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AN EXAMPLE OF STABLE HIGGS BUNDLES WHICH DO NOT SATISFY THE BOGOMOLOV INEQUALITY

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INTRODUCTION

The Bogomolov inequality for semistable vector bundles on smooth complex projective n-folds reads

$$c_2(\mathcal{E})A^{n-2} \geq \frac{r-1}{2r}c_1(\mathcal{E})^2A^{n-2},$$

where A is an ample divisor and $\mathcal{E}$ is an A-semistable vector bundle of rank r. In case $\mathcal{E}$ is A-stable with vanishing $c_1(\mathcal{E})$, the lower bound of this inequality $c_2(\mathcal{E})A^{n-2} \geq 0$ is attained if and only if $\mathcal{E}$ is a flat hermitian bundle associated with an irreducible unitary representation of the fundamental group $\pi_1(X)$, thereby establishing the one-to-one *Kobayashi-Hitchin correspondence* between the stable bundles with vanishing Chern classes and the irreducible unitary representations of $\pi_1(X)$ [2]. The inequality is natural enough to have proofs by several different approaches (geometric invariant theory [1]; characteristic p method [3]; the theory of effective cones on ruled surfaces [8]; differential geometry [2]) and generalizes to bigger classes of semistable bundles, including orbibundles and parabolic bundles.

Another important class of generalised vector bundles is that of Higgs bundles (see [9]), and it is a natural question to ask if the Bogomolov inequality extends also to this class. The inequality is indeed true for standard types of Higgs bundles listed in Section 1 as Examples 0, 1, 2, and for bundles of small ranks 2, 3 as well [7]. Unfortunately, however, this is not the case for Higgs bundles of higher rank. In this note, we construct stable Higgs 4-bundles on surfaces of general type for which the inequality breaks down (Proposition 4 in Section 3). Starting from this examaple, we also find a nontrivially deforming families of stable Higgs 4-bundles with trivial Chern classes or, equivalently, non-trival deformations of irreducible SL(4)-representations of the fundamental group (Section 4).

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1. HIGGS BUNDLES: DEFINITION AND STANDARD EXAMPLES

Let $\mathcal{E}$ be a vector bundle on a complex manifold X and $\theta\colon \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$ an $\mathcal{O}_X$-linear mapping. The pair $(\mathcal{E}, \theta)$ is said to be a *Higgs bundle* if the natural composite map $\theta \wedge \theta\colon \mathcal{E} \to \Omega^2_X \otimes \mathcal{E}$ identically vanishes. Alternatively, $\mathcal{E}$ is a Higgs bundle if an $\mathcal{O}_X$-linear action of the sheaf of the local vector fields Θ_X on $\mathcal{E}$ is given in such a way that $\xi_1(\xi_2(e)) = \xi_2(\xi_1(e)), \forall \xi_i \in \Theta_X, \forall e \in \mathcal{E}$. In other words, a Higgs bundles is a vector bundle equipped with a $\operatorname{Sym}\Theta_X$-module structure, where $\operatorname{Sym}\Theta_X = \bigoplus_{i=0}^{\infty} \operatorname{Sym}^i \Theta_X$ is the symmetric tensor algebra generated by Θ_X.

Higgs subsheaves are, by definition, $\operatorname{Sym}\Theta_X$-submodules. Given an ample divisor A on X, the notion of A-(semi)stability of Higgs bundles is naturally defined.

Historically, Higgs structures were introduced as the moduli of flat connections [5]. Let $\mathcal{E}$ be a vector bundle with an integrable connection $\nabla_0 \colon \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$. Given another integrable connection ∇, the difference $\theta = \nabla - \nabla_0 \colon \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$ obviously gives a Higgs bundle structure and this correspondence translates the moduli of the flat connections on $\mathcal{E}$ into the moduli of Higgs bundle structures.

We give below several standard examples of Higgs bundles.

Example 0. An ordinary vector bundle is viewed as a Higgs bundle with trivial (zero)action of Θ_X.

