Log canonical threshold of the pair: the Grassmannian variety and the Chow form

Yongnam Lee [1]

1. Introduction

The most direct approach to the construction of a moduli space or a compactified moduli space of algebraic varieties is via Geometric Invariant Theory (G.I.T. for short). It is one of the most useful methods to construct a moduli space or a compactified moduli space of algebraic varieties if one knows the effective criteria for stability and semi-stability.

The group $SL(n)$ acts on $V_{d,n} = \mathrm{Sym}^d(V)$, which is the vector space of homogeneous polynomials of degree d in $\mathbb{C}[x_1, \dots, x_n]$. By Hilbert-Mumford criterion, Mumford [15] provides a simple way to decide the stability and the semi-stability of $f \in V_{d,n}$ by the position of nonzero coefficients of f in a $(n-1)$-dimensional polytope. For a higher codimension case, the notion of stability is defined by Chow form. Let X be a subvariety of dimension $r-1$ and of degree d in $\mathbb{P}^{n-1}$. Consider the set $Z(X)$ of all the $(n-r-1)$-dimensional projective subspaces L in $\mathbb{P}^{n-1}$ that intersects X. This is a subvariety in the Grassmannian $G(n-r,n)$ parameterizing all the $(n-r-1)$-dimensional projective subspaces in $\mathbb{P}^{n-1}$. The subvariety $Z(X)$ is a hypersurface of degree d in $G(n-r,n)$. Let $\mathcal{B} = \oplus_{d=0}^{\infty} \mathcal{B}_d$ be the coordinate ring of $G(n-r,n)$ in the Plüker embedding. Then $Z(X)$ is defined by the vanishing of some element $R_X \in \mathcal{B}_d$ which is unique up to a constant factor. This element is called the Chow form of X. A variety X is called Chow semi-stable (resp. Chow stable) if its Chow form is semi-stable (resp. stable) for the natural $SL(n)$-action. Mumford [15] provides a way to decide Chow stability or Chow semi-stability by giving the weighted flag in $H^0(X, \mathcal{O}_X(1))$. Contrary to hypersurfaces in $\mathbb{P}^{n-1}$, there is no simple way to decide Chow stability.

There is an expectation of the restriction of singularities by the notion of stability. A natural question arises, to give a criterion for stability in terms of the nature of the singularities. There are various ways to measure how singularities of a variety are. Let Y be a nonsingular variety and D an effective $\mathbb{Q}$-Cartier divisor of Y. The invariant of the singularities of the pair (Y, D), called the log canonical threshold of Y along D, is an important topic to study the classification of higher dimensional algebraic varieties. It received a lot of attention recently [2], [10], [13], [17], [18].

The aim of this paper is to provide a criterion for Chow stability of X in $\mathbb{P}^{n-1}$ including log canonical threshold of the Chow form $Z(X)$ in the Grassmannian $G = G(n-r,n)$. We prove the following :

Theorem. *Let X be a nondegenerate variety in $\mathbb{P}^{n-1}$. Assume that the dimension of X is $r-1$ and the degree of X is d. Let $(G, Z(X))$ be a pair as above. Then we have the following criterion for Chow stability of X :*

---

[1]The research was supported by Korea Research Foundation Grant (KRF-2002-070-C00003).
2000 *AMS Subject Classification.* 14B05, 14L24, 14J10.

(1) If $\mathrm{lct}(G, Z(X)) \geq \frac{n}{d}$ *then* X *is Chow semi-stable.*
(2) If $\mathrm{lct}(G, Z(X)) > \frac{n}{d}$ *then* X *is Chow stable.*

This result is a generalization of its in [9], and the author was informed that Hacking got the same result independently. The idea that, in great generality, some kind of stability should be equivalent to a type of log canonical property is not new. The main point is that the criterion for stability, and the determination of the log canonical threshold, involve the Newton polyhedron in the same way. On the stability side the criterion is due to Hilbert. On the side of the log canonical threshold, the required statement is made at least in the paper [19]. But our main statement is not in the literature and our main contribution is the interpretation of the stability via the log canonical threshold of the Chow form.

Our theorem has an important application. The problem how to construct a good compactification of a moduli space of surface of general type with fixed numerical invariants was approached by Kollár and Shepherd-Barron [12] via the minimal model program in higher dimensional geometry. It was theoretically clarified by Alexeev [1] with the proof of the bounds conjecture in dimension two. The compactified moduli space should include (possibly reducible) surfaces with some mild singularities. These surfaces are called smoothable stable surfaces. This notion of smoothable stable surface can be generalized to smoothable stable log surfaces similar as the generalization of stable curve to stable pointed curve. For the experiment of the construction of a compatified moduli space consisted of smoothable stable log surfaces, Hassett [7] [8] considered a compactification $\bar{\mathcal{P}}_d^1$ of the family of smooth plane curves of degree d ($d \geq 4$) by using the stable log surfaces and the $\mathbb{Q}$-Gorenstein deformation theory of stable log surfaces. Then he succeeded to prove that $\bar{\mathcal{P}}_4^1$ is isomorphic to the Deligine-Mumford compactification of moduli space of curves of genus 3. In his construction, he considered all possible plane curve singularities appearing on the boundary of the Deligine-Mumford compactification, and then he constructed corresponding stable log surfaces by using local stable reduction theorem. But this is already too complicated to manage if $d \geq 5$. Hacking [6] considered instead the family of compactifications given by moduli space $\bar{\mathcal{P}}_d^\alpha$ of log surfaces (Y, D) where $K_Y + \alpha D$ has semi log canonical singularities and it is ample, where $\frac{3}{d} < \alpha \leq 1$. The compactification is simpler for lower α. He gave a compactification $\bar{\mathcal{P}}_d$ of plane curves of degree d by allowable family of stable pairs of degree d. In their geometric compactifications, a natural question arises, to compare Geometric compactifications via minimal model program with G.I.T. compactifications. By generalization of his notion of stable pairs of degree d to stable pairs of type (r, n, d) via using Grassmannian and Chow form, our theorem implies the following :

