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AN EXAMPLE OF STABLE HIGGS BUNDLES WHICH DO NOT SATISFY THE BOGOMOLOV INEQUALITY

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INTRODUCTION

The Bogomolov inequality for semistable vector bundles on smooth complex projective $n$-folds reads

$$c_2(\mathcal{E})A^{n-2} \geq \frac{r-1}{2r}c_1(\mathcal{E})^2A^{n-2},$$

where $A$ is an ample divisor and $\mathcal{E}$ is an $A$-semistable vector bundle of rank $r$. In case $\mathcal{E}$ is $A$-stable with vanishing $c_1(\mathcal{E})$, the lower bound of this inequality $c_2(\mathcal{E})A^{n-2} \geq 0$ is attained if and only if $\mathcal{E}$ is a flat hermitian bundle associated with an irreducible unitary representation of the fundamental group $\pi_1(X)$, thereby establishing the one-to-one Kobayashi-Hitchin correspondence between the stable bundles with vanishing Chern classes and the irreducible unitary representations of $\pi_1(X)$ [2]. The inequality is natural enough to have proofs by several different approaches (geometric invariant theory [1]; characteristic $p$ method [3]; the theory of effective cones on ruled surfaces [8]; differential geometry [2]) and generalizes to bigger classes of semistable bundles, including orbibundles and parabolic bundles.

Another important class of generalised vector bundles is that of Higgs bundles (see [9]), and it is a natural question to ask if the Bogomolov inequality extends also to this class. The inequality is indeed true for standard types of Higgs bundles listed in Section 1 as Examples 0, 1, 2, and for bundles of small ranks 2, 3 as well [7]. Unfortunately, however, this is not the case for Higgs bundles of higher rank. In this note, we construct stable Higgs 4-bundles on surfaces of general type for which the inequality breaks down (Proposition 4 in Section 3). Starting from this example, we also find a nontrivially deforming families of stable Higgs 4-bundles with trivial Chern classes or, equivalently, non-trivial deformations of irreducible $\text{SL}(4)$-representations of the fundamental group (Section 4).

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1. HIGGS BUNDLES: DEFINITION AND STANDARD EXAMPLES

Let $\mathcal{E}$ be a vector bundle on a complex manifold $X$ and $\theta: \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$ an $\mathcal{O}_X$-linear mapping. The pair $(\mathcal{E}, \theta)$ is said to be a Higgs bundle if the natural composite map $\theta \wedge \theta: \mathcal{E} \to \Omega^2_X \otimes \mathcal{E}$ identically vanishes. Alternatively, $\mathcal{E}$ is a Higgs
bundle if an $\mathcal{O}_X$-linear action of the sheaf of the local vector fields $\Theta_X$ on $\mathcal{E}$ is given in such a way that $\xi_1(\xi_2(e)) = \xi_2(\xi_1(e))$, $\forall \xi_i \in \Theta_X, \forall e \in \mathcal{E}$. In other words, a Higgs bundles is a vector bundle equipped with a $\text{Sym} \Theta_X$-module structure, where $\text{Sym} \Theta_X = \bigoplus_{i=0}^{\infty} \text{Sym}^i \Theta_X$ is the symmetric tensor algebra generated by $\Theta_X$.

Higgs subsheaves are, by definition, $\text{Sym} \Theta_X$-submodules. Given an ample divisor $A$ on $X$, the notion of $A$-(semi)stability of Higgs bundles is naturally defined.

Historically, Higgs structures were introduced as the moduli of flat connections [5]. Let $\mathcal{E}$ be a vector bundle with an integrable connection $\nabla_0 : \mathcal{E} \to \Omega_X^1 \otimes \mathcal{E}$. Given another integrable connection $\nabla$, the difference $\theta = \nabla - \nabla_0 : \mathcal{E} \to \Omega_X^1 \otimes \mathcal{E}$ obviously gives a Higgs bundle structure and this correspondence translates the moduli of the flat connections on $\mathcal{E}$ into the moduli of Higgs bundle structures.

We give below several standard examples of Higgs bundles.