Example 1. Let X be a complex manifold. Then

$$\mathbf{E}^l_X = \bigoplus_{i=0}^{l} \operatorname{Sym}^i \Theta_X$$

is a Higgs bundle, where the action of $\operatorname{Sym}^j \Theta$ is defined by the standard multiplication

$$\operatorname{Sym}^j \Theta \otimes \operatorname{Sym}^i \Theta_X \to \begin{cases} \operatorname{Sym}^{i+j} \Theta_X, & \text{for } 0 \le i+j \le l \\ 0, & \text{for } i+j > l. \end{cases}$$

Given $m \le l$, the sheaf

$$\mathbf{E}^{l,m}_X = \bigoplus_{i=m}^{l} \operatorname{Sym}^i \Theta_X$$

is a Higgs subbundle of $\mathbf{E}^l_X$, and the quotient $\mathbf{E}^l_X / \mathbf{E}^{l,m}_X$ is isomorphic to $\mathbf{E}^m_X$. (Actually a more natural definition of $\mathbf{E}^l_X$ is the quotient $\mathbf{E}^\infty_X / \mathbf{E}^{\infty,l}_X$.)

If $K_X A > 0$ and Θ_X is A-semistable as an ordinary vector bundle [resp. If $K_X A \ge 0$ and Θ_X is A-semistable], then $\mathbf{E}^l_X$ is an A-stable [resp. A-semistable] Higgs bundle. If K_X is ample and $A = K_X$, then the Yau inequality [10]

$$(-1)^n c_2(X) c_1(X)^{n-2} \ge (-1)^n \frac{\dim X - 1}{2 \dim X} c_1(X)^n$$

yields the Bogomolov inequality for the Higgs bundle $\mathbf{E}^l_X$.

Example 2. Given a non-negative integer l, we define the Higgs bundle $\mathbf{F}^l_X$ by

$$\mathbf{F}^l_X = \bigoplus_{i=m}^{l} \operatorname{Sym}^i \Omega^1_X,$$

where the action of $\operatorname{Sym}^j \Theta$ is given by the contraction homomorphism

$$\operatorname{Sym}^j \Theta \otimes \operatorname{Sym}^i \Omega^1_X \to \begin{cases} \operatorname{Sym}^{i-j} \Omega^1_X, & \text{for } j \le i \\ 0, & \text{for } j > i. \end{cases}$$

If $m \le l$, then $\mathbf{F}^m_X \subset \mathbf{F}^l_X$ is naturally a Higgs subbundle. $\mathbf{F}^l_X$ is an A-stable [resp. A-semistable] Higgs bundle if $K_X A > 0$ and $\operatorname{Sym}^l \Omega^1_X$ is A-stable as an ordinary vector bundle [resp. if $K_X A \ge 0$ and Ω^1_X is A-semistable]. When, in addition, $A = K_X$ is ample, the Bogomolov inequality is satisfied by $\mathbf{F}^l_X$.

Example 3. Let $g\colon X \to Y$ be a morphism between complex manifolds and $\mathcal{E}$ a Higgs bundle on Y. The natural homomorphism $\Theta_X \to g^*\Theta_Y$ defines a canonical Higgs bundle structure on $g^*\mathcal{E}$.

Example 4. Given two Higgs bundles $\mathcal{E}_i$, $i = 1, 2$, the tensor bundle $\mathcal{E}_1 \otimes \mathcal{E}_2$ is a Higgs bundle by defining $\theta(e_1 \otimes e_2) = \theta(e_1) \otimes e_2 + e_1 \otimes \theta(e_2)$, $\theta \in \Theta_X$, $e_i \in E_i$. (the tensor Higgs bundle). The dual bundle $\mathcal{E}^\vee$ is a Higgs bundle (the dual Higgs bundle) by $\langle e|\theta(e^\vee)\rangle = -\langle\theta(e)|e^\vee\rangle$. Here $\theta \in \Theta_X$, $e \in \mathcal{E}$, $e^\vee \in \mathcal{E}^\vee$, while $\langle\cdot|\cdot\rangle$ stands for the canonical bilinear pairing. $\mathbf{F}^l_X$ is the dual Higgs bundle of $\mathbf{E}^l_X$, if we give a nondegenerate pairing between $\mathrm{Sym}^m\Theta_X$ and $\mathrm{Sym}^m\Omega^1_X$ by

$$\langle\theta_1 \otimes \cdots \otimes \theta_m|\omega_1 \otimes \cdots \omega_m\rangle = \frac{(-1)^m}{m!} \sum_{\sigma\in\mathfrak{S}_m} \prod_{i=1}^m \omega_i(\theta_{\sigma(i)}).$$

2. Hirzebruch's Kummer covers $X^{(n)}$ attached to the complete quadrilateral line configuration

We briefly review Hirzebruch's construction of Kummer covers of projective plane branching along a complete quadrilateral [4].

