. Note that the number of the vanishing cycles are given by the Milnor number $\mu = \dim R_J$, where R_J is the Jacobian ring of the singularity $UV = W^{\mu+1}$. The vanishing cycles are Lagrangians, and become holomorphic cycles under a suitable hyperkähler rotation. The holomorphic geometry after the rotation is bi-holomorphic to the blown-up geometry of X. If we forget about the role of the B-fields, this describes the mirror symmetry of X. (See e.g. [Huy] for full details of the mirror symmetry via the hyperkähler rotation.) Here we note the intersection form of these cycles are given in both holomorphic and Lagrangian geometry by

$$(C_i \cdot C_j) = (\# L_i \cap L_j) = -\mathcal{C}_{ij} \ ,$$

where $\mathcal{C}_{ij}$ is the Cartan matrix for the root system of $A_{\mu+1}$.

(3-2) **GKZ hypergeometric series.** The minimal resolution $X = \widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$ is a (non-compact) toric variety whose resolution is described by a two dimensional fan Σ with its integral generators for one dimensional cones (see Fig.1);

$$A = \{ \, \nu_0 = (1,0), \nu_1 = (1,1), \cdots, \nu_{\mu+1} = (1, \mu+1) \, \} \ .$$

The half-lines $\overline{o\nu_i}$ $(i = 0, \cdots, \mu+1)$ from the origin $o = (0,0)$ constitute the one dimensional cones of the resolution diagram Σ. In Batyrev's mirror symmetry, the resolution diagram of X, up to flop operations, is identified with the Newton polytope of the defining equation of its mirror Y, i.e., the mirror Y is given by $U^2 + V^2 + f_\Sigma(W) = 0 \subset (\mathbf{C}^*)^3$ with

$$f_\Sigma(W) = a_0 + a_1 W^1 + a_2 W^2 + \cdots + a_{\mu+1} W^{\mu+1} \ .$$

In the case of local mirror symmetry, the meaning of the period integrals of holomorphic two form becomes less clear than the compact cases. However we consider the following integral for a cycle $\gamma \in H_3((\mathbf{C}^*)^3 \setminus (U^2+V^2+f_\Sigma(a;W)=0), \mathbf{Z})$:

$$\Pi_\gamma(a) := \int_\gamma \frac{1}{U^2+V^2+f_\Sigma(a;W)} \frac{dU}{U}\frac{dV}{V}\frac{dW}{W} \ , \tag{3.1}$$

which naturally follows from Batyrev's mirror symmetry. Starting from this integral, and through the parallel calculation done in [HKTY], we can extract the genus zero (local) Gromov-Witten invariants (, see [CKYZ]). In the next section, we will relate this 'period integral' to K. Saito's primitive form (or Gel'fand-Lerayi form) in the deformation theory of singularities. Here we set up a hypergeometric differential equations (GKZ system) satisfied by $\Pi_\gamma(a)$. This GKZ hypergeometric system is also referred to as A-hypergeometric system since it is described by the set A in (3) through the *lattice of relations*,

$$L = \{ \, (l_0, l_1, \cdots, l_{\mu+1}) \,|\, l_0\nu_0 + l_1\nu_1 + \cdots + l_{\mu+1}\nu_{\mu+1} = (0,0) \, \}. \tag{3.2}$$

Using this lattice of relations, the system is written as

$$\Box_l \Pi_\gamma(a) = 0 \ (l \in L) \ , \ \mathcal{Z}_i \Pi_\gamma(a) = 0 \ (i=1,2) \ , \tag{3.3}$$

where

$$\Box_l = \left(\frac{\partial}{\partial a}\right)^{l_+} - \left(\frac{\partial}{\partial a}\right)^{l_-} \ , \ \begin{pmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_2 \end{pmatrix} = \begin{pmatrix} \theta_0+\theta_1+\cdots+\theta_{\mu+1}-1 \\ \theta_1+2\theta_2\cdots+(\mu+1)\theta_{\mu+1} \end{pmatrix}$$

