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Log Crepant Birational Maps and Derived Categories

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1 Introduction

The purpose of this paper is to extend the conjecture stated in the paper [5] to the logarithmic case and prove some supporting evidences. [5] Conjecture 1.2 predicts that birationally equivalent smooth projective varieties have equivalent derived categories if and only if they have equivalent canonical divisors.

According to the experience of the minimal model theory, one has to deal with singular varieties instead of only smooth varieties for the classification of algebraic varieties. Moreover, we should consider not only varieties but also pairs consisting of varieties and divisors on them. These pairs are expected to have some milder singularities, the log terminal singularities.

On the other hand, the theory of derived categories works well under the smoothness assumption of the variety. The reason is that the global dimension is finite only in the case of smooth varieties.

In this paper we shall consider pairs of varieties and $\mathbb{Q}$-divisors which have smooth local coverings (Definition 2.1), and conjecture that, if there is an equivalence of log canonical divisors between birationally equivalent pairs, then there is an equivalence of derived categories (Conjecture 2.2). We note that we need to consider the sheaves on the stacks associated to the pairs instead of the usual sheaves on the varieties in order to have equivalences of derived categories as already notices in [4]. This is a generalization of the conjecture in [5], and includes the case considered in [2]. We note that crepant resolutions in higher dimensions are rare but there are many log crepant partial resolutions.

In §3, we consider the problem on recovering the variety from the category. We prove that some basic birational invariants related to the canonical divisors can be recovered from the derived category (Theorem 3.1). In particular, we prove the converse statement of the conjecture. On the other hand, we remark that the variety itself can be reconstructed from the category of coherent sheaves (Theorem 3.2).

In §4, we consider the toroidal varieties and prove that the conjecture holds in this case (Theorem 4.2). This is a generalization of [4] Theorem 5.2.

We conclude the paper with a remark on the relationship with the noncommutative geometry in Proposition 4.5. We maybe need the moduli theoretic interpretation of the log crepant maps in order to deal with the conjecture in the difficult general case.

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2 Derived equivalence conjecture

We shall consider pairs of varieties with $\mathbb{Q}$-divisors which have local coverings by smooth varieties:

Definition 2.1. Let X be a normal variety, and B an effective $\mathbb{Q}$-divisor on X whose coefficients belong to the standard set $\{1 - 1/n; n \in \mathbb{N}\}$. Assume the following condition.

(*) There exists a quasi-finite and surjective morphism $\pi : U \to X$ from a smooth variety, which may be reducible, such that $\pi^*(K_X + B) = K_U$.

Let $R = (U \times_X U)^{\sim}$ be the normalization of the fiber product. Then the projections $p_i : R \to U$ are etale for $i = 1, 2$, because there is no ramification divisor for p_i.

We define the associated Deligne-Mumford stack $\mathcal{X}$ as a functor

$$\mathcal{X} : (Sch) \to (Groupoid)$$

where (Sch) is the category of schemes and $(Groupoid)$ is the category of groupoids, categories themselves whose morphisms are only isomorphisms. The functor $\mathcal{X}$ is the sheaf with respect to the etale topology which is associated to the presheaf $\mathcal{X}' : (Sch) \to (Groupoid)$ defined in the following way. For any scheme B, an object of the category $\mathcal{X}'(B)$ is an element of the set $U(B) = \text{Hom}(B, U)$, and a morphism of the category $\mathcal{X}'(B)$ is an element of the set $R(B) = \mathrm{Hom}(B, R)$. For a morphism $f : B' \to B$, we have a functor $f^* : \mathcal{X}'(B) \to \mathcal{X}'(B')$ given by $f^* : U(B) \to U(B')$ and $f^* : R(B) \to R(B')$.

The sheafification process corresponds to that of the refinement of the global atlas U, so that the stack $\mathcal{X}$ does not depend on the choice of the covering $\pi : U \to X$ but only on the pair (X, B).