Take general four points $P_1, \ldots, P_4$ on projective plane $\mathbb{P}^2$, and let $L_{ij} = L_{ji}$ denote the line connecting P_i and P_j $(i \neq j)$. The reduced divisor $D = \bigcup L_{ij}$ is the so-called complete quadrilateral consisting of six lines, and the P_i are the triple points of D. The complete quadrilateral D has extra three double points of the form $L_{i_1,i_2} \cap L_{j_1,j_2}$, where $\{i_1, i_2, j_1, j_2\} = \{1, 2, 3, 4\}$. Exactly three singular points of D lies on each L_{ij}, two of which are triple points and one a double point. Thus the Euler number of the nonsingular locus of D is $6 \times (2 - 3) = -6$, while that of D is $-6 + 4 + 3 = 1$. Therefore the Euler number of the complement of D is given by $\mathrm{e}(X \setminus D) = 3 - 1 = 2$.

Let $\mu\colon X \to \mathbb{P}^2$ be the blowing up at the four triple points $P_1, \ldots, P_4$ of D and $E_i \subset X$ the exceptional curve over P_i. X is a Del Pezzo surface of degree 5, with very ample anticanonical divisor $-K_X \sim 3H - \sum E_i$, where H denotes the pullback of the hyperplane of $\mathbb{P}^2$. The effective divisor μ^*D is supported by a reduced effective divisor

$$\tilde{D} \sim \mu^* \sum L_{ij} - 2\sum E_i \sim 6\mu^*H - 2\sum E_i \sim -2K_X$$

with only simple normal crossings. Each E_i contains three singular points of $\tilde{D}$ so that $\tilde{D}$ has exactly $4 \times 3 + 3 = 15$ double points. If $\tilde{L}_{ij} \subset X$ denotes the strict transform of L_{ij}, we have

$$\#\tilde{L}_{ij} \cap \mathrm{Sing}(\tilde{D}) = 3 = \#E_i \cap \mathrm{Sing}(\tilde{D}).$$

Given a positive integer n, there exists a Kummer covering $\pi^{(n)}\colon X^{(n)} \to X$ of degree n^5 branching along $\tilde{D}$ (see Hirzebruch[4]). The function field of $X^{(n)}$ is simply obtained by adjoining the n-th roots $\sqrt[n]{l_{ij}/l_{kl}}$ $(i, j, k, l \in \{1, 2, 3, 4\})$ to $\mathbb{C}(\mathbb{P}^2)$, where l_{ij} is a linear defining equation of the line L_{ij}. $X^{(n)}$ is a smooth projective surface and the local description of X_n is quite simple: if $\tilde{D}$ is locally defined by the equation $x = 0$ or $xy = 0$, then $\pi_n^*\colon \mathcal{O}_X \to \mathcal{O}_{X_n}$ is given by $(x, y) \mapsto (t^n, y)$ or $(x, y) \mapsto (t^n, u^n)$, where (x, y) and (t, u) are local coordinates of X and $X^{(n)}$.

In particular, the inverse image $(\pi^{(n)})^{-1}(p) \subset X^{(n)}$ of a closed point $p \in X$ consists of n^5 [resp. n^4, n^3] points when $p \in X \setminus \tilde{D}$ [resp. $p \in \tilde{D} \setminus \mathrm{Sing}(\tilde{D})$, $p \in \mathrm{Sing}(\tilde{D})$]. The topological Euler number $\mathrm{e}(X^{(n)})$ of $X^{(n)}$ is thus given by

$$\begin{aligned} n^{-5}\mathrm{e}(X^{(n)}) &= \mathrm{e}(X \setminus \tilde{D}) + n^{-1}\mathrm{e}(\tilde{D} \setminus \mathrm{Sing}(\tilde{D})) + n^{-2}\mathrm{e}((\mathrm{Sing}(\tilde{D})) \\ &= 2 + n^{-1} \times (6+4) \times (2-3) + n^{-2} \times 15 = 2 - 10n^{-1} + 15n^{-2}. \end{aligned}$$