**Example 0.** An ordinary vector bundle is viewed as a Higgs bundle with trivial (zero)action of $\Theta_X$.

**Example 1.** Let $X$ be a complex manifold. Then

$$E_X^l = \bigoplus_{i=0}^{l} \text{Sym}^i \Theta_X$$

is a Higgs bundle, where the action of $\text{Sym}^j \Theta$ is defined by the standard multiplication

$$\text{Sym}^j \Theta \otimes \text{Sym}^i \Theta_X \to \begin{cases} \text{Sym}^{i+j} \Theta_X & \text{for } 0 \leq i + j \leq l \\ 0 & \text{for } i + j > l. \end{cases}$$

Given $m \leq l$, the sheaf

$$E_{X,m}^l = \bigoplus_{i=m}^{l} \text{Sym}^i \Theta_X$$

is a Higgs subbundle of $E_X^l$, and the quotient $E_X^l / E_{X,m}^l$ is isomorphic to $E_{X,m}^m$. (Actually a more natural definition of $E_X^l$ is the quotient $E_X^{\infty} / E_X^{l+1}$.)

If $K_X A > 0$ and $\Theta_X$ is $A$-semistable as an ordinary vector bundle [resp. If $K_X A \geq 0$ and $\Theta_X$ is $A$-semistable], then $E_X^l$ is an $A$-stable [resp. $A$-semistable] Higgs bundle. If $K_X$ is ample and $A = K_X$, then the Yau inequality [10]

$$( -1)^n c_2(X) c_1(X)^{n-2} \geq ( -1)^n \frac{\dim X - 1}{2 \dim X} c_1(X)^n$$

yields the Bogomolov inequality for the Higgs bundle $E_X^l$.

**Example 2.** Given a non-negative integer $l$, we define the Higgs bundle $F_X^l$ by

$$F_X^l = \bigoplus_{i=m}^{l} \text{Sym}^i \Omega_X^1,$$

where the action of $\text{Sym}^j \Theta$ is given by the contraction homomorphism

$$\text{Sym}^j \Theta \otimes \text{Sym}^i \Omega_X^1 \to \begin{cases} \text{Sym}^{i-j} \Omega_X^1 & \text{for } j \leq i \\ 0 & \text{for } j > i. \end{cases}$$

If $m \leq l$, then $F_X^{\infty} \subset F_X^l$ is naturally a Higgs subbundle. $F_X^l$ is an $A$-stable [resp. $A$-semistable] Higgs bundle if $K_X A > 0$ and $\text{Sym}^i \Omega_X^1$ is $A$-stable as an ordinary vector bundle [resp. if $K_X A \geq 0$ and $\Omega_X^1$ is $A$-semistable]. When, in addition, $A = K_X$ is ample, the Bogomolov inequality is satisfied by $F_X^l$. 


Example 3. Let $g: X \to Y$ be a morphism between complex manifolds and $E$ a Higgs bundle on $Y$. The natural homomorphism $\Theta_X \to g^* \Theta_Y$ defines a canonical Higgs bundle structure on $g^* E$.

Example 4. Given two Higgs bundles $E_i$, $i = 1, 2$, the tensor bundle $E_1 \otimes E_2$ is a Higgs bundle by defining $\theta(e_1 \otimes e_2) = \theta(e_1) \otimes e_2 + e_1 \otimes \theta(e_2)$, $\theta \in \Theta_X$, $e_i \in E_i$. (the tensor Higgs bundle). The dual bundle $E^\vee$ is a Higgs bundle (the dual Higgs bundle) by $\langle e, \theta(e^\vee) \rangle = -\langle e, \theta(e) \rangle$. Here $\theta \in \Theta_X$, $e \in E$, $e^\vee \in E^\vee$, while $\langle \cdot, \cdot \rangle$ stands for the canonical bilinear pairing. $F_X^t$ is the dual Higgs bundle of $E_X^t$, if we give a nondegenerate pairing between $\text{Sym}^m \Theta_X$ and $\text{Sym}^m \Omega_X^1$ by

$$\langle \theta_1 \otimes \cdots \otimes \theta_m | \omega_1 \otimes \cdots \omega_m \rangle = \frac{(-1)^m}{m!} \sum_{\sigma \in S_m} \prod_{i=1}^m \omega_i(\theta_{\sigma(i)}).$$

2. Hirzebruch’s Kummer covers $X^{(n)}$ attached to the complete quadrilateral line configuration

We briefly review Hirzebruch’s construction of Kummer covers of projective plane branching along a complete quadrilateral [4].