Theorem. *Let* X *be a variety of dimension* $r-1$ *and of degree* d *in* $\mathbb{P}^{n-1}$*. If* $(G(n-r, n), Z(X))$ *be a stable pair of type* (r, n, d) *then* X *is Chow stable in* $\mathbb{P}^{n-1}$*.*

Let X be a variety of dimension $r-1$ and of degree d in $\mathbb{P}^{n-1}$. If X is not Chow stable in $\mathbb{P}^{n-1}$, then $(G(n-r, n), D)$ is not a stable pair of type (r, n, d), therefore, it is not an object corresponding to a point in the compactification of stable pairs.

We work throughout over the complex number field $\mathbb{C}$. The notation here follows Hartshorne's Algebraic Geometry

2. Chow stability

Let $\mathbb{P}(V^*) = \mathbb{P}^{n-1}$. The group $SL(n)$ acts on $V_{d,n} = \mathrm{Sym}^d(V)$, which is the vector space of homogeneous polynomials of degree d in $\mathbb{C}[x_1, \ldots, x_n]$.

The group action $SL(n)$ on $V_{d,n}$ as following:

$$A \cdot f: = f \circ A \text{ for } A \in SL(n) \text{ and } f \in V_{d,n}.$$

Recall G.I.T. [14], [15]. Let $f \in V_{d,n}$. Then f is

- semi-stable if $0 \notin \overline{O^{SL(n)}(f)}$,
- unstable if $0 \in \overline{O^{SL(n)}(f)}$,
- stable if the orbit $O^{SL(n)}(f)$ is closed and the stabilizer $\mathrm{Stab}^{SL(n)}(f)$ is finite.

Each point $f \in V_{d,n}$ defines a hypersurface of degree d in $\mathbb{P}^{n-1}$. There is a simple way to decide the stability of f by using the Hilbert-Mumford criterion [14], [15]. We illustrate the case $n = 3$. The technique for determining stability is essentially same for any n. Represent f as below by a triangle of coefficients, T.

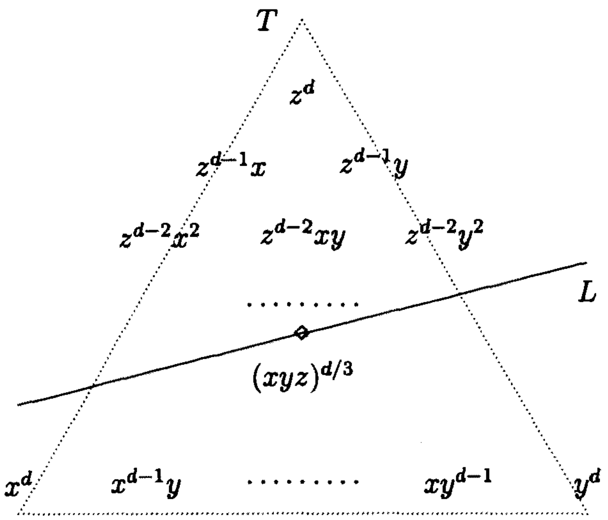

Fig 1. Triangle

We can coordinate this triangle by 3 coordinates i_x, i_y, i_z (the exponents of x, y and z respectively) with $i_x + i_y + i_z = d$. The condition that a line L with equation $ai_x+bi_y+ci_z = 0, (a, b, c) \neq (0, 0, 0)$, should pass through the center is just $a+b+c = 0$; if L also passes through a point with integral coordinates then a, b and c can be chosen integral. Let λ be a one parameter subgroup of $SL(3)$. Then λ can always be diagonalized in a suitable basis:

$$\lambda(t) = \begin{bmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{bmatrix},$$

where $a+b+c=0$. Let $f = \sum_{i_x+i_y+i_z=d} \alpha_{i_x i_y i_z} x^{i_x} y^{i_y} z^{i_z}$ in these coordinates. Then

$$\lambda(t)f = \sum_{i_x+i_y+i_z=d} \alpha_{i_x i_y i_z} t^{a i_x + b i_y + c i_z} x^{i_x} y^{i_y} z^{i_z}.$$

Hence, by Hilbert-Mumford criterion, we have the following :

Proposition 2.1. *Let $f \in V_{d,3}$. Then*
(1) f is unstable if and only if, in some coordinates, all non-zero coefficients of f lie to one side of some L.
(2) f is stable(resp. semi-stable) if an only if, for all coordinates and all L, f has non-zero coordinates on both sides of L (resp. f has non-zero coordinates on both sides of L or has non-zero coefficients on L).

Let X be a subvariety of dimension $r-1$ and of degree d in $\mathbb{P}^{n-1}$. Consider the set $Z(X)$ of all $(n-r-1)$-dimensional projective subspaces L in $\mathbb{P}^{n-1}$ that intersects X. This is a subvariety in the Grassmannian $G(n-r,n)$ parameterizing all the $(n-r-1)$-dimensional projective subspaces in $\mathbb{P}^{n-1}$. The subvariety $Z(X)$ is a hypersurface of degree d in $G(n-r,n)$. Let $\mathcal{B} = \oplus_{d=0}^{\infty} \mathcal{B}_d$ be the coordinate ring of $G(n-r,n)$ in the Plüker embedding. The subvariety $Z(X)$ is defined by the vanishing of some element $R_X \in \mathcal{B}_d$ which is unique up to a constant factor. This element is called the Chow form of X.