Conjecture 2.2. *Let (X, B) and (Y, C) be pairs of quasi-projective varieties with $\mathbb{Q}$-divisors which satisfy the condition (*) in Definition 2.1, and let $\mathcal{X}$ and $\mathcal{Y}$ be the associated stacks. Assume that there are proper birational morphisms $\mu : W \to X$ and $\nu : W \to Y$ from a third variety W such that $\mu^*(K_X + B) = \nu^*(K_Y + C)$. Then there exists an equivalence as triangulated categories $D^b(Coh(\mathcal{X})) \to D^b(Coh(\mathcal{Y}))$.*

The pairs considered in the conjecture are very special kind of log terminal pairs. But our assumption is sifficiently general in dimension 2:

Proposition 2.3. *(1) Let (X, B) be a pair which satisfies the condition (*) in Definition 2.1. Then the pair (X, B) is log terminal.*

(2) Let X be a normal surface, and B an effective $\mathbb{Q}$-divisor on X whose coefficients belong to the standard set $\{1 - 1/n; n \in \mathbb{N}\}$. Assume that the pair (X, B) is log terminal. Then the pair satisfies the condition (*) in Definition 2.1.

Example 2.4. (1) Let $X = \mathbb{C}^2/\mathbb{Z}_8(1, 3)$, and $f : Y \to X$ the minimal resolution. Then $f^*K_X = K_Y + \frac{1}{2}(C_1 + C_2)$ for the exceptional divisors C_1 and C_2; f is log crepant as a morphism from $(Y, \frac{1}{2}(C_1 + C_2))$ to $(X, 0)$. Furthermore, let $g : Z \to Y$ be the blowing up at the point $y_0 = C_1 \cap C_2$. Then $g^*(K_Y + \frac{1}{2}(C_1 + C_2)) = K_Z + \frac{1}{2}(C_1' + C_2')$, where C_i' is the strict transform of C_i for $i = 1, 2$. Thus $f \circ g$ is log crepant as a morphism from $(Z, \frac{1}{2}(C_1' + C_2'))$ to $(X, 0)$.

(2) Let $X = \mathbb{C}^2$, D_i $(i = 1, 2)$ the coordinate curves, and $f : Y \to X$ the weighted blowing up at the origin $x_0 = D_1 \cap D_2$ with weight $(1, 2)$. Then Y has one ordinary double point, and $f^*(K_X + \frac{2}{3}(D_1 + D_2)) = K_Y + \frac{2}{3}(D_1' + D_2')$, where D_i' is the strict transform of D_i for $i = 1, 2$. Thus f is log crepant as a morphism from $(Y, \frac{2}{3}(D_1' + D_2'))$ to $(X, \frac{2}{3}(D_1 + D_2))$.

3 Recovery of varieties from categories

The variety cannot be recovered from the derived category, but the canonical divisor can because the Serre functor is categorical:

Theorem 3.1. *Let (X,B) and (Y,C) be pairs of projective varieties with $\mathbb{Q}$-divisors which satisfy the condition (*) in Definition 2.1, and let $\mathcal{X}$ and $\mathcal{Y}$ be the associated stacks. Assume that there is an equivalence as triangulated categories:*

$$F : D^b(Coh(\mathcal{X})) \to D^b(Coh(\mathcal{Y})).$$

Then the following hold:

(1) $\dim X = \dim Y$.

(2) $K_X + B$ (resp. $-(K_X + B)$) is nef if and only if $K_Y + C$ (resp. $-(K_Y + C)$) is nef. Moreover, if this is the case, then the numerical Kodaira dimensions are equal

$$\nu(X, \pm(K_X + B)) = \nu(Y, \pm(K_Y + C)).$$

(3) If $K_X + B$ or $-(K_X + B)$ is big, then there are birational morphisms $\mu : W \to X$ and $\nu : W \to Y$ from a third projective variety W such that $\mu^*(K_X + B) = \nu^*(K_Y + C)$.

(4) There is an isomorphism of big canonical rings as graded $\mathbb{C}$-algebras

$$\bigoplus_{m \in \mathbf{Z}} H^0(X, \llcorner m(K_X + B) \lrcorner) \to \bigoplus_{m \in \mathbf{Z}} H^0(Y, \llcorner m(K_Y + C) \lrcorner).$$

The variety can be recovered from the category of sheaves:

Theorem 3.2. *Let (X,B) and (Y,C) be pairs of projective varieties with $\mathbb{Q}$-divisors which satisfy the condition (*) in Definition 2.1, and let $\mathcal{X}$ and $\mathcal{Y}$ be the associated stacks. Assume that there is an equivalence as abelian categories:*

$$F : Coh(\mathcal{X}) \to Coh(\mathcal{Y}).$$

Then there exists an isomorphism $f : X \to Y$ such that $f_*B = C$. Moreover, if X is smooth and $B = 0$, then there exists an invertible sheaf L on Y such that F is isomorphic to the functor f_*^L defined by $f_*^L(a) = f_*a \otimes L$.

4 Toroidal case

We confirm our conjecture in the case of toroidal varieties.