Take general four points $P_1, \ldots, P_4$ on projective plane $\mathbb{P}^2$, and let $L_{ij} = L_{ji}$ denote the line connecting $P_i$ and $P_j$ ($i \neq j$). The reduced divisor $D = \bigcup L_{ij}$ is the so-called complete quadrilateral consisting of six lines, and the $P_i$ are the triple points of $D$. The complete quadrilateral $D$ has extra three double points of the form $L_{i_1,i_2} \cap L_{j_1,j_2}$, where $\{i_1, i_2, j_1, j_2\} = \{1, 2, 3, 4\}$. Exactly three singular points of $D$ lies on each $L_{ij}$, two of which are triple points and one a double point. Thus the Euler number of the nonsingular locus of $D$ is $6 \times (2 - 3) = -6$, while that of $D$ is $-6 + 4 + 3 = 1$. Therefore the Euler number of the complement of $D$ is given by $e(\mathbb{X} \setminus D) = 3 - 1 = 2$.

Let $\mu: X \to \mathbb{P}^2$ be the blowing up at the four triple points $P_1, \ldots, P_4$ of $D$ and $E_i \subset X$ the exceptional curve over $P_i$. $X$ is a Del Pezzo surface of degree 5, with very ample anticanonical divisor $-K_X \sim 3H - \sum E_i$, where $H$ denotes the pullback of the hyperplane of $\mathbb{P}^2$. The effective divisor $\mu^* D$ is supported by a reduced effective divisor

$$\tilde{D} \sim \mu^* \sum L_{ij} - 2 \sum E_i \sim 6 \mu^* H - 2 \sum E_i \sim -2K_X$$

with only simple normal crossings. Each $E_i$ contains three singular points of $\tilde{D}$ so that $\tilde{D}$ has exactly $4 \times 3 + 3 = 15$ double points. If $\tilde{L}_{ij} \subset X$ denotes the strict transform of $L_{ij}$, we have

$$\# \tilde{L}_{ij} \cap \text{Sing}(\tilde{D}) = 3 = \# E_i \cap \text{Sing}(\tilde{D}).$$

Given a positive integer $n$, there exists a Kummer covering $\pi^{(n)}: X^{(n)} \to X$ of degree $n^5$ branching along $\tilde{D}$ (see Hirzebruch[4]). The function field of $X^{(n)}$ is simply obtained by adjoining the $n$-th roots $\sqrt[n]{l_{ij}/l_{kl}}$ ($i, j, k, l \in \{1, 2, 3, 4\}$) to $\mathbb{C}(\mathbb{P}^2)$, where $l_{ij}$ is a linear defining equation of the line $L_{ij}$. $X^{(n)}$ is a smooth projective surface and the local description of $X_n$ is quite simple: if $D$ is locally defined by the equation $x = 0$ or $zy = 0$, then $\pi_n^*: \mathcal{O}_X \to \mathcal{O}_{X_n}$ is given by $(x, y) \mapsto (t^n, y)$ or $(x, y) \mapsto (t^n, u^n)$, where $(x, y)$ and $(t, u)$ are local coordinates of $X$ and $X^{(n)}$. 

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In particular, the inverse image \((\pi^{(n)})^{-1}(p) \subset X^{(n)}\) of a closed point \(p \in X\) consists of \(n^5\) [resp. \(n^4, n^3\)] points when \(p \in X \setminus \bar{D}\) [resp. \(p \in \bar{D} \setminus \text{Sing}(\bar{D}), p \in \text{Sing}(\bar{D})\)]. The topological Euler number \(e(X^{(n)})\) of \(X^{(n)}\) is thus given by

\[
n^{-5}e(X^{(n)}) = e(X \setminus \bar{D}) + n^{-1}e(\bar{D} \setminus \text{Sing}(\bar{D})) + n^{-2}e((\text{Sing}(\bar{D}))
= 2 + n^{-1} \times (6 + 4) \times (2 - 3) + n^{-2} \times 15 = 2 - 10n^{-1} + 15n^{-2}.
\]