If $u = (u_{ij}) \in (\mathbb{P}^{n-1})^*$ write H_u for the hyperplane $\sum_{i=1}^{n} u_i X_i = 0$ where X_i, $i = 1, \ldots, n$ are coordinates on $\mathbb{P}^{n-1}$. Then

$$[X \cap H_u^{(1)} \cap \ldots \cap H_u^{(r)} \neq \emptyset] \Leftrightarrow [R_X(u_i^{(1)}, \ldots, u_i^{(r)}) = 0].$$

The coordinate ring $\oplus_{d=0}^{\infty} \mathcal{B}_d$ = subring of $\mathbb{C}[\ldots, U_i^{(j)}, \ldots]$ generated by the Plücker coordinates $P_{i_1,\ldots,i_r}$ = determinant of $r \times r$ maximal minors of $(U_{i_\ell}^{(j)})$, $i_1 < \ldots < i_r$.

A variety X is called Chow semi-stable (resp. Chow stable) if its Chow form is semi-stable (resp. stable) for the natural $SL(n)$-action. Contrary to hypersurfaces in $\mathbb{P}^{n-1}$, there is no simple way to decide Chow stability.

Choose one parameter subgroup (1-PS for short)

$$\lambda(t) = \begin{bmatrix} t^{r_1} & & & 0 \\ & t^{r_2} & & \\ & & \ddots & \\ 0 & & & t^{r_n} \end{bmatrix} \cdot t^{-k}$$

k chosen so that this is a 1-PS of $SL(n)$, i.e. $k = \frac{\sum r_i}{n}$. Define an ideal sheaf $\mathcal{I}_F \subset \mathcal{O}_{X \times \mathbb{A}^1}$ by $\mathcal{I}_F\,[\mathcal{O}_X(1) \otimes \mathcal{O}_{\mathbb{A}^1}]$ = subsheaf generated by $\{t^{r_i} X_i\}$, $i = 1, \ldots, n$. The subscheme $Z = \mathcal{O}_{X \times \mathbb{A}^1}/\mathcal{I}_F$ is concentrated over $0 \in \mathbb{A}^1$ and the support of $\mathcal{I}_F$ lies over the section $X_n = 0$ in X.

Consider the weighted flag:

$$V_1 = (X_2 = \ldots = X_n = 0) \subset V_2 = (X_3 = \ldots = X_n = 0) \subset \ldots \subset V_{n-1} = (X_n = 0)$$

where V_i has the weight r_i.

Denote by e_F the multiplicity $e_{\mathcal{O}_{X\times\mathbb{A}^1}(1)}(\mathcal{I}_F) = e_{\mathcal{O}_X(1)\times\mathcal{O}_{m\mathbb{A}^1}}(\mathcal{I}_F)$. Then Chow stability is determined by the multiplicity e_F :

Theorem 2.2. [15] *Let X be a variety of dimension $r-1$ and of degree d in $\mathbb{P}^{n-1}$. Fix a weighted flag $F = \{(V_i, r_i)\}$ in $H^0(X, \mathcal{O}_X(1))$. Then the following are equivalent:*
(i) X is Chow semi-stable (resp. Chow stable) in $\mathbb{P}^{n-1}$ with respect to F.
(ii) $e_F \leq \frac{rd}{n}\sum_{i=1}^n r_i$ (resp. $e_F < \frac{rd}{n}\sum_{i=1}^n r_i$).

3. Chow stability criterion including log canonical thresholds

Let Y be a nonsingular variety and D an effective $\mathbb{Q}$-Cartier divisor of Y. The invariant of the singularities of the pair (Y, D), called the log canonical threshold of Y along D, is an important topic to study the classification of higher dimensional algebraic varieties.

The notion of discrepancy is the fundamental measure of the singularities of (Y, D). The usual definitions in the theory of singularities of pairs, for which we refer to [10] or [11].

Definition. Let (Y, D) be a pair as above, $Z \subset Y$ a closed subscheme. The log canonical threshold of (Y, D) along Z is defined by
$\mathrm{lct}_Z(Y, D) := \sup\{c \in \mathbb{Q}_+ \mid (Y, cD)$ is log canonical in an open neighborhood of $Z\}$.
When $Z = Y$, we write it as $\mathrm{lct}(Y, D)$ by deleting Z from the notion.

The log canonical threshold of the pair can be computed by using a log resolution of the pair or by assigning the weights to the variables. Let (Y, D) be a pair as above. Then $\mathrm{lct}(Y, D) = \inf\{\mathrm{lct}_y(Y, D) \mid y \in Y\}$.

Let $p : W \to Y$ be a proper birational morphism. Write

$$K_W = p^*K_Y + \sum a_i E_i, \quad \text{and} \quad p^*D = \sum b_i E_i.$$

Then

$$\mathrm{lct}_Z(Y, D) \leq \min_{p(E_i)\cap Z\neq\emptyset}\left\{\frac{a_i + 1}{b_i}\right\}.$$

Equality holds if $\sum E_i$ is a divisor with normal crossing only. In particular, $\mathrm{lct}_Z(Y, D) \in \mathbb{Q}$.

In general, it is hard to construct a log resolution explicitly. An efficient way of computation of log canonical threshold is in the weighted case:

Theorem 3.1. [10] *Let f be a holomorphic function near $0 \in \mathbb{C}^n$ and $D = (f = 0)$. Assign positive integer weights $w(x_i)$ to the variables x_i, and let $w(f)$ be the weighted multiplicity of f (the lowest weight of the monomial occurring in f). Then*

$$\mathrm{lct}_0(\mathbb{C}^n, D) \leq \min\left\{1, \frac{\sum w(x_i)}{w(f)}\right\}.$$

And the equality holds if the weighted homogeneous leading term f_w of f has an isolated critical point at the origin or if $f_w(x_1^{w(x_1)}, \dots, x_n^{w(x_n)}) = 0 \subset \mathbb{P}^{n-1}$ is smooth, or f is quasi-homogeneous.

