RESTRICTED VOLUMES AND THE AUGMENTED BASE LOCUS

MIRCEA MUSTAŢĂ

This is the content of my talk at the Kinosaki meeting in October, 2003. All new results are based on the joint work [ELMNP1] with L. Ein, R. Lazarsfeld, M. Popa and M. Nakamaye.

1. BASIC DEFINITIONS

We work in the following set-up. Let $X$ be a smooth complex projective variety. We denote by $N^1(X)$ the group of line bundles on $X$ modulo numerical equivalence (denoted $\equiv$), and by $N^1(X)_\mathbb{Q}$ and $N^1(X)_\mathbb{R}$ the corresponding vector spaces obtained by tensoring with $\mathbb{Q}$ and $\mathbb{R}$, respectively. Recall that $N^1(X)$ is a finitely generated free abelian group. $N^1(X)_\mathbb{R}$ is a finite dimensional vector space, on which we consider the euclidean topology.

If $L \in \text{Pic}(X)$, then its asymptotic base locus is

$$B(L) := \ldots \bigcup \text{Bs}(L^m)_{\text{red}} = \text{Bs}(L^q)_{\text{red}},$$

if $q$ is divisible enough. As it is clear that $B(L^m) = B(L)$, we may define the asymptotic base locus of a Q-divisor in the obvious way.

One problem about this definition is that it is not numerically invariant: take on a curve of genus $g \geq 1$ a non-torsion line bundle $L \in \text{Pic}^0(X)$. By definition $L \equiv \mathcal{O}_X$, but it is clear that $B(L) = X$ and $B(\mathcal{O}_X) = \emptyset$. See Example 1.2 in [ELMNP2] for two big numerically equivalent line bundles with different asymptotic base loci.

Another problem with the above definition is that it works only for Q-divisors. The following notion introduced by Nakamaye in [Na1] remedies both these problems.

**Definition 1.1.** If $D$ is an $\mathbb{R}$-divisor, then

$$B_+(D) := \bigcap_{A} B(D - A),$$

where the intersection is over all ample divisors $A$ such that $D - A$ is a $\mathbb{Q}$-divisor.

**Remark 1.2.** The following properties are easy to prove from definition:

1. There is a neighbourhood $U_D$ of the origin in $N^1(X)_\mathbb{R}$ such that if $A$ is ample, with class in $U_D$ and with $D - A$ is a $\mathbb{Q}$-divisor, then $B_+(D) = B(D - A)$.
2. If $D \equiv E$, then $B_+(D) = B_+(E)$.
3. If $D$ is a $\mathbb{Q}$-divisor, then $B(D) \subseteq B_+(D)$.
4. $B_+(D) \neq X$ if and only if $D$ is big.
5. $B_+(D) = \emptyset$ if and only if $D$ is ample.
Understanding the augmented base locus is important for many applications. For example, here is a question of Vojta motivated by hyperbolicity problems.

**Question 1.3.** ([Voj]) Let $E$ be a vector bundle on $X$, and let $\pi : \mathbb{P}(E) \to X$ be the corresponding projective bundle. If $E$ is big (i.e. if $\mathcal{O}(1)$ is big on $\mathbb{P}(E)$) and Bogomolov semistable, then $\pi(B_+(\mathcal{O}(1))) \neq X$.

We will be concerned with giving a numerical description of $B_+(D)$ for a divisor $D$. Suppose first that $D$ is nef. Note that if $V$ is a subvariety of $X$ such that $D|_V$ is not big, then $V \subseteq B_+(D)$. Indeed, otherwise there is a decomposition

$$D = A + E,$$

with $A$ ample and with $E$ effective and such that $V \not\subseteq \text{Supp}(E)$. We deduce that $D|_V = A|_V + E|_V$, and $E|_V$ is effective. Therefore $D|_V$ is big, a contradiction.

The main result of [Nal] says that this phenomenon accounts for all the irreducible components of $B_+(D)$. More precisely, we have

**Theorem 1.4.** ([Nal]) Let $D$ be a nef $\mathbb{Q}$-divisor on $X$. Then

$$B_+(D) = \bigcup_{D|_V \neq \text{big}} V.$$

Our main goal is to generalize the above theorem to the case of an arbitrary divisor (allowing also $\mathbb{R}$-coefficients).

Note first that if $D$ is nef, then $D|_V$ is big if and only if $(D|_V)^{\dim V} > 0$. Suppose now that $D$ is general (we may assume $D$ big, because otherwise $B_+(D) = X$). Assume for the moment that $D$ is a $\mathbb{Q}$-divisor which has a Zariski decomposition on $X$ with rational coefficients. We mean that we can write $D = P + N$, where $P$ and $N$ are $\mathbb{Q}$-divisors, with $P$ nef, $N$ effective, and such that if $m$ is divisible enough, then the induced map

$$H^0(X, \mathcal{O}(mP)) \hookrightarrow H^0(X, \mathcal{O}(mD))$$

is an isomorphism. In this case it is easy to see that

$$B_+(D) = \text{Supp}(N) \cup B_+(D).$$

By the above theorem of Nakamaye, it follows that we need an invariant associated to every subvariety $V \subseteq X$ and to each divisor $D$, such that if we have a decomposition as above, then the invariant is equal to $(P|_V)^{\dim V}$.

### 2. Restricted Volumes and $B_+(D)$

We proceed now to define the invariant used to describe $B_+(D)$. The case when $V = X$ is well-known. We recall the definition and the basic properties, for which we refer to [Laz].