On the other hand, we calculate \(K_{X^{(n)}}\) as

\[
K_{X^{(n)}} \sim \pi^{(n)*}(K_X + (1 - n^{-1})\bar{D}) \sim (1 - 2n^{-1})\pi^{(n)*}(-K_X),
\]
and hence

\[
\frac{c_1(X^{(n)})^2}{c_1(O_{X^{(n)}})^2} = 5n^5(1 - 2n^{-1})^2.
\]

The surface \(X_n\) has ample canonical divisor if \(n \geq 3\).

When \(n = 5\), we have \(c_2(X_5) = 5^4 \times 9, c_2(X_5) = e(X_5) = 5^4 \times 3\), meaning that \(X_5\) is a surface of general type which attains the upper bound of the Miyaoka-Yau inequality \(K^2 \leq 3c_2\).

The Del Pezzo surface \(X\) carries five linear pencils \(|2H - \sum E_i|, |H - E_1|, \ldots, |H - E_4|\), defining five surjective morphisms from \(X\) onto \(\mathbb{P}^1\). Each of these morphisms has exactly three fibres contained in the branch locus \(\bar{D}\). For instance, for the morphism associated with \(|2H - \sum E_i|\), the three curves \(\tilde{L}_{12} + \tilde{L}_{34}, \tilde{L}_{13} + \tilde{L}_{24}\) and \(\tilde{L}_{14} + \tilde{L}_{23}\) are such fibres, and so are \(\tilde{L}_{1i} + E_i, i = 2, 3, 4\) for \(|H - E_1|\).

Upstairs on \(X^{(n)}\), there are thus five morphisms \(f_0^{(n)}, \ldots, f_4^{(n)}\) onto the curve \(C^{(n)}\), an \(n^2\)-sheeted Kummer cover of \(\mathbb{P}^1\) branching at three points, 0, 1 and \(\infty\), say. The pullback line bundle \(L_j^{(n)} = f_j^{(n)*}\omega_{C^{(n)}}\) is an invertible subsheaf of \(\Omega_{X^{(n)}}\).

We easily check that \(L_j^{(n)}\) is saturated in \(\Omega_{X^{(n)}}^1\) and that

\[
L_0^{(n)} \sim (1 - 3n^{-1})\pi^{(n)*}(2H - \sum E_i),
\]
\[
L_i^{(n)} \sim (1 - 3n^{-1})\pi^{(n)*}(H - E_i), \quad i = 1, \ldots, n.
\]

Ishida [6] determined the irregularity of \(X^{(n)}\) by showing that the natural map \(\bigoplus_j f_j^*H^0(C^{(n)}, \Omega_{C^{(n)}}^1) \to H^0(X^{(n)}, \Omega_{X^{(n)}}^1)\) is an isomorphism (for instance, \(q(X^{(5)}) = 5g(C^{(5)}) = 30\)).

In view of the definitions of \(X^{(n)}\) and \(C^{(n)}\), the family \((X^{(n)}, C^{(n)}, f_i^{(n)})\) form a partially ordered tower: there are natural Kummer covers \(\pi^{(mn)}_{(n)} : X^{(mn)} \to X^{(n)}\) of degree \(m^5\) and \(p^{(mn)}_{(n)} : C^{(mn)} \to C^{(n)}\) such that the diagram

\[
\begin{array}{ccc}
X^{(mn)} & \xrightarrow{\pi^{(mn)}_{(n)}} & X^{(n)} \\
\downarrow f_i^{(mn)} & & \downarrow f_i^{(n)} \\
C^{(mn)} & \xrightarrow{p^{(mn)}_{(n)}} & C^{(n)}
\end{array}
\]

commutes.
3. CONSTRUCTION OF A STABLE HIGGS 4-BUNDLE \( \mathcal{H} \)

Let the notation be as in the previous section.

We construct a Higgs 4-bundle \( \mathcal{H} \) on \( X^{(15)} \) as a subsheaf of \( \pi_{(5)}^{(15)}*F^2_{X^{(5)}} \) (see Section 1, Examples 2 and 3).