Example 3.2. Let $f = y^2 - x^4$ and let $D = (f = 0)$ in $\mathbb{C}^2$.
(1) By blowing up two times, we have a log resolution $p : W \to \mathbb{C}^2$ and

$$K_W = p^* K_{\mathbb{C}^2} + E_1 + 2E_2, \ \ p^* D = p^{*-1} D + 2E_1 + 4E_2.$$

Hence we have $\operatorname{lct}_0(\mathbb{C}^2, D) = \min\{\frac{0+1}{1}, \frac{1+1}{2}, \frac{2+1}{4}\} = \frac{3}{4}$.
(2) Assign weights $w(x) = 1$ and $w(y) = 2$, then $w(f) = 4$. Hence we have $\operatorname{lct}_0(\mathbb{C}^2, D) = \frac{w(x)+w(y)}{w(f)} = \frac{3}{4}$.

Let D be a hypersurface in $\mathbb{P}^{n-1}$. The stability of D in $\mathbb{P}^{n-1}$ can be determined by the singularities of the pair $(\mathbb{P}^{n-1}, D)$.

Theorem 3.3. *Let D be a hypersurface of degree d in $\mathbb{P}^{n-1}$. Then we have the following criterion for stability of D :*
(1) If $\operatorname{lct}(\mathbb{P}^{n-1}, D) \geq \frac{n}{d}$ then D is semi-stable.
(2) If $\operatorname{lct}(\mathbb{P}^{n-1}, D) > \frac{n}{d}$ then D is stable.

Proof. The detailed proof of the case of the pair $(\mathbb{P}^2, D)$ is given in [9]. Since our proof goes basically same way, we give a sketch of proof. Note that $\operatorname{lct}_p(\mathbb{P}^{n-1}, D)$ is lower semi-continuous. Let $D = (f = 0)$. Choose a point $p \in D$ such that $\operatorname{lct}_p(\mathbb{P}^{n-1}, D) = \operatorname{lct}(\mathbb{P}^{n-1}, D)$.

Assume that D is not semi-stable. By a linear change of coordinates $x_1, \dots, x_n$ and by Hilbert-Mumford criterion, we may assume that $x_n(p) = 1$, $x_i(p) = 0$ for $i = 1, \dots, n-1$ and we have the following:
(1) every monomial $x_1^{i_1} \dots x_{n-1}^{i_{n-1}} x_n^{i_n}$ in f satisfies $i_1 + \dots + i_{n-1} \leq \frac{n-1}{n} d$, i.e. $i_n \geq \frac{d}{n}$,
(2) there are non-negative integers $k_1, \dots, k_{n-1}$ and a negative integer k_n such that $k_1 + \dots + k_n = 0$ and $k_1 i_1 + \dots + k_n i_n > 0$ for every monomial $x_1^{i_1} \dots x_{n-1}^{i_{n-1}} x_n^{i_n}$ in f.

Therefore we have

$$k_1 i_1 + \dots + k_{n-1} i_{n-1} > (-k_n) i_n \geq (-k_n) \frac{d}{n}.$$

Let $\bar{f}(x_1, \dots, x_{n-1}) = f(x_1, \dots, x_{n-1}, 1)$. Assign the weights the the variables x_i, $w(x_i) = k_i$ for $i = 1, \dots, n-1$, it implies that

$$\operatorname{lct}_p(\mathbb{P}^{n-1}, D) \leq \frac{k_1 + \dots + k_{n-1}}{w(\bar{f})} < \frac{-k_n}{(-k_n)\frac{d}{n}} = \frac{n}{d}.$$

It proves (1). The proof of (2) is the same as above. □

Remark 3.4. The condition $\operatorname{lct}(\mathbb{P}^{n-1}, D) > \frac{n}{d}$ can be expressed in other way. Note that

$$\operatorname{lct}(\mathbb{A}^n, \operatorname{Cone}(D)) = \min\left\{\frac{n}{d}, \operatorname{lct}(\mathbb{P}^{n-1}, D)\right\}.$$

Therefore the following are equivariant:
(1) $\operatorname{lct}(\mathbb{P}^{n-1}, D) > \frac{n}{d}$.

(2) The pair $(\mathbb{A}^n, \mathrm{Cone}(D))$ has the worst singularity at 0, i.e. if we define $t = \mathrm{lct}(\mathbb{A}^n, \mathrm{Cone}(D))$ then the non log terminal locus of the pair $(\mathbb{A}^n, t\,\mathrm{Cone}(D)) = \{0\}$.

Remark 3.5. The converse of Theorem 3.3 is also true in most of cases. Let $D = (f = 0)$ be a semi-stable (resp. stable) hypersurface of degree d in $\mathbb{P}^{n-1}$. Choose a point p so that $\mathrm{lct}_p(\mathbb{P}^{n-1}, D) = \mathrm{lct}(\mathbb{P}^{n-1}, D)$. Then by a linear change of coordinates $x_1, \ldots x_n$ and by Hilbert-Mumford criterion, we may assume that $x_n(p) = 1$, $x_i(p) = 0$ for $i = 1, \ldots, n-1$, and there is a monomial $x_{i_1} \ldots x_n^{i_n}$ of f such that
(1) $i_1 + \ldots + i_{n-1} \geq \frac{n-1}{n} d$,
(2) $k_1 + \ldots + k_n = 0$,
(3) $k_1 i_1 + \ldots + k_n i_n \leq 0$ (resp. $k_1 i_1 + \ldots + k_n i_n < 0$).