with $l = l_+ - l_-$ and $\theta_k = a_k \frac{\partial}{\partial a_k}$. From the formal solutions of this system in [GKZ], and setting $w(x) = a_0\Pi_\gamma(a)$, it is easy to write down our $w(x, \frac{J}{2\pi i})$ (see [Hos]). Another important aspect of this system is that there is a natural toric compactification $\mathcal{M}_{Sec(\Sigma)}$ of the parameter space $\{(a_0,\cdots,a_{\mu+1}) \in (\mathbf{C}^*)^{\mu+2}/(\mathbf{C}^*)^2\}$, in terms of the *secondary fan* $Sec(\Sigma)$, where the quotient by $(\mathbf{C}^*)^2$ corresponds to the linear (scaling) operators $\mathcal{Z}_1$ and $\mathcal{Z}_2$. This compactification plays an important role in the applications of mirror symmetry to Gromov-Witten invariants, since the large radius limit appears as an intersection point of the boundary toric divisors. Connecting the GKZ system to K. Saito's differential equations in singularity theory, we see that this compactification will also provide a natural way to compactify the deformation space of the singularity theory which is local in nature (, see (4-1) for a brief description of the deformation space).

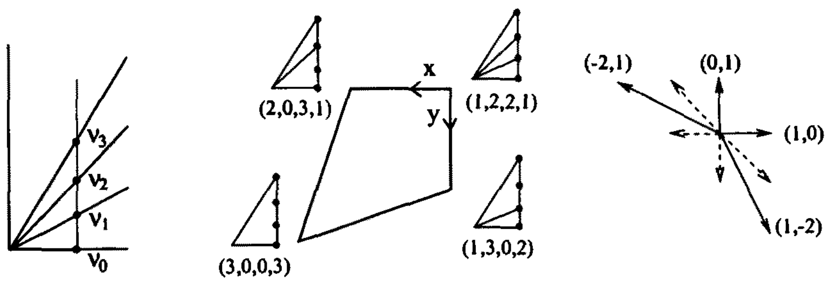

Fig.1. *The resolution diagram (left), the secondary polytope (middle), and the secondary fan $Sec(\Sigma)$ (right) for $\mu = 2$. The secondary polytope has its apexes parametrized by the (regular) triangulations of the polytope as shown. For a triangulation T, the corresponding apex is determined by a vector $v_T = (\varphi_T(\nu_0), \varphi_T(\nu_1), \varphi_T(\nu_2), \varphi_T(\nu_3))$ with $\varphi_T(\nu_i) = \sum_{\nu_i \prec \sigma} Vol(\sigma)$. As see in this example ($\mu = 2$), the*

convex hull of these vectors lies on $L_{\mathbf{R}} = L \otimes \mathbf{R}$. Normal cones of the secondary polytope determines the secondary fan.

(3-3) Example ($\mu = 2$). Here we present an example in stead of giving general formulas valid for any μ. For $\mu = 2$, the cohomology-valued hypergeometric series is simple and takes the following form;

$$w(x, y; \frac{J_1}{2\pi i}, \frac{J_2}{2\pi i}) = w(x, y; \rho_1, \rho_2)|_{\rho_1 = \frac{J_1}{2\pi i}, \rho_2 = \frac{J_2}{2\pi i}} ,$$

where $w(x, y; \rho_1, \rho_2) = \sum_{n,m \geq 0} c(n + \rho_1, m + \rho_2) x^{n+\rho_1} y^{m+\rho_2}$ with

$$c(n, m) = 1/\big(\Gamma(1+n)\Gamma(1-2n+m)\Gamma(1+n-2m)\Gamma(1+m)\big) .$$

J_1 and J_2 are semi-ample classes which are dual to the exceptional curves C_1 and C_2 (, i.e. the toric divisors D_{ν_1} and D_{ν_2}), respectively. The local parameters $x := \frac{a_0 a_1}{a_1^2}, y := \frac{a_1 a_3}{a_2^2}$ are depicted in Fig.1. Here we remark that the secondary polytope in Fig.1 sit in the scalar extension $L_{\mathbf{R}}$ of the L lattice, and the summation in $\sum_{n,m \geq 0} c(m, m) x^m y^n$ is in fact that over the integral points inside the normal cone from the vertex $v_T = (1, 2, 2, 1)$. We may recognize this in a relation $x^n y^m = a^{nl^{(1)} + ml^{(2)}}$ with $l^{(1)} = (1, -2, 1, 0)$, $l^{(2)} = (0, 1, -2, 1)$. Since the ring $H^{even}(X, \mathbf{Z})$ is generated by $1, J_1, J_2$, we have the expansion;

$$w(\vec{x}, \frac{\vec{J}}{2\pi i}) = 1 + w_1(x, y) J_1 + w_2(x, y) J_2 ,$$

with $w_1(x, y) = \frac{1}{2\pi i} \log x + \cdots$, $w_2(x, y) = \frac{1}{2\pi i} \log y + \cdots$. The mirror map is defined from the relations,