On the other hand, we calculate $K_{X^{(n)}}$ as

$$K_{X^{(n)}} \sim \pi^{(n)*}(K_X + (1-n^{-1})\tilde{D}) \sim (1-2n^{-1})\pi^{(n)*}(-K_X),$$

and hence

$$c_1(X^{(n)})^2 = c_1(\Omega^1_{X^{(n)}})^2 = 5n^5(1-2n^{-1})^2.$$

The surface X_n has ample canonical divisor if $n \geq 3$.

When $n = 5$, we have $c_1^2(X_5) = 5^4 \times 9$, $c_2(X_5) = e(X_5) = 5^4 \times 3$, meaning that X_5 is a surface of general type which attains the upper bound of the Miyaoka-Yau inequality $K^2 \leq 3c_2$.

The Del Pezzo surface X carries five linear pencils $|2H - \sum E_i|$, $|H - E_1|$, ... , $|H - E_4|$, defining five surjective morphisms from X onto $\mathbb{P}^1$. Each of these morphisms has exactly three fibres contained in the branch locus $\tilde{D}$. For instance, for the morphism associated with $|2H - \sum E_i|$, the three curves $\tilde{L}_{12} + \tilde{L}_{34}$, $\tilde{L}_{13} + \tilde{L}_{24}$ and $\tilde{L}_{14} + \tilde{L}_{23}$ are such fibres, and so are $\tilde{L}_{1i} + E_i$, $i = 2, 3, 4$ for $|H - E_1|$.

Upstairs on $X^{(n)}$, there are thus five morphisms $f_0^{(n)}, \dots, f_4^{(n)}$ onto the curve $C^{(n)}$, an n^2-sheeted Kummer cover of $\mathbb{P}^1$ branching at three points, 0, 1 and ∞, say. The pullback line bundle $\mathcal{L}_j^{(n)} = f_j^{(n)*}\omega_{C^{(n)}}$ is an invertible subsheaf of $\Omega^1_{X^{(n)}}$. We easily check that $\mathcal{L}_j^{(n)}$ is saturated in $\Omega^1_{X^{(n)}}$ and that

$$\begin{aligned} \mathcal{L}_0^{(n)} &\sim (1-3n^{-1})\pi^{(n)*}(2H - \textstyle\sum E_i), \\ \mathcal{L}_i^{(n)} &\sim (1-3n^{-1})\pi^{(n)*}(H - E_i), \quad i = 1, \dots, n. \end{aligned}$$

Ishida [6] determined the irregularity of $X^{(n)}$ by showing that the natural map $\bigoplus_j f_j^* \mathrm{H}^0(C^{(n)}, \Omega^1_{C^{(n)}}) \to \mathrm{H}^0(X^{(n)}\Omega^1_{X^{(n)}})$ is an isomorphism (for instance, $q(X^{(5)}) = 5g(C^{(5)}) = 30$).

In view of the definitions of $X^{(n)}$ and $C^{(n)}$, the family $(X^{(n)}, C^{(n)}, f_i^{(n)})$ form a partially ordered tower: there are natural Kummer covers $\pi^{(mn)}_{(n)} : X^{(mn)} \to X^{(n)}$ of degree m^5 and $p^{(mn)}_{(n)} : C^{(mn)} \to C^{(n)}$ such that the diagram

$$\begin{array}{ccc} X^{(mn)} & \xrightarrow{\pi^{(mn)}_{(n)}} & X^{(n)} \\ {\scriptstyle f_i^{(mn)}}\downarrow & & \downarrow{\scriptstyle f_i^{(n)}} \\ C^{(mn)} & \xrightarrow{p^{(mn)}_{(n)}} & C^{(n)} \end{array}$$

commutes.

3. Construction of a stable Higgs 4-bundle $\mathcal{H}$

Let the notation be as in the previous section.