Let $\bar{f}(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, 1)$. Assign the weights $w(x_i) = k_i$. Then the weight $w(\bar{f}) \leq k_1 i_1 + \ldots + k_{n-1} i_{n-1} \leq (-k_n) i_n$. Assume that the weighted homogeneous leading term $\bar{f}_w$ of $\bar{f}$ in $\mathbb{P}^{n-2}$ has an isolated critical point at the origin or if $\bar{f}_w(x_1^{w(x_1)}, \ldots, x_{n-1}^{w(x_{n-1})}) = 0 \subset \mathbb{P}^{n-2}$ is smooth. Then we have

$$\mathrm{lct}_p(\mathbb{P}^{n-1}, D) \geq \frac{k_1 + \ldots + k_{n-1}}{w(\bar{f})} \geq \frac{k_1 + \ldots + k_{n-1}}{(-k_n) i_n} \geq \frac{n}{d} \text{ (resp. } \mathrm{lct}_p(\mathbb{P}^{n-1}, D) > \frac{n}{d}).$$

Example 3.6. Let D be a 3 times nonsingular conic plane curve $C = (xz + y^2)$, i.e. $D = 3C$. Then $(\mathbb{P}^2, D)$ is semi-stable but $\mathrm{lct}(\mathbb{P}^2, D) = \frac{1}{3}$. Let $\bar{f}(x, y) = f(x, y, 1) = x + y^2$ and assign the weights $w(x) = 2, w(y) = 1$. Then $\bar{f}_w = (x^2 + y^2)^3 = 0$ in $\mathbb{P}^1$ does not give distinct points.

Theorem 3.3 can be generalized to the pair of Grassmannian variety and Chow form. Let X be a variety of dimension $r - 1$ in $\mathbb{P}^{n-1}$. Assume that the degree of X is d and X is nondegenerate. Chow form R_X determines a hypersurface $Z(X)$ in the Grassmannian variety $G = G(n - r, n)$ parameterizing all the $(n - r - 1)$-dimensional projective subspaces in $\mathbb{P}^{n-1}$.

$$Z(X) \subset G = G(n - r, n).$$

Theorem 3.7. *Let X be a nondegenerate variety in $\mathbb{P}^{n-1}$. Assume that the dimension of X is $r - 1$ and the degree of X is d. Let $(G, Z(X))$ be a pair as above. Then we have the following criterion for Chow stability of X :*
(1) If $\mathrm{lct}(G, Z(X)) \geq \frac{n}{d}$ then X is Chow semi-stable.
(2) If $\mathrm{lct}(G, Z(X)) > \frac{n}{d}$ then X is Chow stable.

Proof. Consider the product $\tilde{X} = X \times \mathbb{P}^{r-1}$ as a subvariety of $\mathbb{P}(\mathbb{C}^n \otimes (\mathbb{C}^r)^*)$ via the Segre embedding. Identify $\mathbb{C}^n \otimes (\mathbb{C}^r)^*$ with the space $\mathrm{Mat}(r, n)$ of $r \times n$-matrices and consider the projection

$$\mathrm{Mat}(r, n) \supset S(r, n) \xrightarrow{p} G = G(n - r, n)$$

where $S(r, n)$ is the subset of $\mathrm{Mat}(r, n)$ with full rank. By this identification, the equation of dual variety $\tilde{X}^\vee$ in $\mathbb{P}^{nr-1}$ is the same as the equation $\tilde{R}_X$ lifted by R_X. This identification implies that

$$\tilde{X}^\vee = \text{projectivization of the closure of } p^{-1}(Z(X)) \text{ [5]}.$$

Assume that $Z(X)$ is not Chow semi-stable in G. By the functorial properties [16], $p^{-1}(Z(X))$ is not semi-stable in $S(r,n)$. And it implies that $\tilde{X}^\vee$ is not semi-stable in $\mathbb{P}^{nr-1}$. By the proof of Theorem 3.3,

$$\inf_{y\in U} \operatorname{lct}_y(\mathbb{P}^{nr-1}, \tilde{X}^\vee) < \frac{nr}{\deg \tilde{X}^\vee} = \frac{n}{d}$$

where U be the projectivization of $S(r,n)$ in $\mathbb{P}^{nr-1}$. And

$$\inf_{y\in U} \operatorname{lct}_y(\mathbb{P}^{nr-1}, \tilde{X}^\vee) = \operatorname{lct}(S(r,n), p^{-1}(Z(X))) = \operatorname{lct}(G, Z(X))$$

because $S(r,n)$ is a $GL(r)$-bundle over G. It proves (1). The proof of (2) is the same as above. □

Example 3.8. Let $X = p_1 \cup \ldots \cup p_d$ be d points in $\mathbb{P}^{n-1}$. Then X is Chow semi-stable (resp. Chow stable), i.e. the Chow form $Z(X)$ is semi-stable (resp. Chow stable) in $G(n-1,n) = (\mathbb{P}^{n-1})^*$, if and only if for every proper linear subspace W of $\mathbb{P}^{n-1}$ (cf. [4])

$$\#\{i | p_i \in W\} \leq \frac{d}{n}(\dim W + 1) \text{ (resp. } <).$$

By the following easy lemma, this is the same condition as

$$\operatorname{lct}((\mathbb{P}^{n-1})^*, Z(X)) \geq \frac{n}{d} \text{ (resp. } >).$$

Lemma 3.9. Let Y be a nonsingular variety of dimension m. Let D be a union of nonsingular divisors $D_1, \ldots, D_d$ of Y. Assume that the scheme theoretic intersection Z of $D_1, \ldots, D_d$ is a nonsingular variety of dimension k, and that $D_1, \ldots, D_d$ meet transversally at Z. Then $\operatorname{lct}(Y, D) = \frac{m-k}{d}$.

Proof. The proof is obtained by blowing up of Z in Y. □

Example 3.10. Let $X = \ell_1 \cup \ldots \cup \ell_d$ be d lines in $\mathbb{P}^3$. Then X is Chow semi-stable if and only if it satisfies the following (cf. [4]):
(1) no more than $\frac{d}{2}$ lines intersects at one point,
(2) no more than $\frac{d}{2}$ lines coincides and no more than $m - 2t$ lines intersects a line which is repeated t times,
(3) no more than $\frac{d}{2}$ lines are coplanar.