$$q_1 := e^{2\pi i w_1(x,y)} = x(1 + g_1(x, y)) \ , \ q_2 := e^{2\pi i w_2(x,y)} = y(1 + g_2(x, y)) \ , \tag{3.4}$$

where $g_1(x, y), g_2(x, y)$ represent powerseries of x and y. Then $t_1 := \frac{1}{2\pi i} \log q_1 (= w_1(x, y))$ and $t_2 := \frac{1}{2\pi i} \log q_2 (= w_2(x, y))$ are the complexified Kähler moduli and measure the volumes of the exceptional curves C_1, and C_2, respectively. The inverse relation $x = x(q_1, q_2), y = y(q_1, q_2)$ of (3.4) is often referred to as the mirror map, and has the following properties:

Proposition 3.1.
1) *The mirror map $x = x(q_1, q_2), y = y(q_1, q_2)$ is rational of the form;*

$$x = \frac{q_1(1 + q_2 + q_1 q_2)}{(1 + q_1 + q_1 q_2)^2} \ , \ y = \frac{q_2(1 + q_1 + q_1 q_2)}{(1 + q_1 + q_1 q_2)^2} \ , \tag{3.5}$$

and is expressed by $x = \frac{a_0 a_2}{a_1^2}, y = \frac{a_1 a_3}{a_2^2}$ with a_i's determined through

$$a_0 + a_1 W + a_2 W^2 + a_3 W^3 = (1 + W)(1 + q_1 W)(1 + q_1 q_2 W) .$$

2) *The discriminant of the GKZ system (3.3) consists of three components; $x = 0, y = 0$ and $dis(x, y) = 0$ with*

$$dis(x, y) = 1 - 4x - 4y + 18xy - 27x^2y^2 = \frac{(1 - q_1)^2(1 - q_2)^2(1 - q_1 q_2)^2}{(1 + q_1 + q_1 q_2)^2(1 + q_2 + q_1 q_2)^2} .$$

It is easy to see that $x = 0, y = 0$ are the toric boundary divisors whose intersection point define the large complex structure. Over the zeros of the discriminant $dis(x, y)$, we see vanishing cycles in $f_\Sigma(a, W) + U^2 + V^2 = 0 \subset (\mathbf{C}^*)^3$. In fact, in the holomorphic picture, $q_1 = 1 (q_2 = 1)$ represents a vanishing volume limit of the exceptional curve $C_1(C_2)$. After a hyperkähler rotation, these vanishing volumes are viewed as the corresponding vanishing of the Lagrangian cycles L_1, L_2. We remark that the above Proposition 3.1 and the interpretation done for $\mu = 2$ case generalizes to arbitrary μ in a straightforward way.

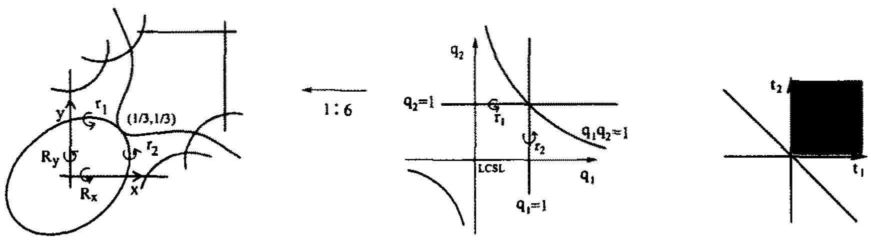

Fig.2. *The discriminant $dis(x,y) = 0$ in $\hat{\mathcal{M}}_{Sec(\Sigma)}$ (left), the mirror map to q_1q_2-plane (3.5)(middle), and the complexified Kähler moduli t_1, t_2 with the complexified Kähler cone (right). The discriminant is an elliptic curve with a node at $(x,y) = (\frac{1}{3}, \frac{1}{3})$, which is mapped to $(q_1, q_2) = (1,1)$. The mirror map is $1:6$ at generic (x,y). Over the discriminant, it is $1:3$ and represented by the three lines $q_1 = 1, q_2 = 1, q_1q_2 = 1$ in the q-plane.*