We construct a Higgs 4-bundle $\mathcal{H}$ on $X^{(15)}$ as a subsheaf of $\pi_{(5)}^{(15)*}\mathbf{F}^2_{X^{(5)}}$ (see Section 1, Examples 2 and 3).

Let $C_{ij}^{(n)}, F_k^{(n)} \subset X^{(n)}$ ($1 \le i < j \le 4$, $1 \le k \le 4$) be the inverse images of $\tilde{L}_{ij}, E_k \subset X$ with reduced scheme structures. Each of them is a union of n^3 copies of a curve isomorphic to $C^{(n)}$. $C_{ij}^{(n)}$ is contained in n^3 fibres of $f_i^{(n)}, f_j^{(n)}$ and $f_0^{(n)}$, whereas it is union of sections of $f_l^{(n)}$ for $l \in \{1,2,3,4\} \setminus \{i,j\}$. Similarly, $F_i^{(n)}$ is a union of sections of $f_i^{(n)}$, $f_0^{(n)}$ and contained in fibres of the other three projections to $C^{(n)}$.

Let $n=3$. Then $C^{(3)}$ is an elliptic curve. Fix a basis η of $H^0(C^{(3)}, \Omega^1_{C^{(3)}}) \simeq \mathbb{C}$, and put $\eta_i = f_i^{(3)*}\eta \in H^0(X^{(3)}, \Omega^1_{X^{(3)}})$.

Proposition 1. *The 1-forms η_0 and $\omega = \eta_0 - \eta_1 - \eta_2 - \eta_3 - \eta_4$ are non-zero and sit in the subsheaves*

$$\operatorname{Ker}(\Omega^1_{X^{(3)}} \to \bigoplus_{ij} \Omega^1_{C_{ij}^{(3)}})$$

and

$$\operatorname{Ker}(\Omega^1_{X^{(3)}} \to \bigoplus_{i=1}^{4} \Omega^1_{F_i^{(3)}}),$$

respectively.

Proof. This immediately follows from the following two facts.

1. (1) f_0 maps each $\tilde{L}_{ij}$ to a single point on $\mathbb{P}^1$ or, equivalently, $f_0^{(3)}$ maps each $C_{ij}^{(3)}$ to finitely many points of $C^{(3)}$;
2. (2) $f_i : E_j \to \mathbb{P}^1$ is either an isomorphism ($i=0$ or $i=j$) or a constant map ($j \ne i = 1,2,3,4$) or, equivalently, $f_i^{(3)} : F_j^{(3)} \to C^{(3)}$ restricted to each irreducible component is either an isomorphism or a constant map.

Indeed, viewed as 1-forms on $C_{ij}^{(3)} \simeq C^{(3)}$, $\eta_0 = \eta_i = \eta_j = 0$, $\eta_l = \eta$, $l \ne 0, i, j$, so that $\eta_0|_{C_{ij}^{(3)}} = 0$, $\omega|_{C_{ij}^{(3)}} = -2\eta \ne 0$. On $F_i^{(3)}$, we have $\eta_0 = \eta_i = \eta$, $\eta_j = 0$, $j \ne 0, i$, so that $\eta_0|_{F_i^{(3)}} = \eta \ne 0$, $\omega|_{F_i^{(3)}} = 0$.

Corollary 2. *In the same notation as above,*

$$\pi_{(3)}^{(3n)*}\eta_0 \in \Omega^1_{X^{(3n)}}(-(n-1)\sum_{1\le i<j\le 4} C_{ij}^{(3n)}),$$

$$\pi_{(3)}^{(3n)*}\omega \in \Omega^1_{X^{(3n)}}(-(n-1)\sum_{i=1}^{4} F_i^{(3n)}).$$

Proof. Let (x,y) be a local coordinate at a general point of $C_{ij}^{(3)}$ such that $C_{ij}^{(3)}$ is defined by $x=0$. $\pi_{(3)}^{(3n)}$ is then given by $(t,u) \mapsto (x,y) = (t^n, u)$. Proposition 1 asserts that η_0 is of the form $\alpha dx + x\beta dy$, $\alpha, \beta \in \mathcal{O}$, so that $\pi_{(3)}^{(3n)*}\eta_0$ is of the form $nt^{n-1}\alpha dt + t^n \beta du$. The second statement for ω is similarly proved.