If $\operatorname{lct}(G(2,4), Z(X)) \geq \frac{4}{d}$ then X is Chow semi-stable. But the conditions (1), (2), (3) do not imply $\operatorname{lct}(G(2,4), Z(X)) \geq \frac{4}{d}$. If we translate the conditions (1) and (3) into the conditions in Chow form, then we have the following:
(1) no more than $\frac{d}{2}$ hyperplanes meets at quadric surface induced by the intersection of G with two hyperplanes (the set of lines through at one point in $\mathbb{P}^3$),
(3) no more than $\frac{d}{2}$ hyperplanes meets at $\mathbb{P}^2$ (the set of lines in the coplane).
These imply that $\operatorname{lct}(G(2,4), Z(X)) \geq \frac{4-2}{\frac{d}{2}} = \frac{4}{d}$ by Lemma 3.9. But the second condition gives $\operatorname{lct}(G(2,4), Z(X)) \geq \frac{2}{d}$.

4. Log canonical thresholds of Chow forms

Let X be a nonsingular variety of dimension $r-1$ in $\mathbb{P}^{n-1}$. Assume that the degree of X is d and X is nondegenerate. Furthermore, we assume that the dual variety $X^\vee$ of X in $(\mathbb{P}^{n-1})^*$ is a hypersurface. Let $(G, Z(X))$ be a pair of Grassmannian variety and Chow form as before. Let $\tilde{X} = X \times \mathbb{P}^{r-1}$ in $\mathbb{P}^{nr-1}$ via the Segre embedding (cf. the proof of Theorem 3.7). By the construction, we have the inequality $\mathrm{lct}((\mathbb{P}^{n-1})^*, X^\vee) \leq \mathrm{lct}((\mathbb{P}^{nr-1})^*, \tilde{X}^\vee)$. And by the proof of Theorem 3.7, we have the inequality $\mathrm{lct}((\mathbb{P}^{nr-1})^*, \tilde{X}^\vee) \leq \mathrm{lct}(G, Z(X))$. Therefore we have the following:

Proposition 4.1. *Let $X, \tilde{X}, G, Z(X)$ be varieties as above. Then we have the following inequality:* $\mathrm{lct}((\mathbb{P}^{n-1})^*, X^\vee) \leq \mathrm{lct}(G, Z(X))$.

Example 4.2. Let X be a rational normal curve of degree d in $\mathbb{P}^d$. Then the dual variety $X^\vee$ in $(\mathbb{P}^d)^*$ is the classical discriminant (cf. [5]). Let $f(x) = \sum_{i=0}^d a_i x^{d-i}$. The classical discriminant $\Delta(f) = R(f, f')$ vanishes when $f(x)$ has multiple root, i.e. $f(x)$ has a multiple root if and only if $(a_0, \ldots, a_d) \in X^\vee \subset (\mathbb{P}^d)^*$.

By the definition of $\Delta(f)$, it has at worst singularity when $f(x)$ has a d-multiple root. Let $p = (1, 0, \ldots, 0)$. The discriminant $\Delta(f) = \Delta(a_0, \ldots, a_d)$ is a homogeneous polynomial in the a_i of degree $2d-2$. In addition, it satisfies the quasi-homogeneity condition:

$$\Delta(\lambda^0 a_0, \lambda^1 a_1, \ldots, \lambda^d a_d) = \lambda^{d(d-1)} \Delta(a_0, a_1, \ldots, a_d).$$

Assign the weights $w(a_i) = i$. Then

$$\mathrm{lct}((\mathbb{P}^d)^*, X^\vee) = \mathrm{lct}_p((\mathbb{P}^d)^*, X^\vee) = \frac{1 + \ldots + d}{d(d-1)} = \frac{1}{2}\frac{d+1}{d-1}.$$

Example 4.3. Let X be a rational normal curve of degree d in $\mathbb{P}^d$. Consider the product $\tilde{X} = X \times \mathbb{P}^1$ as a subvariety of $\mathbb{P}^{2d+1}$ via the Segre embedding. Then the dual variety $(\tilde{X})^\vee$ in $(\mathbb{P}^{2d+1})^*$ is the classical resultant (cf. [5]). Let $f(x) = \sum_{i=0}^d a_i x^{d-i}$, $g(x) = \sum_{i=0}^d b_i x^{d-i}$. The classical resultant $R(f, g)$ vanishes when f and g has a d-multiple common root.

By the definition of $R(f, g)$ it has at worst singularity when f and g has a d-multiple common root. Let $p = (1, 0, \ldots, 0, 1, 0, \ldots, 0)$. The classical resultant $R(f, g)$ is homogeneous of degree d in the a_i and in the b_i. In addition, it satisfies the following quasi-homogeneity:

$$R(\lambda^0 a_0, \ldots, \lambda^d a_d, \lambda^0 b_0, \ldots, \lambda^d b_d) = \lambda^{d^2} R(a_0, \ldots, a_d, b_0, \ldots, b_d).$$

Assign the weights $w(a_i) = i$, $w(b_i) = i$. Then

$$\begin{aligned} \mathrm{lct}((\mathbb{P}^{2d+1})^*, \tilde{X}^\vee) &= \mathrm{lct}_p((\mathbb{P}^{2d+1})^*, \tilde{X}^\vee) \\ &= \min\left\{1, \tfrac{1+\ldots+d+1\ldots+d}{d^2}\right\} \\ &= \min\left\{1, \tfrac{d(d+1)}{d^2}\right\}. \end{aligned}$$

So $\mathrm{lct}((\mathbb{P}^{2d+1})^*, \tilde{X}^\vee) = 1$.