Definition 4.1. Let X be a normal variety and $\bar{B}$ a reduced divisor. The pair $(X, \bar{B})$ is said to be *toroidal* if, for each $x \in X$, there exists a toric variety (P_x, Q_x) with a point t, called a *toric chart*, such that the completion $(\hat{X}_x, \hat{\bar{B}}_x)$ at x is isomorphic to the completion $((\hat{P}_x)_t, (\hat{Q}_x)_t)$ at t. A toroidal pair is said to be *quasi-smooth* if it has only quotient singularities. Such a pair always satisfies the condition (*) in Definition 2.1.

A morphism $f : (X, \bar{B}) \to (Y, \bar{C})$ of toroidal pairs is said to be *toroidal* if, for any point $x \in X$ and any toric chart $(P'_y, Q'_y; t')$ at $y = f(x) \in Y$, there exists a toric chart $(P_x, Q_x; t)$ at x and a toric morphism $g_x : (P_x, Q_x) \to (P'_y, Q'_y)$ such that $g_x(t) = t'$ and the completion $\hat{f}_x : (\hat{X}_x, \hat{\bar{B}}_x) \to (\hat{Y}_y, \hat{\bar{C}}_y)$ is isomorphic to the completion $(\hat{g_x})_t : ((\hat{P}_x)_t, (\hat{Q}_x)_t) \to ((\hat{P}'_y)_{t'}, (\hat{Q}'_y)_{t'})$ under the toric chart isomorphisms.

A toroidal morphism $f : (X, B) \to (Y, C)$ is said to be a *divisorial contraction* if it is a projective birational morphism and such that the exceptional locus is a prime divisor. A pair of toroidal morphisms $\phi : (X, B) \to (Z, D)$ and $\psi : (Y, C) \to (Z, D)$ is said to be a *flip* if they are projective birational morphisms such that the codimensions of their exceptional loci are at least 2, the composite map $(\psi)^{-1} \circ \phi$ is not an isomorphism, and that the relative Picard numbers $\rho(X/Z)$ and $\rho(Y/Z)$ are equal to 1.

Theorem 4.2. *Let X and Y be quasi-smooth toroidal varieties, B and C be effective toroidal $\mathbb{Q}$-divisors on X and Y, respectively, whose coefficients are contained in the set $\{1 - 1/r | r \in \mathbb{Z}_{>0}\}$, and let $\mathcal{X}$ and $\mathcal{Y}$ be smooth Deligne-Mumford stacks associated to the pairs (X, B) and (Y, C), respectively. Assume that one of the following holds.*

(1) $X = Y$ and $K_X + B \ge K_Y + C$.

(2) There is a toroidal divisorial contraction $f : X \to Y$ such that $C = f_*B$ and

$$K_X + B \ge f^*(K_Y + C).$$

(3) There is a toroidal flip $f : X \to Z \leftarrow Y$ such that $C = f_*B$ and

$$\mu^*(K_X + B) \ge \nu^*(K_Y + C)$$

for common toroidal resolutions $\mu : W \to X$ and $\nu : W \to Y$ with $f = \nu \circ \mu^{-1}$.

(4) There is a toroidal divisorial contraction $f : Y \to X$ such that $B = f_*C$ and

$$f^*(K_X + B) \ge K_Y + C.$$

Setting $Z = Y$ in the cases (1) and (2) and $Z = X$ in the case (4), let $\mathcal{W} = (\mathcal{X} \times_Z \mathcal{Y})^{\sim}$ be the normalization of the fiber product with the natural morphisms $\tilde{\mu} : \mathcal{W} \to \mathcal{X}$ and $\tilde{\nu} : \mathcal{W} \to \mathcal{Y}$. Then the functor

$$F = \tilde{\mu}_* \tilde{\nu}^* : D^b(Coh(\mathcal{Y})) \to D^b(Coh(\mathcal{X})).$$

is fully faithful. Moreover, if the inequality between the log canonical divisors become an equality, then F is an equivalence of triangulated categories.