Let \( \mathcal{C}^{(n)}_{ij}, \mathcal{F}^{(n)}_k \subseteq X^{(n)} \) \( (1 \leq i < j \leq 4, 1 \leq k \leq 4) \) be the inverse images of \( \tilde{L}_{ij}, E_k \subseteq X \) with reduced scheme structures. Each of them is a union of \( n^3 \) copies of a curve isomorphic to \( C^{(n)} \). \( C^{(n)}_{ij} \) is contained in \( n^3 \) fibres of \( f^{(n)}_1, f^{(n)}_j \) and \( f^{(n)}_0 \), whereas it is union of sections of \( f^{(n)}_i \) for \( i \in \{1,2,3,4\} \setminus \{i,j\} \). Similarly, \( \mathcal{F}^{(n)}_i \) is a union of sections of \( f^{(n)}_i \), \( f^{(n)}_0 \) and contained in fibres of the other three projections to \( C^{(n)} \).

Let \( n = 3 \). Then \( C^{(3)} \) is an elliptic curve. Fix a basis \( \eta \) of \( H^0(C^{(3)}, \Omega^1_{C^{(3)}}) \cong \mathbb{C} \), and put \( \eta_i = f^{(3)*}_i \eta \in H^0(X^{(3)}, \Omega^1_{X^{(3)}}) \).

**Proposition 1.** The 1-forms \( \eta_0 \) and \( \omega = \eta_0 - \eta_1 - \eta_2 - \eta_3 - \eta_4 \) are non-zero and sit in the subsheaves

\[
\ker(\Omega^1_{X^{(3)}} \to \bigoplus_{ij} \Omega^1_{C^{(3)}_{ij}})
\]

and

\[
\ker(\Omega^1_{X^{(3)}} \to \bigoplus_{i=1}^4 \Omega^1_{F^{(3)}_i})
\]

respectively.

**Proof.** This immediately follows from the following two facts.

1. \( f_0 \) maps each \( \tilde{L}_{ij} \) to a single point on \( \mathbb{P}^1 \), or, equivalently, \( f^{(3)}_0 \) maps each \( C^{(3)}_{ij} \) to finitely many points of \( C^{(3)} \);
2. \( f_i : E_j \to \mathbb{P}^1 \) is either an isomorphism \((i = 0 \text{ or } i = o')\) or a constant map \((j \neq i = 1, 2, 3, 4) \) or, equivalently, \( f^{(3)}_i : F^{(3)}_j \to C^{(3)} \) restricted to each irreducible component is either an isomorphism or a constant map.

Indeed, viewed as 1-forms on \( C^{(3)}_{ij} \cong C^{(3)} \), \( \eta_0 = \eta_i = \eta_j = 0, \eta_l = \eta, l \neq 0, i, j, \) so that \( \eta_0|_{C^{(3)}_{ij}} = 0, \omega|_{C^{(3)}_{ij}} = -2\eta \neq 0 \). On \( F^{(3)}_i \), we have \( \eta_0 = \eta_i = \eta, \eta_j = 0, j \neq 0, i, \) so that \( \eta_0|_{F^{(3)}_i} = \eta \neq 0, \omega|_{F^{(3)}_i} = 0 \).

**Corollary 2.** In the same notation as above,

\[
\pi_{(3)}^{(3n)*}\eta_0 \in \Omega^1_{X^{(3n)}}(- (n - 1) \sum_{1 \leq i < j \leq 4} C^{(3n)}_{ij}),
\]

\[
\pi_{(3)}^{(3n)*}\omega \in \Omega^1_{X^{(3n)}}(- (n - 1) \sum_{i=1}^4 F^{(3n)}_i).
\]

