代数幾何学シンポジウム記録
2003年度 pp.58-62

RESTRICTED VOLUMES AND THE AUGMENTED BASE LOCUS

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This is the content of my talk at the Kinosaki meeting in October, 2003. All new results are based on the joint work [ELMNP1] with L. Ein, R. Lazarsfeld, M. Popa and M. Nakamaye.

1. Basic definitions

We work in the followng set-up. Let X be a smooth complex projective variety. We denote by $N^1(X)$ the group of line bundles on X modulo numerical equivalence (denoted $\equiv$), and by $N^1(X)_{\mathbb{Q}}$ and $N^1(X)_{\mathbb{R}}$ the corresponding vector spaces obtained by tensoring with $\mathbb{Q}$ and $\mathbb{R}$, respectively. Recall that $N^1(X)$ is a finitely generated free abelian group. $N^1(X)_{\mathbb{R}}$ is a finite dimensional vector space, on which we consider the euclidean topology.

If $L \in \mathrm{Pic}(X)$, then its asymptotic base locus is

$$\mathrm{B}(L) := \bigcap_m \mathrm{Bs}(L^m)_{\mathrm{red}} = \mathrm{Bs}(L^q)_{\mathrm{red}},$$

if q is divisible enough. As it is clear that $\mathrm{B}(L^m) = \mathrm{B}(L)$, we may define the asymptotic base locus of a $\mathbb{Q}$-divisor in the obvious way.

One problem about this definition is that it is not numerically invariant: take on a curve of genus $g \geq 1$ a non-torsion line bundle $L \in \mathrm{Pic}^0(X)$. By definition $L \equiv \mathcal{O}_X$, but it is clear that $\mathrm{B}(L) = X$ and $\mathrm{B}(\mathcal{O}_X) = \emptyset$. See Example 1.2 in [ELMNP2] for two big numerically equivalent line bundles with different asymptotic base loci.

Another problem with the above definition is that it works only for $\mathbb{Q}$-divisors. The following notion introduced by Nakamaye in [Na1] remedies both these problems.

Definition 1.1. If D is an $\mathbb{R}$-divisor, then

$$\mathrm{B}_+(D) := \bigcap_A \mathrm{B}(D - A),$$

where the intersection is over all ample divisors A such that $D - A$ is a $\mathbb{Q}$-divisor.

Remark 1.2. The following properties are easy to prove from definition:

(1) There is a neighbourhood $\mathcal{U}_D$ of the origin in $N^1(X)_{\mathbb{R}}$ such that if A is ample, with class in $\mathcal{U}_D$ and with $D - A$ is a $\mathbb{Q}$-divisor, then
$$\mathrm{B}_+(D) = \mathrm{B}(D - A).$$
(2) If $D \equiv E$, then $\mathrm{B}_+(D) = \mathrm{B}_+(E)$.
(3) IF D is a $\mathbb{Q}$-divisor, then $\mathrm{B}(D) \subseteq \mathrm{B}_+(D)$.
(4) $\mathrm{B}_+(D) \neq X$ if and only if D is big.
(5) $\mathrm{B}_+(D) = \emptyset$ if and only if D is ample.

Understanding the augmented base locus is important for many applications. For example, here is a question of Vojta motivated by hyperbolicity problems.

Question 1.3. ([Voj]) Let E be a vector bundle on X, and let $\pi : \mathbb{P}(E) \longrightarrow X$ be the corresponding projective bundle. If E is big (i.e. if $\mathcal{O}(1)$ is big on $\mathbb{P}(E)$) and Bogomolov semistable, then $\pi(\mathrm{B}_+(\mathcal{O}(1))) \neq X$.

We will be concerned with giving a numerical description of $\mathrm{B}_+(D)$ for a divisor D. Suppose first that D is nef. Note that if V is a subvariety of X such that $D|_V$ is not big, then $V \subseteq \mathrm{B}_+(D)$. Indeed, otherwise there is a decomposition

$$D = A + E,$$

with A ample and with E effective and such that $V \not\subseteq \mathrm{Supp}(E)$. We deduce that $D|_V = A|_V + E|_V$, and $E|_V$ is effective. Therefore $D|_V$ is big, a contradiction.

The main result of [Na1] says that this phenomenon accounts for all the irreducible components of $B_+(D)$. More precisely, we have

Theorem 1.4. ([Na1]) *Let D be a nef $\mathbb{Q}$-divisor on X. Then*

$$\mathrm{B}_+(D) = \bigcup_{D|_V \neq \mathrm{big}} V.$$

Our main goal is to generalize the above theorem to the case of an arbitrary divisor (allowing also $\mathbb{R}$-coefficients).

Note first that if D is nef, then $D|_V$ is big if and only if $(D|_V^{\dim V}) > 0$. Suppose now that D is general (we may assume D big, because otherwise $\mathrm{B}_+(D) = X$). Assume for the moment that D is a $\mathbb{Q}$-divisor which has a Zariski decomposition on X with rational coefficients. We mean that we can write $D = P + N$, where P and N are $\mathbb{Q}$-divisors, with P nef, N effective, and such that if m is divisible enough, then the induced map

$$H^0(X, \mathcal{O}(mP)) \hookrightarrow H^0(X, \mathcal{O}(mD))$$

is an isomorphism. In this case it is easy to see that

$$\mathrm{B}_+(D) = \mathrm{Supp}(N) \cup B_+(D).$$

By the above theorem of Nakamaye, it follows that we need an invariant associated to every subvariety $V \subseteq X$ and to each divisor D, such that if we have a decomposition as above, then the invariant is equal to $(P|_V^{\dim V})$.