In the rest of this subsection, we take a close look at the mirror map (3.5), and summarize its monodromy property. Let us first note that the moduli space $\mathcal{M}_{Sec(\Sigma)}$ is a two dimensional singular toric variety, and may be desingularized to $\hat{\mathcal{M}}_{Sec(\Sigma)} = Bl_4(\mathbf{P}^1 \times \mathbf{P}^1)$ after blowing up four points (, see the dashed lines in Fig.1). Then the discriminant $dis(x,y) = 0$ describes a nodal elliptic curve written in Fig.2. There, the mirror map (3.5) is depicted. Note that the mirror map form (q_1, q_2) to (x,y) is six to one at generic points because (3.5) is invariant under the reflections;

$$r_1 : q_1 \to 1/q_1 \ , \ q_2 \to q_1q_2 \quad , \quad r_2 : q_2 \to q_1q_2 \ , \ q_1 \to 1/q_2 \ . \tag{3.6}$$

which satisfy $r_1^2 = r_2^2 = (r_1r_2)^3 = 1$ and thus generate the symmetric group of order 3. By making an analytic continuation of the hypergeometric series, we can verify the above invariance group actions (3.6) as the monodromy actions for the loops r_1 and r_2 depicted in Fig.2. Together with the monodromy matrices R_x, R_y about the large complex structure limit, we summarize the monodromy generators with respect to a basis ${}^t(1, w_1(x,y), w_2(x,y)) = {}^t(1, \frac{1}{2\pi i}\log q_1, \frac{1}{2\pi i}\log q_2)$;

$$r_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} , \ r_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} ; \ R_x = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \ R_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} .$$

Proposition 3.2. *The mirror map (3.5) (precisely its generalization to arbitrary μ) uniformizes the solutions of the GKZ system (3.3) up to the shifts $w_i(x_1, \cdots, x_\mu) \mapsto w_i(x_1, \cdots, x_\mu) + 1$ $(i = 1, \cdots, \mu)$. More precisely, the mirror coordinate $q_1, \cdots, q_\mu$ uniformizes the solutions with the symmetric group S_μ.*

The above result seems to be interesting from the viewpoint of the uniformization of hypergeometric series[Yo], since our GKZ system provides an infinite number of examples for which multi-valued hypergeometric series are uniformizable. However the result itself is not surprising, as we will show in the next section that our

GKZ system has a close relation to K. Saito's differential equations for which the monodromy property is well-studied.

4. K.Saito's differential equations

In the deformation theory of a singularity, we have the notion of so-called *primitive form* which satisfies a set of differential equations, K. Saito's differential equations[Sa]. The primitive form is also referred to as Gel'fand-Leray form[AGV]. Here we briefly introduce the primitive form and K. Saito's differential equations, and then connect them to the 'period integral' (3.1) and the GKZ system (3.3).

(4-1) **K.Saito's system.** Let us first note that, in the deformation theory of a singularity, the polynomial equation $f_\Sigma(a,W)+U^2+V^2=0$ will be considered in $\mathbf{C}^3$ and the parameters are set to $a_{\mu+1}=1, a_\mu=0$ by a coordinate change of W. Namely we take the defining equation of the form $f_\Sigma(a,W)+U^2+V^2=a_0+F_1(a,W,U,V)$ with

$$F_1(a,W,U,V)=a_1W+\cdots+a_{\mu-1}W^\mu+W^{\mu+2}+U^2+V^2\ ,$$

and regard the parameters $a_0,a_1,\cdots,a_{\mu-1}$ as giving a deformation of the singularity $W^{\mu+1}+U^2+V^2=0\subset\mathbf{C}^2$ at the origin. Since the parameter a_0 plays a distinguished role from the others, we set the local parameters $(a_1,\ldots,a_{\mu-1})$ as a coordinate of $T:=\mathbf{C}^{\mu-1}$. The full parameters $(a_0,a_1,\cdots,a_{\mu-1})$ will be regarded as a coordinate of $S=\mathbf{C}\times T$. We consider the total space $\mathfrak{X}$ with coordinate $(W,U,V,a_1,\cdots,a_{\mu-1})$. Then we have a natural map $\varphi:\mathfrak{X}\to S$ by $(W,U,V,a_1,\cdots,a_{\mu-1})\mapsto((-F_1(a,W,U,V),a_1,\cdots,a_{\mu-1})$. This map plays important roles in describing the deformation of the singularity. Consider a sheaf $\Omega^p_{\mathfrak{X}/T}$ of germs of relative holomorphic p forms for $\mathfrak{X}\to T$. We may consider the following sheaves on S,

$$\mathcal{H}^{(0)}=\varphi_*\Omega^3_{\mathfrak{X}/T}/dF\wedge d(\varphi_*\Omega^2_{\mathfrak{X}/T})\ ,\ \ \mathcal{H}^{(-1)}=\varphi_*\Omega^2_{\mathfrak{X}/T}/(dF\wedge\varphi_*\Omega^1_{\mathfrak{X}/T}+d(\varphi_*\Omega^1_{\mathfrak{X}/T}))\ .$$