**Proof.** Let \((x, y)\) be a local coordinate at a general point of \( C^{(3)}_{ij} \) such that \( C^{(3)}_{ij} \) is defined by \( x = 0 \). \( \pi^{(3n)}_{(3)} \) is then given by \((t, u) \mapsto (x, y) = (t^n, u) \). Proposition 1 asserts that \( \eta_0 \) is of the form \( \alpha dx + x \beta dy, \alpha, \beta \in \mathcal{O}, \) so that \( \pi_{(3)}^{(3n)*}\eta_0 \) is of the form \( nt^{n-1} \alpha dt + t^n \beta du \). The second statement for \( \omega \) is similarly proved.
$\Omega^1_{X^{(3n)}}$ contains $\pi^{(3n)\ast}_n \Omega^1_{X^{(n)}}$ as well as $\pi^{(3)\ast}_n \Omega^1_{X^{(3)}}$. At a general point $p$ of each component of the ramification locus, the former subsheaf is generated by $i^2 dt, du$, while the latter by $i^{n-1} dt, du$. In particular, when $n > 3$,

$\pi^{(3n)\ast}_n \Omega^1_{X^{(3)}} \subset \pi^{(3n)\ast}_n \Omega^1_{X^{(n)}} \subset \Omega^1_{X^{(3n)}}$.

The following assertion follows from the above local description of $\pi^{(3n)\ast}_n \Omega^1_{X^{(n)}}$ together with the proof of Corollary 2.

**Corollary 3.** Fix $n \geq 4$. Then

$$\pi^{(3n)\ast}_n \eta_0 \in (\pi^{(3n)\ast}_n \Omega^1_{X^{(n)}})(-(n - 3)(\sum_{1 \leq i < j \leq 4} C^{(3n)}_{ij} + \sum_{i=1}^4 F^{(3n)}_i)),$$

$$\pi^{(3n)\ast}_n \omega \in (\pi^{(3n)\ast}_n \Omega^1_{X^{(n)}})(-(n - 3)i=1^4 \sum_{i=1}^4 F^{(3n)}_i).$$

Hence

$$\pi^{(3n)\ast}_n \eta_0 \omega \in \pi^{(3n)\ast}_n \text{Sym}^2 \Omega^1_{X^{(n)}}(n - 3)(\sum_{1 \leq i < j \leq 4} C^{(3n)}_{ij} + \sum_{i=1}^4 F^{(3n)}_i))$$

$$= \pi^{(3n)\ast}_n \text{Sym}^2 \Omega^1_{X^{(n)}}(-2(n - 3)/3n) \pi^{(3)\ast}_n (3H - \sum F^{(3n)}_i))$$

$$= \pi^{(3n)\ast}_n \text{Sym}^2 \Omega^1_{X^{(n)}}(-2(n - 3)/(3n - 2) K_{X^{(n)}}).$$

Put $n = 5$. Then we get the following

**Proposition 4.** (1) $\pi^{(15)\ast}_5 \text{Sym}^2 \Omega^1_{X^{(5)}} \supset \mathcal{L} = \mathcal{O}((\frac{4}{5}) \pi^{(15)\ast}_5 K_{X^{(5)}})$.

(2) $\mathcal{H} = \mathcal{L} \oplus \pi^{(15)\ast}_5 \Omega^1_{X^{(5)}} \oplus \mathcal{O}$ is a Higgs subsheaf of

$$\pi^{(15)\ast}_5 \mathbf{F}^2_{X^{(5)}} = \pi^{(15)\ast}_5 (\text{Sym}^2 \Omega^1_{X^{(5)}} \oplus \Omega^1_{X^{(5)}} \oplus \mathcal{O}_{X^{(5)}})).$$

(3) $c_1(\mathcal{H}) = 13^9 \pi^{(15)\ast}_5 K_{X^{(5)}}$, $c_2(\mathcal{H}) = \pi^{(15)\ast}_5 (c_2(\Omega^1_{X^{(5)}}) + \frac{4}{3} K^2_{X^{(5)}})^2 = \frac{7}{9} \pi^{(15)\ast}_5 K^2_{X^{(5)}}$, so that

$$\frac{c_1^2(\mathcal{H})}{c_2(\mathcal{H})} = \frac{169}{63} = \frac{8}{3} \times \frac{169}{168} > \frac{8}{3}.$$  

(4) The Higgs 4-bundle $\mathcal{H}$ is $\pi^{(15)\ast}_5 K_{X^{(5)}}$-stable.