Let X be a nonsingular variety of dimension $r-1$ in $\mathbb{P}^{n-1}$. Assume that the degree of X is d and X is nondegenerate. Let $(G, Z(X))$ be a pair as before. Consider the incidence variety $W = \{(x, L) \in X \times G \mid x \in L\}$ in $\mathbb{P}^{n-1} \times G$.

$$\begin{array}{ccccc} W & \subset & \mathbb{P}^{n-1} \times G & \rightarrow & \mathbb{P}^{n-1} \\ \downarrow & & \downarrow & & \\ L \in Z(X) & \subset & G & & \end{array}$$

Incidence variety W is nonsingular, therefore the singularities of $Z(X)$ at L are related to the points of $X \cap L$ in $L = \mathbb{P}^{n-r-1}$. In particular, the multiplicity of $Z(X)$ is determined [3]:

$$\operatorname{mult}_L(Z(X)) = \sum \operatorname{mult}(X \cap L, x_i)\,.$$

Let $m = \max_{L \in Z(X)} \operatorname{mult}_L(Z(X))$. Consider a subscheme

$$Y = \{L \in Z(X) \mid \operatorname{mult}_L(Z(X)) = m\}.$$

By upper semi-continuity of $\operatorname{mult}_L(Z(X))$, Y is a finite union of subvarieties Y_i of $Z(X)$. Let $\ell = \max\{\dim(Y_i)\}$. Then it is easily obtained that

$$\operatorname{lct}(G, Z(X)) \leq \frac{\dim G - \ell}{m}.$$

Choose a linear plane $\Lambda = \mathbb{P}^{n-r-2}$ with $X \cap \Lambda = \emptyset$. Consider a linear projection π_Λ of X by Λ:

$$\begin{array}{ccc} X & \subset & \operatorname{Bl}_\Lambda \mathbb{P}^{n-1} \\ & & \downarrow \pi_\Lambda \\ \bar{X} & \subset & \mathbb{P}^r. \end{array}$$

The pair $(\mathbb{P}^r, \bar{X})$ can be realized as a subset of the pair $(G, Z(X))$ by the following: The projective space $\mathbb{P}^r$ parameterizes all linear subspaces $L = \mathbb{P}^{n-r-1}$ containing Λ and $\bar{X} = \{L \in G \mid L \cap X \neq 0,\ \Lambda \subset L\}$.

By the above argument, $\operatorname{mult}_L(\bar{X}) = \sum \operatorname{mult}_{x_i \in X \cap L}(X \cap L)$ where $L \in \bar{X}$. Assume that $\operatorname{mult}_L(\bar{X}) = m$ and that $\operatorname{lct}(\mathbb{P}^r, \bar{X}) \geq c$. Then $m \leq \frac{r}{c}$. So we have the following:

Proposition 4.4. *Let $(\mathbb{P}^r, \bar{X})$ be the pair and π_Λ be the map as above. Let $\bar{X}_k$ be the closed set of points $\bar{x} \in \bar{X}$ such that the scheme-theoretic length of the fiber $\pi_\Lambda^{-1}(\bar{x})$ is at least k. Assume that* $\operatorname{lct}(\mathbb{P}^r, \bar{X}) = 1$. *Then*
(i) $\bar{X}_{r+1}$ is empty.
(ii) $\bar{X}_k$ has dimension at most $r-k$.

Example 4.5. Let X be a curve represented by the divisor class $(d-1, 1)$, $d \geq 5$ in a nonsingular quadric surface Q in $\mathbb{P}^3$.

There is a one-dimensional family of $d-1$ secant lines L in X, and there is no k $(3 \leq k \leq d-2)$ secant line in X. Since the dimension of G is four and there is a one-dimensional family of $d-1$ secant lines, $\operatorname{lct}(G, Z(X)) \leq \frac{3}{d-1}$.

By the adjunction formula, the genus of X is zero. So X is linearly semi-stable, and it is Chow semi-stable [15]. And by a generic projection of X from a point, $\operatorname{lct}(\mathbb{P}^2, \bar{X}) = 1$.

5. Compactifications of the family of Chow forms

For the experiment of the construction of a compatified moduli space consisted of smoothable stable log surfaces, Hassett [7] [8] considered a compactification $\bar{\mathcal{P}}_d^1$ of the family of smooth plane curves of degree d ($d \geq 4$). Then he succeeded to prove that $\bar{\mathcal{P}}_4^1$ is isomorphic to the Deligine-Mumford compactification of moduli space of curves of genus 3. In his construction, he considered all possible plane curve singularities appearing on the boundary of the Deligine-Mumford compactification, and then he constructed corresponding stable log surfaces by using local stable reduction theorem. But this is already too complicated to manage if $d \geq 5$. Hacking [6] considered instead the family of compactifications given by moduli space $\bar{\mathcal{P}}_d^\alpha$ of log surfaces (Y, D) where $K_Y + \alpha D$ has semi log canonical singularities and it is ample, where $\frac{3}{d} < \alpha \leq 1$. The compactification is simpler for lower α. He gave a compactification $\bar{\mathcal{P}}_d$ of plane curves of degree d by allowable family of stable pairs of degree d. In their geometric compactifications, a natural question arises, to compare Geometric compactifications via minimal model program with G.I.T. compactifications.

Definition. A log variety (Y, D) has semi log canonical singularities if
1. Y satisfies Serre's condition S_2,
2. Y has normal crossing singularities in codimension one,
3. $K_Y + D$ is $\mathbb{Q}$-Cartier, and for any birational morphism $\varphi : Z \to Y$ from a normal $\mathbb{Q}$-Gorenstein variety Y we have

$$K_Z \equiv \varphi^*(K_Y + D) + \sum a_i E_i$$

where all $a_i \geq -1$.