A primitive form ζ is an element in $H^0(S,\mathcal{H}^{(0)})$ satisfying certain conditions (see [Sa] for details). In stead of ζ, hereafter, we consider its image $\mathcal{U}_0$ in $H^0(S,\mathcal{H}^{(-1)})$ under an isomorphisms $\mathcal{H}^{(0)}\cong\mathcal{H}^{(-1)}$, which we may write explicitly as

$$\mathcal{U}_0(a)=Res_{\{a_0+F=0\}}\left(\frac{dW\wedge dU\wedge dV}{a_0+F(a,W,U,V)}\right)\ .$$

In this from, the primitive form is called also as Gel'fand-Leray form in the study of oscillating integrals (, see e.g. [AGV]). We note that a similarity of $\mathcal{U}_0(a)$ to our 'period integral' (3.1), although, in (3.1), we do not set $a_{\mu+1}=1, a_\mu=0$ but consider torus actions $(\mathbf{C}^*)^2$ in stead.

K.Saito's system is defined as a set of differential equations satisfied by the primitive form $\mathcal{U}_0(a)=\mathcal{U}_0(a_0,\cdots,a_{\mu-1})$;

$$\begin{aligned} P_{ij}\mathcal{U}_0(a) &= \left\{\frac{\partial^2}{\partial a_i\partial a_j}-\nabla_{\frac{\partial}{\partial a_i}}\frac{\partial}{\partial a_j}-(\frac{\partial}{\partial a_i}*\frac{\partial}{\partial a_j})\frac{\partial}{\partial a_0}\right\}\mathcal{U}_0(a)=0\\ Q(\tfrac{\partial}{\partial a_i})\mathcal{U}_0(a) &= \left\{w(\frac{\partial}{\partial a_i})\frac{\partial}{\partial a_0}-N(\frac{\partial}{\partial a_i})+\frac{3}{2}\frac{\partial}{\partial a_i}\right\}\mathcal{U}_0(a)=0\ ,\end{aligned}$$

see [Sa] for detailed definitions. This system is defined for general setting of the deformation theory of singularity and also known that it is a holonomic system. The following proposition is shown in Appendix by Ambai in [Oda], not only for our $A_{\mu+1}$ case but also for other D-E type singularities:

Proposition 4.1. *Let $\beta_0(a), \cdots, \beta_\mu(a)$ be the roots of $\psi_0(W) := a_0 + a_1 W + \cdots + a_{\mu-1}W^\mu + W^{\mu+1}$ which in satisfy $\beta_0(a) + \cdots + \beta_\mu(a) = 0$. Then the space of the solutions of K.Saito's system is generated by*

$$1 \, , \, \beta_0(a) - \beta_1(a) \, , \, \cdots \, , \, \beta_{\mu-1}(a) - \beta_\mu(a) \, . \tag{4.1}$$

The system has a regular singularity at the discriminant locus

$$dis(a) = \Pi_{1 \le i,j \le \mu}(\beta_i(a) - \beta_j(a))^2 = 0 \, ,$$

and the monodromy group about the discriminant coincides with the symmetric group S_μ acting as the permutations among the roots $\beta_i(a)$.

As we have noted in (3-1), μ vanishing cycles appears in the deformation. Now it is easy to deduce that the above solutions $\alpha_i := \beta_i(a) - \beta_{i-1}(a)$ represent the integrals $\int_\gamma \mathcal{U}_0(a)$ over the corresponding vanishing cycles. In fact, it is known that there is a residue pairing $I(\alpha_i, \alpha_j)$ among the solutions which reproduce the intersection pairing $\# L_i \cap L_j$ among the vanishing cycles.