2. Restricted volumes and $\mathrm{B}_+(D)$

We proceed now to define the invariant used to describe $\mathrm{B}_+(D)$. The case when $V = X$ is well-known. We recall the definition and the basic properties, for which we refer to [Laz].

Definition 2.1. If L is a line bundle on X, and if $\dim X = n$, then its *volume* is defined by

$$\mathrm{vol}(X, L) := \limsup_{m \to \infty} \frac{n! \cdot h^0(X, L^m)}{m^n}.$$

The volume has the following properties:

• $\mathrm{vol}(X, L) > 0$ if and only if L is big.

• $\mathrm{vol}(X, L^m) = m^n \cdot \mathrm{vol}(X, L)$. Therefore the definition can be extended in the obvious way to $\mathbb{Q}$-divisors.

• If L is nef, then $\mathrm{vol}(X, L) = (L^n)$.

• If D and E are numerically equivalent $\mathbb{Q}$-divisors, then $\mathrm{vol}(X, D) = \mathrm{vol}(X, E)$.

• $\mathrm{vol}(X, -)$ can be extended as a continuous function to $N^1(X)_{\mathbb{R}}$.

• If $D = P + N$ is a Zariski decomposition as above, then $\mathrm{vol}(X, D) = \mathrm{vol}(X, P) = (P^n)$.

Probably the most important property of volumes is given by the following theorem due to Fujita, and with a different proof, to Demailly, Ein and Lazarsfeld. It says that the volume can be approximated by intersection numbers of ample divisors. More precisely, we have

Theorem 2.2. ([Fuj],[DEL]) *If D is a big divisor on X, then*

$$\mathrm{vol}(X, D) = \sup_{\pi^* D = A + E} (A^n),$$

where the supremum is over all proper birational morphisms $\pi : X' \longrightarrow X$, with X' smooth, and over all expressions $D = A + E$, where A is ample and E is effective.

We define now a similar notion relative to a subvariety $V \subseteq X$, inspired by ideas of Tsuji.

Definition 2.3. Let $V \subseteq X$ be a subvariety with $\dim V = r$. If L is a line bundle on X, then the *restricted volume* of L along V is

$$\mathrm{vol}_X(V, L) := \limsup_{m \to \infty} \frac{r! \cdot h^0_X(V, L^m)}{m^r},$$

where $h^0_X(V, L^m)$ is the dimension of the image of the natural map $H^0(X, L^m) \longrightarrow H^0(V, L^m|_V)$.

A few properties follow easily from the definition:

• $\mathrm{vol}_X(V, L^m) = m^r \cdot \mathrm{vol}_X(V, L)$, so the definition extends in the obvious way to $\mathbb{Q}$-divisors.

• If L is ample, then $\mathrm{vol}_X(V, L) = (L|_V^r)$.

• If $V \not\subseteq \mathrm{B}_+(D)$, then we can write $D = A + E$, with A ample and E effective, such that $V \not\subseteq \mathrm{Supp}(E)$. It follows that

$$\mathrm{vol}_X(V, D) \geq \mathrm{vol}_X(V, A) = (A|_V^r).$$

The philosophy is that properties of the usual volume function for big divisors generalize to properties of the restricted volumes for divisors D such that $V \not\subseteq \mathrm{B}_+(D)$. All these properties follow from the following extension of Fujita's theorem. The proof is based on the approach to Fujita's theorem from [DEL], using the Subadditivity Theorem for asymptotic multiplier ideals.

Theorem 2.4. *Let D be a $\mathbb{Q}$-divisor, and let V be an r-dimensional subvariety of X. If $V \not\subseteq \mathrm{B}_+(D)$, then*

$$\mathrm{vol}_X(V,D) = \sup_{\pi^*D=A+E} (A|_{\widetilde{V}}^r),$$

where the supremum is over all proper birational morphisms $\pi : X' \longrightarrow X$ (with X' smooth) which are isomorphisms at the generic point of V, and over all expressions $\pi^*D = A+E$, where A is ample, E is effective, and $\widetilde{V} \not\subseteq \mathrm{Supp}(E)$ ($\widetilde{V}$ is the proper transform of V).

One can use the above theorem to deduce other properties of the restricted volumes:

- If D and E are numerically equivalent $\mathbb{Q}$-divisors, and if $V \not\subseteq \mathrm{B}_+(D)$, then $\mathrm{vol}_X(V,D) = \mathrm{vol}_X(V,E)$.

- $\mathrm{vol}_X(V,-)$ can be extended as a continuous function to

$$\{D \in \mathrm{Big}(X)_{\mathbb{R}} | V \not\subseteq \mathrm{B}_+(D)\}.$$

- If D is nef, and if $V \not\subseteq \mathrm{B}_+(D)$, then

$$\mathrm{vol}_X(V,D) = (D|_V^r).$$

- If $D = P + N$ is a Zariski decomposition as before, and if $V \not\subseteq \mathrm{B}_+(D)$, then $\mathrm{vol}_X(V,D) = \mathrm{vol}_X(V,P) = (P|_V^r)$.

The following is our main result. It shows that restricted volumes describe the irreducible components of the augmented base locus. Moreover, it gives also a continuity statement. The proof is based on ideas of Nakamaye from [Na2].

Theorem 2.5. *If D is a big $\mathbb{R}$-divisor on X, and if V is an irreducible component of $\mathrm{B}_+(D)$, then*

$$\lim_{D' \to D} \mathrm{vol}_X(V,D') = 0,$$

where the limit is over $\mathbb{Q}$-divisors D' whose classes go the the class of D.

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