**Proof.** Corollary 3 is rephrased into (1), which in turn yields (2). (3) follows from direct computation. In order to show (4), we check that the average degree of a saturated Higgs subsheaf of $\mathcal{H}$ is strictly smaller than $(13/36) \pi^{(15)\ast}_5 K^2_{X^{(5)}}$, the average degree of $\mathcal{H}$. At a general point $q$ of $C^{(3)}_{ij}$, the product $\eta_0 \omega$ is of the form $\alpha dx dy$, where $\alpha \in \mathcal{O}^\times$ and $(x, y)$ is a local coordinate. Hence $\Theta_{X^{(3)}} \mathcal{O} \eta_0 \omega = \Omega^1_{X^{(3)}}$ around $q$. Then it is obvious that, at a general point $p \in X^{(15)}$, $\Theta_{X^{(15)}} \mathcal{L}_p = (\pi^{(15)\ast}_5 \Theta_{X^{(5)}}) \mathcal{L}_p = (\pi^{(15)\ast}_5 \Theta_{X^{(5)}}) \mathcal{L}_p = (\pi^{(15)\ast}_5 \Theta_{X^{(5)}}) \mathcal{L}_p = (\pi^{(15)\ast}_5 \Theta_{X^{(5)}}) \mathcal{L}_p = (\pi^{(15)\ast}_5 \Theta_{X^{(5)}}) \mathcal{L}_p$. This shows that a proper Higgs subsheaf of $\mathcal{H}$ must be contained in $\pi^{(15)\ast}_5 \mathbf{F}^1_{X^{(5)}} = \pi^{(15)\ast}_5 (\Omega^1_{X^{(5)}} \oplus \mathcal{O}_{X^{(5)}})$, and the assertion follows from the semistability of $\mathbf{F}^1_{X^{(5)}}$ of average degree $(1/3) \pi^{(15)\ast}_5 K^2_{X^{(5)}}$.  

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4. Stable Higgs 4-bundles with vanishing Chern classes

Starting from the 4-bundle $\mathcal{H}$ described in the previous section, we can construct many stable Higgs 4-bundles with trivial Chern classes.

Recall that $\mathcal{H}$ is the direct sum $\mathcal{L} \oplus \pi^{(15)}_*(\Omega^1_{X(5)} \oplus \mathcal{O})$. Take line bundles $\mathcal{L}_1, \mathcal{L}_2$ such that $\mathcal{L}_1 \subset \mathcal{L}, \mathcal{L}_2 \supset \mathcal{O}$. Then $\mathcal{H}' = \mathcal{L}_1 \oplus \pi^{(15)}_*(\Omega^1_{X(5)} \oplus \mathcal{L}_2$ is naturally a Higgs bundle. $\mathcal{H}'$ is stable if

1. $c_1(\mathcal{L}_1)\pi^{(15)}_*(K_{X(5)}) > \frac{1}{2}(\pi^{(15)}_*(K_{X(5)} + c_1(\mathcal{L}_2)))\pi^{(15)}_*(K_{X(5)})$, and
2. $c_1(\mathcal{L}_2)\pi^{(15)}_*(K_{X(5)}) < \frac{1}{2}(\pi^{(15)}_*(K_{X(5)}))^2$.

Choose $\mathcal{L}_i$ to be of the form $\mathcal{O}(t_i\pi^{(15)}_*(K_{X(5)}))$, where $t_i \in \mathbb{Q}$, $t_1 < 4/9, t_2 > 0$. Such $\mathcal{L}_i$'s make sense if we replace $X^{(5)}$ by a suitable ramified cover $Y = X^{(51)}$, where $l$ is a sufficiently divisible positive integer. Thus we consider the vector bundle $\mathcal{H}' = \mathcal{L}_1 \oplus \pi^{(5)}_*(\Omega^1_{X(5)} \oplus \mathcal{L}_2$ on $Y$, where $\pi^{(5)}_Y : Y \to X^{(5)}$ is the projection.