(4-2) **GKZ system for $\mathcal{U}(a)$.** As it is briefly sketched above, the primitive form $\mathcal{U}_0(a)$ is parametrized by $(a_0, \cdots, a_{\mu-1}) \in S$ and provides a way to describe the deformation of singularity near the origin. As remarked there, we may consider a natural torus actions $(\mathbf{C}^*)^2$ in stead of setting $a_\mu = 0, a_{\mu+1} = 1$ as in $\mathcal{U}_0(a)$. With this slight change of the parameter setting ('gauge'), we may connect K.Saito's system to a GKZ system. Let us define a period integral of the primitive form

$$\Pi'_\gamma(a) := \int_\gamma \mathcal{U}(a) = \int_\gamma Res_{f_\Sigma(W)+U^2+V^2=0}\left(\frac{dW \wedge dU \wedge dV}{f_\Sigma(a, W) + U^2 + V^2}\right) \, , \tag{4.2}$$

where γ is a two cycle of $f_\Sigma(a, W) + U^2 + V^2 = 0 \subset \mathbf{C}^3$ and $f_\Sigma(a, W) := a_0 + a_1 W + \cdots + a_{\mu+1} W^{\mu+2}$. The period integral above has a similar form to that in (3.1), and thus satisfies a GKZ system which is similar to (3.3). The only difference appears in the scaling properties expressed by the linear operators $\mathcal{Z}_i$.

Proposition 4.2. *1) The period integral (4.2) satisfies*

$$\Box_l \Pi'_\gamma(a) = 0 \; (l \in L) \, , \quad \mathcal{Z}'_i \Pi'_\gamma(a) = 0 \; (i = 1, 2) \, , \tag{4.3}$$

where the operators $\Box_l$ and the lattice L are the same as in (3.3), and $Z'_i \, (i = 1, 2)$ are given by

$$\begin{pmatrix} \mathcal{Z}'_1 \\ \mathcal{Z}'_2 \end{pmatrix} = \begin{pmatrix} \theta_0 + \theta_1 + \cdots + \theta_{\mu+1} \\ \theta_1 + 2\theta_2 \cdots + (\mu+1)\theta_{\mu+1} - 1 \end{pmatrix}$$

2) The system (4.3) above is reducible of rank $\mu + 1$ with its irreducible part of rank μ. The μ independent solutions of the irreducible part are given by

$$\beta_1(a) - \beta_0(a), \cdots, \beta_\mu(a) - \beta_{\mu-1}(a) \, ,$$

where $\beta_i(a)$'s are roots of $\psi(W) = a_0 + a_1 W + \cdots + a_{\mu+1} W^{\mu+2} = 0$.

Derivation of 1) above is straightforward, but it should be noted that K.Saito's system is replaced by a different but a simple GKZ system. The numbers of independent solutions of a GKZ system is given by the volume of a relevant polytope[GKZ], which in our case $Vol(\Sigma) = \mu + 1$. It is straightforward to verify that the system is reducible observing a factorization of a differential operator in our GKZ system when expressing $\Box_l$ operators in an affine coordinate of $\mathcal{M}_{Sec(\Sigma)}$. (It is known that the GKZ systems in mirror symmetry are often reducible in this way, see [HKTY].)

The solutions for the irreducible part may be determined by using the following property(, see Appendix by Ambai in [Oda]):

Proposition 4.3. *The root $\beta(a)$ $(=\beta_0(a),\cdots,\beta_\mu(a))$ of $\psi(W)=0$ satisfies*

$$\frac{\partial\beta}{\partial a_i}=-\frac{\beta^{i-1}}{\psi'(\beta)}\ ,\quad \frac{\partial^2\beta}{\partial a_i\partial a_j}=\frac{1}{\psi'(\beta)}\frac{d}{dx}\left(\frac{x^{i+j-2}}{\psi'(x)}\right)\Big|_{x=\beta}\ . \tag{4.4}$$

Using the relations (4.4) again and also the property in Proposition 3.1, 1), we can obtain the solutions of the GKZ system (3.3), precisely $w(x)=a_0\Pi_\gamma(a)$:

Proposition 4.4. *1) The independent solutions of the GKZ system (3.3), with multiplied by a_0, i.e. $w(x)=a_0\Pi_\gamma(a)$, are given by*

$$1\quad \log\beta_1(a)-\log\beta_0(a)\ ,\ \cdots\ ,\ \log\beta_\mu(a)-\log\beta_{\mu-1}(a)\ .$$

2) Up to a suitable analytic continuation, these solutions are related to the expansion $w(x,\frac{J}{2\pi i})=1+\sum_{k=1}^{\mu}w_k(a)J_k$ near the large complex structure by

$$2\pi i\, w_k(x)=-\log\beta_k(a)+\log\beta_{k-1}(a)\ \ (k=1,\cdots,\mu).$$

The above form of the solutions may be connected to the piecewise linear functions on the fan Σ[Ba2]. Since we have established a relation of the GKZ system (3.3) of $\widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$ to K.Saito's system for primitive form, for which integral monodromy property is known in Proposition 4.1, it is clear that we have the uniformization property of the mirror map stated in Proposition 3.2.