**Definition 2.1.** If $L$ is a line bundle on $X$, and if $\dim X = n$, then its *volume* is defined by

$$\text{vol}(X, L) := \limsup_{m \to \infty} \frac{n! \cdot h^0(X, L^m)}{m^n}.$$
The volume has the following properties:

- \( \text{vol}(X, L) > 0 \) if and only if \( L \) is big.
- \( \text{vol}(X, L^m) = m^n \cdot \text{vol}(X, L) \). Therefore the definition can be extended in the obvious way to \( \mathbb{Q} \)-divisors.
- If \( L \) is nef, then \( \text{vol}(X, L) = (L^n) \).
- If \( D \) and \( E \) are numerically equivalent \( \mathbb{Q} \)-divisors, then \( \text{vol}(X, D) = \text{vol}(X, E) \).
- \( \text{vol}(X, -) \) can be extended as a continuous function to \( N^1(X)_{\mathbb{R}} \).
- If \( D = P + N \) is a Zariski decomposition as above, then \( \text{vol}(X, D) = \text{vol}(X, P) = (P^n) \).

 Probably the most important property of volumes is given by the following theorem due to Fujita, and with a different proof, to Debarre, Ein and Lazarsfeld. It says that the volume can be approximated by intersection numbers of ample divisors. More precisely, we have

**Theorem 2.2.** ([Fujita],[DEL]) If \( D \) is a big divisor on \( X \), then

\[
\text{vol}(X, D) = \sup_{\pi^* D = A + E} (A^n),
\]

where the supremum is over all proper birational morphisms \( \pi : X' \to X \), with \( X' \) smooth, and over all expressions \( D = A + E \), where \( A \) is ample and \( E \) is effective.

We define now a similar notion relative to a subvariety \( V \subseteq X \), inspired by ideas of Tsuji.

**Definition 2.3.** Let \( V \subseteq X \) be a subvariety with \( \text{dim } V = r \). If \( L \) is a line bundle on \( X \), then the restricted volume of \( L \) along \( V \) is

\[
\text{vol}_V(X, L) := \limsup_{m \to \infty} \frac{r! \cdot h^0_X(V, L^m)}{m^r},
\]

where \( h^0_X(V, L^m) \) is the dimension of the image of the natural map \( H^0(X, L^m) \to H^0(V, L^m|_V) \).

A few properties follow easily from the definition:

- \( \text{vol}_V(X, L^m) = m^r \cdot \text{vol}_V(X, L) \), so the definition extends in the obvious way to \( \mathbb{Q} \)-divisors.
- If \( L \) is ample, then \( \text{vol}_V(X, L) = (L|_V^r) \).
- If \( V \not\subseteq \text{B}_+(D) \), then we can write \( D = A + E \), with \( A \) ample and \( E \) effective, such that \( V \not\subseteq \text{Supp}(E) \). It follows that

\[
\text{vol}_V(X, D) \geq \text{vol}_V(X, A) = (A|_V^r).
\]

The philosophy is that properties of the usual volume function for big divisors generalize to properties of the restricted volumes for divisors \( D \) such that \( V \not\subseteq \text{B}_+(D) \). All these properties follow from the following extension of Fujita's theorem. The proof is
based on the approach to Fujita's theorem from [DEL], using the Subadditivity Theorem for asymptotic multiplier ideals.

**Theorem 2.4.** Let $D$ be a $\mathbb{Q}$-divisor, and let $V$ be an $r$-dimensional subvariety of $X$. If $V \not\subseteq B_+(D)$, then

$$\text{vol}_X(V, D) = \sup_{\pi^* D = A + E} (A|_V),$$

where the supremum is over all proper birational morphisms $\pi : X' \to X$ (with $X'$ smooth) which are isomorphisms at the generic point of $V$, and over all expressions $\pi^* D = A + E$, where $A$ is ample, $E$ is effective, and $\tilde{V} \subseteq \text{Supp}(E)$ ($\tilde{V}$ is the proper transform of $V$).

One can use the above theorem to deduce other properties of the restricted volumes:

- If $D$ and $E$ are numerically equivalent $\mathbb{Q}$-divisors, and if $V \not\subseteq B_+(D)$, then $\text{vol}_X(V, D) = \text{vol}_X(V, E)$.
- $\text{vol}_X(V, -)$ can be extended as a continuous function to

$$\{D \in \text{Big}(X)_\mathbb{R} | V \not\subseteq B_+(D)\}.$$

- If $D$ is nef, and if $V \not\subseteq B_+(D)$, then

$$\text{vol}_X(V, D) = (D|_V).$$

- If $D = P + N$ is a Zariski decomposition as before, and if $V \not\subseteq B_+(D)$, then $\text{vol}_X(V, D) = \text{vol}_X(V, P) = (P|_V)$.

The following is our main result. It shows that restricted volumes describe the irreducible components of the augmented base locus. Moreover, it gives also a continuity statement. The proof is based on ideas of Nakamaye from [Na2].

**Theorem 2.5.** If $D$ is a big $\mathbb{R}$-divisor on $X$, and if $V$ is an irreducible component of $B_+(D)$, then

$$\lim_{D' \to D} \text{vol}_X(V, D') = 0,$$

where the limit is over $\mathbb{Q}$-divisors $D'$ whose classes go the the class of $D$.

**REFERENCES**


RESTRICTED VOLUMES AND THE AUGMENTED BASE LOCUS

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY, ONE OXFORD STREET, CAMBRIDGE, MA 02138, USA

E-mail address: mirceamustata@yahoo.com