$\Omega^1_{X^{(3n)}}$ contains $\pi_{(n)}^{(3n)*}\Omega^1_{X^{(n)}}$ as well as $\pi_{(3)}^{(3n)*}\Omega^1_{X^{(3)}}$. At a general point p of each component of the ramification locus, the former subsheaf is generated by t^2dt, du, while the latter by $t^{n-1}dt, du$. In particular, when $n > 3$,

$$\pi_{(3)}^{(3n)*}\Omega^1_{X^{(3)}} \subset \pi_{(n)}^{(3n)*}\Omega^1_{X^{(n)}} \subset \Omega^1_{X^{(3n)}}.$$

The following assertion follows from the above local description of $\pi_{(n)}^{(3n)*}\Omega^1_{X^{(n)}}$ together with the proof of Corollary 2.

Corollary 3. *Fix $n \geq 4$. Then*

$$\pi_{(3)}^{(3n)*}\eta_0 \in (\pi_{(n)}^{(3n)*}\Omega^1_{X^{(n)}})(-(n-3)\sum_{1\leq i<j\leq 4} C_{ij}^{(3n)}),$$

$$\pi_{(3)}^{(3n)*}\omega \in (\pi_{(n)}^{(3n)*}\Omega^1_{X^{(n)}})(-(n-3)\sum_{i=1}^{4} F_i^{(3n)}).$$

Hence

$$\begin{aligned}\pi_{(3)}^{(3n)*}\eta_0\omega &\in \pi_{(n)}^{(3n)*}\mathrm{Sym}^2\Omega^1_{X^{(n)}}(-(n-3)(\sum_{1\leq i<j\leq 4} C_{ij}^{(3n)} + \sum_{i=1}^{4} F_i^{(3n)}))\\ &= (\pi_{(n)}^{(3n)*}\mathrm{Sym}^2\Omega^1_{X^{(n)}})(-\frac{2(n-3)}{3n}\pi^{(3n)*}(3H - \sum E_i))\\ &= \pi_{(n)}^{(3n)*}\mathrm{Sym}^2\Omega^1_{X^{(n)}}(-\frac{2(n-3)}{3(n-2)}K_{X^{(n)}}).\end{aligned}$$

Put $n = 5$. Then we get the following

Proposition 4. (1) $\pi_{(5)}^{(15)*}\mathrm{Sym}^2\Omega^1_{X^{(5)}} \supset \mathcal{L} = \mathcal{O}(\frac{4}{9}\pi_{(5)}^{(15)*}K_{X^{(5)}})$.

(2) $\mathcal{H} = \mathcal{L} \oplus \pi_{(5)}^{(15)*}\Omega^1_{X^{(5)}} \oplus \mathcal{O}$ *is a Higgs subsheaf of*

$$\pi_{(5)}^{(15)*}\mathbf{F}^2_{X^{(5)}} = \pi_{(5)}^{(15)*}(\mathrm{Sym}^2\Omega^1_{X^{(5)}} \oplus \Omega^1_{X^{(5)}} \oplus \mathcal{O}_{X^{(5)}}).$$

(3) $c_1(\mathcal{H}) = \frac{13}{9}\pi_{(5)}^{(15)*}K_{X^{(5)}}$, $c_2(\mathcal{H}) = \pi_{(5)}^{(15)*}(c_2(\Omega^1_{X^{(5)}})+\frac{4}{9}K^2_{X^{(5)}})^2 = \frac{7}{9}\pi_{(5)}^{(15)*}K^2_{X^{(5)}}$, *so that*

$$\frac{c_1^2(\mathcal{H})}{c_2(\mathcal{H})} = \frac{169}{63} = \frac{8}{3} \times \frac{169}{168} > \frac{8}{3}.$$