5. Central charge formula and G-Hilb$\mathbf{C}^2$

In the last two sections, we have looked the local mirror symmetry of $\widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$ paying our attentions to the monodromy property of the associated GKZ system. Here we come back to our claim for the central charge formula (2.4) in this case.

Let us first recall that the non-compact Calabi-Yau manifold $X=\widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$ is given as the Hilbert scheme of points on $\mathbf{C}^2$, G-Hilb $\mathbf{C}^2$. Here G-Hilb $\mathbf{C}^n$ is defined for a finite subgroup $G\subset SL(n,\mathbf{C})$ and consists of zero dimensional subschemes Z in $\mathbf{C}^n$ of length equal $|G|$ such that G acts on Z and $H^0(\mathcal{O}_Z)$ is the regular representation of G. The following results for $n=2$ are due to Gonzalez-Sprinberg and Verdier[GV] and their generalizations to $n=3$ are known in [Na][IN][BKR]. We summarize the relevant results for our case $G=\mathbf{Z}_{\mu+1}$:

1) The K-group $K(X)$ of algebraic vector bundles are generated by the so-called tautological bundles $\mathcal{F}_0,\mathcal{F}_1,\cdots,\mathcal{F}_\mu$, where the subscripts refer to the one-dimensional representations of $\mathbf{Z}_{\mu+1}$. From this we have
$$H^{even}(X,\mathbf{Z})=\mathbf{Z}c_1(\mathcal{F}_0)+\mathbf{Z}c_1(\mathcal{F}_1)+\cdots+\mathbf{Z}c_1(\mathcal{F}_\mu)\ .$$
2) Let $K^c(X)$ be the K-group of the complexes of algebraic vector bundles which are exact off $\pi^{-1}(0)$ where $\pi:X\to\mathbf{C}^2$. Then there exist a complete pairing $K^c(X)\times K(X)\to\mathbf{Z}$, and the dual basis $S_0,S_1,\cdots,S_\mu$ of $K^c(X)$ satisfying
$$\langle ch(S_i),ch(\mathcal{F}_j)\rangle:=\int_X ch(S_i)ch(\mathcal{F}_j)Todd_X=\delta_{ij}\ . \tag{5.1}$$
3) The dual bases $S_k(k\neq 0)$ are given by $S_k=\mathcal{O}_{C_k}(-1)$ with $C_k\cong\mathbf{P}^1$ being the exceptional curves. And they satisfy
$$\chi(S_i,S_j):=\int_X ch(S_i^\vee)ch(S_j)Todd_X=\mathcal{C}_{ij}\ \ (1\leq i,j\leq\mu), \tag{5.2}$$

where C_{ij} is the Cartan matrix of the root system $A_{\mu+1}$.

Now with these properties about the geometry of X, let us recall that our Conjecture 2.2;

$$w(x_1,\cdots,x_r;\frac{J_1}{2\pi i},\cdots,\frac{J_r}{2\pi i})=\sum_{i,j}\int_{mir(E_i)}\Omega(Y_x)\,\chi^{ij}\,ch(E_j)\;.$$

One should note that this is a conjecture for X a compact Calabi-Yau manifold. It is rather clear how to modify this relation for our non-compact X if we notice that the expression $\sum_j \chi^{ij}ch(E_j)$ satisfy $\int_X ch(E_k^\vee)\sum_j \chi^{ij}ch(E_j)=\delta^i{}_k$, i.e., provides a dual to $ch(E_k)$. From this, we claim the central charge formula for X non-compact cases as

$$w(x_1,\cdots,x_r;\frac{J_1}{2\pi i},\cdots,\frac{J_r}{2\pi i})=\sum_k\int_{mir(S_k)}\Omega(Y_x)\,ch(\mathcal{F}_k)\;, \tag{5.3}$$

where the holomorphic two form $\Omega(Y_x)$ should be understood as a holomorphic two-form associated to the integrand of (3.1) which is similar to the primitive form in (4.2). Now, since we verify $c_1(\mathcal{F}_k)=J_k$ $(k\geq 1)$ by an explicit evaluation, we have $w(x,\frac{J}{2\pi i})=1+\sum_k w_k(x)J_k$. Through these relations, we have the central charge of the sheaf $S_k=\mathcal{O}_{C_k}(-1)$ by

$$Z_t(S_k)=-\int_{mir(S_k)}\Omega(Y_x)=-w_k(x)=\frac{1}{2\pi i}\big(\log\beta_{k-1}(a)-\log\beta_k(a)\big)\;. \tag{5.4}$$

In this form, we may refine the mirror symmetry described in (3-1) to the claim that the mirror image $mir(S_k)$ of the sheaf $S_k=\mathcal{O}_{C_k}(-1)$ is the vanishing cycle whose 'period' is given by $Z_t(S_k)$ above.