The Chern classes of the vector bundle $\mathcal{H}'$ are:

$$c_1(\mathcal{H}') = (t_1 + t_2 + 1)\pi^{(5)}_*(K_{X(5)}),$$
$$c_2(\mathcal{H}') = \left(t_1t_2 + t_1 + t_2 + \frac{1}{3}\right)\left(\pi^{(5)}_*(K_{X(5)})\right)^2.$$

Thus the condition

$$3c_1^2(\mathcal{H}') = 8c_2(\mathcal{H}')$$

is given by the quadratic equation

$$3t_1^2 - 2t_1t_2 + 3t_2^2 - 2t_1 - 2t_2 + \frac{1}{3} = 0,$$

a solution of which is $(t_1, t_2) = (1/3, 0)$. Hence there are infinitely many rational solutions of the quadratic equation, and the stability condition is a non-empty open condition on those solutions (the rational point $(1/3, 0)$ lies on the boundary of the region given by the stability condition (1) and in the interior of the one given by (2)). For instance,

$$(t_1, t_2) = \left(\frac{25}{3 \cdot 19}, \frac{2}{3^2 \cdot 19}\right)$$

is a solution with

$$(c_1(\mathcal{H}'), c_2(\mathcal{H}')) = \left(\frac{8 \cdot 31}{9 \cdot 19}\pi^{(5)}_*(K_{X(5)}), \frac{3 \cdot 8 \cdot 31^2}{9^2 \cdot 19^2}(\pi^{(5)}_*(K_{X(5)}))^2\right).$$

We thus conclude that there are 4-bundles $\mathcal{H}'$ such that the normalized bundles $\mathcal{G} = \mathcal{H}'(-\frac{1}{4}c_1(\mathcal{H}'))$ are stable Higgs 4-bundles with trivial Chern classes. By a theorem of Simpson [9], $\mathcal{G}$ is a flat vector bundle induced by an irreducible representation $\pi_1(Y) \to \text{SL}(4)$.

On $Y = X^{(51)}$, the (integral) divisor $(\frac{2}{3} - t_1)(\pi^{(5)}_*(K_{X(5)}))$ is linearly equivalent to a sum of the fibres $f^*_i\pi^{(5)}_a$, where $f^*_i : Y \to C^{(51)}$ is the projection and $\pi_a \in C^{(51)}$ (indeed, $-2K_X \sim \sum_{i=0}^4 f^*_i\mathcal{O}_{P^1}(1)$ on the Del Pezzo surface $X$, and the divisor in
question is a rational multiple of the pullback of $-2K_X$). If we replace $p_{i\alpha}$ by another point $q_{i\alpha} \sim p_{i\alpha} + \tau_{i\alpha}, \tau_{i\alpha} \in \text{Pic}^0(C^{(15)})$, we get an effective invertible sheaf

$$\mathcal{O}_Y \left( \sum_{i,\alpha} f_i^Y \cdot q_{i\alpha} \right) \simeq \mathcal{O}_Y \left( \left( \frac{4}{9} - t_1 \right) \pi_Y \cdot K_X + \sum_{i,\alpha} f_i^Y \cdot \tau_{i\alpha} \right) = \mathcal{L} \otimes \mathcal{L}^{-1} \left( \sum_{i} f_i^Y \cdot \tau_{i\alpha} \right).$$

This isomorphism induces an injection

$$\mathcal{L}_1(-\tau) = \mathcal{L}_1(-\sum_{i,\alpha} f_i^Y \cdot \tau_{i\alpha}) \hookrightarrow \mathcal{L} \subset \pi_Y \cdot \text{Sym}^2 \Omega^1_X(1),$$

where $\tau = \sum f_i^Y \cdot \tau_{i\alpha}$ nontrivially moves in Pic$^0(Y)$. Putting

$$\mathcal{H}_\tau' = \mathcal{L}_1(-\tau) \oplus \pi_Y(15) \cdot \Omega^1_X(5) \oplus \mathcal{L}_2, \quad \mathcal{G}_\tau = \mathcal{H}_\tau'(-\frac{1}{4}c_1(\mathcal{H}_\tau')),$$

we obtain a deforming family of stable Higgs bundles $\mathcal{G}_\tau$ with $c_1 = 0 \in \text{Pic}(Y)$ and $c_2 = 0 \in \text{H}^4(Y, \mathbb{Z})$. By Simpson's theorem [9], it gives rise to a deformation of irreducible representations $\pi_1(Y) \rightarrow \text{SL}(4)$ parametrized by a product of several copies of $C^{(15)}$.

REFERENCES