A stable log variety is the pair (Y, D) where
1. Y is a connected projective variety and D a reduced Weil divisor on Y,
2. (condition on singularities) the pair (Y, D) has semi log canonical singularities,
3. (numerical condition) $K_Y + D$ is ample.

A log variety (Y, D) is called a stable pair of type (r, n, d) if Y is a proper connected variety and D an effective Weil divisor with the following properties:
1. There is an $\epsilon > 0$ such that $K_Y + (\frac{n}{d} + \epsilon)D$ has semi log canonical singularities and it is ample.
2. $dK_Y + nD \sim 0$ ($\frac{d}{n}K_Y + D \sim 0$ if $n|d$).
3. There is a $\mathbb{Q}$-Gorenstein smoothing to a pair of Grassmannian and Chow form (i.e. there is a deformation $\mathcal{Y}$ of Y over a discrete valuation ring T with smooth general fiber such that $K_{\mathcal{Y}/T}$, $\mathcal{D}/T$ are $\mathbb{Q}$-Cartier, and whose general fiber is $(G(n-r, n), Z(X))$ where $Z(X)$ is the Chow form of a variety X of dimension $r-1$ and of degree d in $\mathbb{P}^{n-1}$).

A family of stable pair of type $(2, 3, d)$ was studied by Hacking [6]. Let $\bar{\mathcal{P}}_d(S) = \{(\mathcal{Y}, \mathcal{D})/S \,|\, \text{allowable family of stable pairs of type (2,3,d)}\}$. $(\mathcal{Y}, \mathcal{D})/S$ is called an allowable family if $\omega_{\mathcal{Y}/S}^{[i]}$, $\mathcal{O}_{\mathcal{Y}}(\mathcal{D})^{[i]}$ commute with base change for all i. He proved that $\bar{\mathcal{P}}_d$ is a separated proper Deligne-Mumford stack and $\bar{\mathcal{P}}_d$ is smooth if $3 \nmid d$.

Let $D = Z(X)$ be a Chow form of a variety X of dimension $r-1$ and of degree d in $\mathbb{P}^{n-1}$, and let $(G(n-r,n), D)$ be a stable pair of type (r,n,d). By the definition of stable pair of (r,n,d), $K_{G(n-r,n)}+(\frac{n}{d}+\epsilon)D$ is log canonical, i.e. $\mathrm{lct}(G(n-r,n),D) > \frac{n}{d}$. Then Theorem 3.7 implies that D is the Chow form of a Chow stable variety X in $\mathbb{P}^{n-1}$.

Theorem 5.1. *Let X be a variety of dimension $r-1$ and of degree d in $\mathbb{P}^{n-1}$. If $(G(n-r,n), Z(X))$ be a stable pair of type (r,n,d) then X is Chow stable in $\mathbb{P}^{n-1}$.*

Acknowledgements

The author acknowledges the invitation and the hospitality of the Department of Mathematics, University of Michigan in January of 2003, where the work was started. He would like to thank professor I. Dolgachev for many helpful discussions during his stay. Also he would like to thank professors Y. Kiem, S. Kwak, and J. Park for some useful discussions.

References

[1] V. Alexeev, Boundness and K^2 for log surfaces, Internat. J. Math. **5** (1994), 779-810.

[2] T. de Fernex, L. Ein, M. Mustaţă, Bounds for log canonical thresholds with applications to birational rigidity, Math. Res. Lett. **10** (2003), 219-236.

[3] A. Dimca, Milnor numbers and multiplicities of dual varieties. Rev. Roumaine Math. Pures Appl. 31 (1986), no. 6, 535-538.

[4] I. Dolgachev, *Lectures on invariant theory*, London Math. Soc. Lect. Notes Ser. vol. 296, 2003.

[5] I. Gelfand, M. Kapranov, A. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Birkhäuser, 1994.

[6] P. Hacking, *A compactification of the space of plane curves*, Ph.D. Thesis, Cambridge University, 2001.

[7] B. Hassett, Stable log surfaces and limits of quartic plane curves, Manuscripta Math. **100** (1999), 469-497.

[8] B. Hassett, Local stable reduction of plane curve singularities, J. Reine Angew. Math. **520** (2000), 169-194.

[9] H. Kim, Y. Lee, Log canonical thresholds of semistable plane curves, To appear in Math. Proc. Cambridge Phil. Soc.

[10] J. Kollár, Singularity of pairs, Proc. Symp. Pure Math. **62.1** (1997), 221-287.

[11] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, vol. 134, Cambridge Tracts in Math., 1998.

[12] J. Kollár, Shepherd-Barron, Threefolds and deformations of surface singularities, Invent. Math. **91** (1988), 299-338.

[13] T. Kuwata, On log canonical thresholds of reduced plane curves, Amer. J. Math. **121** (1999), 701-721.

[14] J. Le Potier, *Lectures on vector bundles*, Cambridge Studies in Advanced Math. vol. 54, Cambridge University Press, 1997.

[15] D. Mumford, Stability of projective varieties, L'Ens. Math. **23** (1977), 39-110.

[16] D. Mumford, J. Fogarty, *Geometric invariant theory*, 2nd ed. Springer-Verlag, 1982.

[17] M. Mustaţă, Singularities of pairs via jet schemes, J. Amer. Math. Soc. **15** (2002), 599-615.

[18] M. Mustaţă, Jet schemes of locally complete intersection canonical singularities, With an appendix by D. Eisenbud and E. Frenkel, Invent. Math. **145** (2001), 397-424.

[19] A. N. Varchenko, Newton polyhedra and estimates of oscilatory integrals, Funkcional. Anal. i Prilozen **10(3)** (1976), 13-38.

Department of Mathematics
Sogang University
Sinsu-dong, Mapo-gu
Seoul 121-742, Korea

email : ynlee@sogang.ac.kr