(4) *The Higgs 4-bundle $\mathcal{H}$ is $\pi_{(5)}^{(15)*}K_{X^{(5)}}$-stable.*

Proof. Corollary 3 is rephrased into (1), which in turn yields (2). (3) follows from direct computation. In order to show (4), we check that the avarage degree of a saturated Higgs subsheaf of $\mathcal{H}$ is strictly smaller than $(13/36)\pi_{(5)}^{(15)*}K^2_{X^{(5)}}$, the average degree of $\mathcal{H}$. At a general point q of $C_{ij}^{(3)}$, the product $\eta_0\omega$ is of the form $\alpha dxdy$, where $\alpha \in \mathcal{O}^\times$ and (x, y) is a local coordinate. Hence $\Theta_{X^{(3)}}\mathcal{O}\eta_0\omega = \Omega^1_{X^{(3)}}$ around q. Then it is obvious that, at a general point $p \in X^{(15)}$, $\Theta_{X^{(15)},p} = (\pi_{(5)}^{(15)*}\Theta_{X^{(5)}})_p = (\pi^{(15)*}\Theta_X)_p$, $\Theta_{X^{(15)}}\mathcal{L}_p = \pi_{(5)}^{(15)*}\Omega^1_{X^{(5)},p}$. This shows that a proper Higgs subsheaf of $\mathcal{H}$ must be contained in $\pi_{(5)}^{(15)*}\mathbf{F}^1_{X^{(5)}} = \pi_{(5)}^{(15)*}(\Omega^1_{X^{(5)}} \oplus \mathcal{O}_{X^{(5)}})$, and the assertion follows from the semistability of $\mathbf{F}^1_{X^{(5)}}$ of average degree $(1/3)\pi_{(5)}^{(15)*}K^2_{X^{(5)}}$.

4. Stable Higgs 4-bundles with vanishing Chern classes

Starting from the 4-bundle $\mathcal{H}$ described in the previous section, we can construct many stable Higgs 4-bundles with trivial Chern classes.

Recall that $\mathcal{H}$ is the direct sum $\mathcal{L} \oplus \pi_{(5)}^{(15)*}\Omega^1_{X^{(5)}} \oplus \mathcal{O}$. Take line bundles $\mathcal{L}_1, \mathcal{L}_2$ such that $\mathcal{L}_1 \subset \mathcal{L}$, $\mathcal{L}_2 \supset \mathcal{O}$. Then $\mathcal{H}' = \mathcal{L}_1 \oplus \pi_{(5)}^{(15)*}\Omega^1_{X^{(5)}} \oplus \mathcal{L}_2$ is naturally a Higgs bundle. $\mathcal{H}'$ is stable if

(1) $c_1(\mathcal{L}_1)\pi_{(5)}^{(15)*}K_{X^{(5)}} > \frac{1}{3}(\pi_{(5)}^{(15)*}K_{X^{(5)}} + c_1(\mathcal{L}_2))\pi_{(5)}^{(15)*}K_{X^{(5)}}$, and

(2) $c_1(\mathcal{L}_2)\pi_{(5)}^{(15)*}K_{X^{(5)}} < \frac{1}{2}(\pi_{(5)}^{(15)*}K_{X^{(5)}})^2$.

Choose $\mathcal{L}_i$ to be of the form $\mathcal{O}(t_i\pi_{(5)}^{(15)*}K_{X^{(5)}})$, where $t_i \in \mathbb{Q}$, $t_1 < 4/9, t_2 > 0$. Such L_i's make sense if we replace $X^{(15)}$ by a suitable ramified cover $Y = X^{(15l)}$, where l is a sufficiently divisible positive integer. Thus we consider the vector bundle $\mathcal{H}' = \mathcal{L}_1 \oplus \pi_{(5)}^{Y*}\Omega^1_{X^{(5)}} \oplus \mathcal{L}_2$ on Y, where $\pi_{(5)}^{Y} : Y \to X^{(5)}$ is the projection.