Corollary 4.3. *Let X and Y be quasi-smooth projective toric varieties, let B and C be effective toric $\mathbb{Q}$-divisors on X and Y, respectively, whose coefficients are contained in the set $\{1-1/r | r \in \mathbb{Z}_{>0}\}$, and let $\mathcal{X}$ and $\mathcal{Y}$ be smooth Deligne-Mumford stacks attached to the pairs (X, B) and (Y, C), respectively. Let $f : X - \to Y$ be a toric proper birational map which is log crepant in the sense that*

$$g^*(K_X + B) = h^*(K_Y + C)$$

for toric proper birational morphisms from a common toric variety $g : Z \to X$ and $h : Z \to Y$ such that $f = h \circ g^{-1}$. Then there is an equivalence of triangulated categories

$$F : D^b(Coh(\mathcal{Y})) \to D^b(Coh(\mathcal{X})).$$

Remark 4.4. (1) We note that the Fourier-Mukai functors for both divisorial contractions and flips are of the same type consisting of pull-backs and push-downs. Indeed, divisorial contractions and flips are of the same kind of operations from the view point of the Minimal Model Program though they look very different geometrically.

(2) In the situation of Theorem 4.2 (1), suppose that there is a third $\mathbb{Q}$-divisor D on X with standard coefficients such that $B \geq C \geq D$. Let $\mathcal{Z}$ be the stack corresponding to the pair (X, D). Then there are fully faithful functors $F_1 : D^b(\text{Coh}(\mathcal{Z})) \to D^b(\text{Coh}(\mathcal{Y}))$, $F_2 : D^b(\text{Coh}(\mathcal{Y})) \to D^b(\text{Coh}(\mathcal{X}))$ and $F_3 : D^b(\text{Coh}(\mathcal{Z})) \to D^b(\text{Coh}(\mathcal{X}))$ as proved there. But we have $F_3 \not\cong F_2 \circ F_1$ in general.

(3) Let $X = \mathbb{C}^2$, $\bar{B} = B_1 + B_2$ the union of the two coordinate lines, $f : Y \to X$ the blowing up at the singular point $B_1 \cap B_2$ of $\bar{B}$, and $\bar{C} = B_1' + B_2' + C_3$ the union of the strict transforms and the exceptional divisor. Let $\mathcal{X}_n$ and $\mathcal{Y}_n$ be the stacks associated to the pairs $(X, \frac{1}{2n}(B_1 + B_2))$ and $(Y, \frac{1}{2n}(B_1' + B_2') + \frac{1}{n}C_3)$, respectively. Since $f^*(K_X + \frac{1}{2n}(B_1 + B_2)) = K_Y +$

$\frac{1}{2n}(B_1' + B_2') + \frac{1}{n}C_3$, we have equivalences $\Phi_n : D^b(\mathrm{Coh}(\mathcal{X}_n)) \to D^b(\mathrm{Coh}(\mathcal{Y}_n))$ for each n by Theorem 4.2 (2). On the other hand, we have fully faithful functors $\Psi_{X,nn'} : D^b(\mathrm{Coh}(\mathcal{X}_n)) \to D^b(\mathrm{Coh}(\mathcal{X}_{n'}))$ and $\Psi_{Y,nn'} : D^b(\mathrm{Coh}(\mathcal{Y}_n)) \to D^b(\mathrm{Coh}(\mathcal{Y}_{n'}))$ for $n < n'$ by Theorem 4.2 (1). We have to be careful that the following diagram is not commutative

$$\begin{tikzcd} D^b(\mathrm{Coh}(\mathcal{X}_n)) \arrow[r, "\Phi_n"] \arrow[d, "\Psi_{X,nn'}"'] & D^b(\mathrm{Coh}(\mathcal{Y}_n)) \arrow[d, "\Psi_{Y,nn'}"] \\ D^b(\mathrm{Coh}(\mathcal{X}_{n'})) \arrow[r, "\Phi_{n'}"] & D^b(\mathrm{Coh}(\mathcal{Y}_{n'})). \end{tikzcd}$$

The reason is that the Serre functors, defined by using different invertible sheaves, are not compatible.

We conclude this paper with a remark on the non-commutative geometry as in [9]. We consider the situation of Theorem 4.2 (3) under the additional assumption that $\mu^*(K_X + B) = \nu^*(K_Y + C)$.

Although the set of all invertible sheaves on $\mathcal{Y}$ in the range (??) is infinite, there are only finitely many ismomorphism classes. Let P_Y be the direct sum of these representatives, and let $A_Y = \mathrm{Hom}(P_Y, P_Y)$ be the non-commutative ring of endomorphisms. We denote by $\mathrm{Mod}(A_Y)$ the abelian category of finitely generated right A_Y-modules.

Proposition 4.5. *There is an equivalence of triangulated categories:*

$$D^b(\mathit{Coh}(\mathcal{Y})) \cong D^b(\mathit{Mod}(A_Y)).$$

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