Finally, we note that the sheaves S_k are spherical and thus define self-equivalences of $D^b(Coh(X))$, the Seidel-Thomas twists summarized in section 2. In [ST, Proposition 3.19], it is shown that these spherical objects, which form the so-called (A_μ)-configuration, generate a weak braid group action on $D^b(Coh(X))$. This braid group action should be mirrored to the corresponding Dhen twists (Picard- Lefschetz transformations) in the symplectic side. In our (5.4), we see this mirror correspondence in the linear transformations on the central charges.

6. Conclusion and discussions

We have given an interpretation for a cohomology-valued hypergeometric series, which was reported in [Hos]. Giving an interpretation for $w(x,\frac{J}{2\pi i})$ (Conjecture thm:conj) as the central charge formula which naturally appears in homological mirror symmetry, we have provided supporting evidences for the conjecture in the case of local mirror symmetry of $X=\widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$.

As a byproduct, we have found that K.Saito's differential equations satisfied by the primitive forms may be replaced by a suitable (resonant) GKZ system whose solutions are easy to be setup[HKTY][HLY].

As addressed after Conjecture 2.2, our cohomology-valued hypergeometric series (or the central charge formula) connects two different 'monodromy' properties, Fourier-Mukai transforms and the monodromy transforms of hypergeometric series (Dhen twists in symplectic mapping class group). We see that the latter monodromy property arises associated with the discriminant locus in $\mathcal{M}_{Sec(\Sigma)}$. As shown in section 3 for $X = \widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}}$, the discriminant splits into several irreducible components in the q-coordinate and the monodromy transform around each irreducible component is identified with a suitable twist functors. From these facts, it is conceivable that the group of self-equivalences of $D^b(Coh(X))$, i.e. Auteq $D^b(X)$, is generated by these 'monodromy' transformations up to the shift functors,

$$\mathrm{Auteq}(\widehat{\mathbf{C}^2/\mathbf{Z}_{\mu+1}})/\{\, [k] \mid k \in \mathbf{Z}\,\} = \langle R_1, \cdots, R_\mu, T_{\mathcal{O}(-C_1)}, \cdots, T_{\mathcal{O}(-C_\mu)} \rangle\ ,$$

where R_k represents the functor tensoring the tautological line bundle $\mathcal{F}_k$ and $T_{\mathcal{O}(-C_k)}$ is the Seidel-Thomas twist.

Although in this note we have restricted our attention to two dimensional cases, the claimed relation (5.3), for example, generalizes to three dimensional G-Hilb $\mathbf{C}^3$[IN][BKR][CI] with G abelian. It is actually our motivation to see the homological mirror symmetry in such a special kind of non-compact (toric) Calabi-Yau threefolds. From the hypergeometric series side, the monodromy calculations become suddenly tedious for multi-valuable series. We can perform, however, some explicit calculations for lower number of parameters providing consistency check for the conjecture (5.3), which will be reported in the forthcoming publication. From the holomorphic side, we have many possibilities other than G-Hilb $\mathbf{C}^3$ for the Calabi-Yau resolution of the singularity $\mathbf{C}^3/G$, and it is known that we have a variety of Fourier-Mukai transforms on $D^b(X)$ for these different resolutions[BKR][CI]. Making a (homological) mirror picture for these is strongly desired.

Finally, as for the compact Calabi-Yau (hypersurfaces), several supporting evidences for our Conjecture 2.2 are presented in [Hos], especially for dimensions one and two. However, for example, the observation (1.4) made in three dimension is still need clarification. Namely it is not clear that the specific form of the 'charges' $1, J - \frac{c_1(X_5)J}{12} - \frac{11}{2}, \frac{J^2}{5}, \frac{J^3}{5}$ come from an integral, symplectic basis of the K-group $K(X_5)$. From the SYZ construction we expect that $K(X_5)$ is generated by the structure sheaf $\mathcal{O}_X$, the skyscraper sheaf $\mathcal{O}_p$, and additional sheaves $\mathcal{E}$ and $\mathcal{F}$ which, respectively, have their support on a divisor and a curve (i.e. $D4$ and $D2$ branes).

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