The Chern classes of the vector bundle $\mathcal{H}'$ are:

$$c_1(\mathcal{H}') = (t_1 + t_2 + 1)\pi_{(5)}^{Y*}K_{X^{(5)}},$$
$$c_2(\mathcal{H}') = \left(t_1t_2 + t_1 + t_2 + \frac{1}{3}\right)\left(\pi_{(5)}^{Y*}K_{X^{(5)}}\right)^2.$$

Thus the condition

$$3c_1^2(\mathcal{H}') = 8c_2(\mathcal{H}')$$

is given by the quadratic equation

$$3t_1^2 - 2t_1t_2 + 3t_2^2 - 2t_1 - 2t_2 + \frac{1}{3} = 0,$$

a solution of which is $(t_1, t_2) = (\frac{1}{3}, 0)$. Hence there are infinitely many rational solutions of the quadratic equation, and the stability condition is a non-empty open condition on those solutions (the rational point $(1/3, 0)$ lies on the boundary of the region given by the stability condition (1) and in the interior of the one given by (2)). For instance,

$$(t_1, t_2) = \left(\frac{25}{3\cdot 19}, \frac{2}{3^2\cdot 19}\right)$$

is a solution with

$$(c_1(\mathcal{H}'), c_2(\mathcal{H}')) = \left(\frac{8\cdot 31}{9\cdot 19}\pi_{(5)}^{Y*}K_{X^{(5)}}, \frac{3\cdot 8\cdot 31^2}{9^2\cdot 19^2}(\pi_{(5)}^{Y*}K_{X^{(5)}})^2\right).$$

We thus conclude that there are 4-bundles $\mathcal{H}'$ such that the normalized bundles $\mathcal{G} = \mathcal{H}'(-\frac{1}{4}c_1(\mathcal{H}'))$ are stable Higgs 4-bundles with trivial Chern classes. By a theorem of Simpson [9], $\mathcal{G}$ is a flat vector bundle induced by an irreducible representation $\pi_1(Y) \to \mathrm{SL}(4)$.

On $Y = X^{(15l)}$, the (integral) divisor $(\frac{4}{9} - t_1)(\pi_{(5)}^{Y*}K_{X^{(5)}})$ is linerly equivalent to a sum of the fibres $f_i^{Y*}p_{i\alpha}$, where $f_i^Y : Y \to C^{(15l)}$ is the projection and $p_{i\alpha} \in C^{(15l)}$ (indeed, $-2K_X \sim \sum_{i=0}^4 f_i^*\mathcal{O}_{\mathbb{P}^1}(1)$ on the Del Pezzo surface X, and the divisor in question is a rational multiple of the pullback of $-2K_X$). If we replace $p_{i\alpha}$ by another point $q_{i\alpha} \sim p_{i\alpha} + \tau_{i\alpha}$, $\tau_{i\alpha} \in \mathrm{Pic}^0(C^{(15l)})$, we get an effective invertible sheaf

$$\mathcal{O}_Y(\sum_{i,\alpha} f_i^{Y*} q_{i\alpha}) \simeq \mathcal{O}_Y((\frac{4}{9} - t_1)\pi_{(5)}^{Y*} K_{X^{(5)}} + \sum_{i,\alpha} f_i^{Y*}\tau_{i\alpha})) = \mathcal{L} \otimes \mathcal{L}_1^{-1}(\sum f_i^{Y*}\tau_{i\alpha}).$$

This isomorphism induces an injection

$$\mathcal{L}_1(-\tau) = \mathcal{L}_1(-\sum_{i,\alpha} f_i^{Y*}\tau_{i\alpha}) \hookrightarrow \mathcal{L} \subset \pi_{(5)}^{Y*}\mathrm{Sym}^2\Omega^1_{X^{(5)}},$$

where $\tau = \sum f_i^{Y*}\tau_{i\alpha}$ nontrivially moves in $\mathrm{Pic}^0(Y)$. Putting

$$\mathcal{H}'_\tau = \mathcal{L}_1(-\tau) \oplus \pi_{(5)}^{(15)*}\Omega^1_{X^{(5)}} \oplus \mathcal{L}_2, \quad \mathcal{G}_\tau = \mathcal{H}'_\tau(-\frac{1}{4}c_1(\mathcal{H}'_\tau)),$$

we obtain a deforming family of stable Higgs bundles $\mathcal{G}_\tau$ with $c_1 = 0 \in \mathrm{Pic}(Y)$ and $c_2 = 0 \in \mathrm{H}^4(Y, \mathbb{Z})$. By Simpson's theorem [9], it gives rise to a deformation of irreducible representations $\pi_1(Y) \to \mathrm{SL}(4)$ parametrized by a product of several copies of $C^{(15l)}$.

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