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A CHARACTERISTIC p ANALOG OF MULTIPLIER IDEALS AND ITS APPLICATIONS

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The notion of test ideals plays a very important role in the theory of tight closure introduced by Hochster and Huneke [HH1]. The test ideal $\tau(R)$ of a commutative ring R of characteristic $p > 0$ is the ideal generated by all test elements of R and equal to the annihilator ideal of all tight closure relations in R. Recently, K. Yoshida and I [HY] introduced a generalization of tight closure, which we call $\mathfrak{a}^t$-tight closure associated to given any ideal $\mathfrak{a} \subseteq R$ and any rational number $t \geq 0$, and defined the ideal $\tau(\mathfrak{a}^t)$ to be the annihilator ideal of all $\mathfrak{a}^t$-tight closure relations in R.

Our work [HY] is motivated by the correspondence of test ideals and multiplier ideals. Multiplier ideals have recently arisen as objects with highly interesting applications, and can be defined in various settings via resolution of singularities in characteristic zero ([E], [La]). Among them is the multiplier ideal $\mathcal{J}(\mathfrak{a}^t)$ associated to an ideal $\mathfrak{a}$ and a rational number $t \geq 0$. Precisely speaking, the multiplier ideal that corresponds to the test ideal $\tau(R)$ is the one associated to the unit ideal $\mathfrak{a} = R$. In most applications, however, the usefulness of multiplier ideals is performed by considering multiplier ideals associated to various ideals, and this is the reason why we introduced the ideal $\tau(\mathfrak{a}^t)$.

It turns out that the multiplier ideal $\mathcal{J}(\mathfrak{a}^t)$ in a normal $\mathbb{Q}$-Gorenstein ring of characteristic zero coincides, after reduction to characteristic $p \gg 0$, with the ideal $\tau(\mathfrak{a}^t)$. Also in fixed characteristic $p > 0$, the ideals $\tau(\mathfrak{a}^t)$ have several nice properties similar to those of multiplier ideals $\mathcal{J}(\mathfrak{a}^t)$; e.g., an analog of Skoda's theorem ([HT], cf. [Li]) and the subadditivity theorem in regular local rings ([HY], cf. [DEL]). It is notable that the above properties of the ideals $\tau(\mathfrak{a}^t)$ are proved quite algebraically via characteristic p methods.

The purpose of this note is to give a brief overview of the theory of $\mathfrak{a}^t$-tight closure and the ideal $\tau(\mathfrak{a}^t)$ developed in [HY] (see also [HT]), and give some applications arising from the relationship with multiplier ideals [H]. Namely, we give new proofs to Smith's result [S] on global generation of adjoint bundles in characteristic $p > 0$ and results on uniform behavior of symbolic powers of an ideal in a regular local ring obtained in [ELS] and [HH2].

1. Definitions and basic notions on $\mathfrak{a}$-tight closure

We review a generalization of tight closure from [HY]. The reader is referred to Hochster and Huneke [HH1] and Huneke [Hu] for the original notion of tight closure.

Throughout this section, R will denote a Noetherian ring of prime characteristic $p > 0$. We denote by R° the set of elements of R not in any minimal prime of R, and by $F\colon R \to R$ the Frobenius map sending $z \in R$ to $z^p \in R$. We always use the letter q for a power p^e of p. The ring R viewed as an R-module via the e-times iterated Frobenius map $F^e\colon R \to R$ is denoted by ${}^eR$. If R is reduced, $F^e\colon R \to {}^eR$

is identified with the natural inclusion map $R \hookrightarrow R^{1/q}$. We say that R is *F-finite* if ${}^1R$ (or $R^{1/p}$) is a finitely generated R-module.

Let M be an R-module. For each $e \in \mathbb{N}$, we denote $\mathbb{F}^e(M) := {}^eR \otimes_R M$ and regard it as an R-module by the action of $R = {}^eR$ from the left. Then we have the induced e-times iterated Frobenius map $F^e \colon M \to \mathbb{F}^e(M)$. The image of $z \in M$ via this map is denoted by $z^q := F^e(z) \in \mathbb{F}^e(M)$. For an R-submodule N of M, we denote by $N_M^{[q]}$ the image of the induced map $\mathbb{F}^e(N) \to \mathbb{F}^e(M)$. If I is an ideal of R, then $I^{[q]} := I_R^{[q]}$ is the ideal generated by the qth powers of elements of I.

Definition 1.1. Let $\mathfrak{a}$ be an ideal of a Noetherian ring R of characteristic $p > 0$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$ and let $N \subseteq M$ be R-modules. Given a rational number $t \geq 0$, the $\mathfrak{a}^t$*-tight closure* of N in M, denoted by $N_M^{*\mathfrak{a}^t}$, is defined to be the submodule of M consisting of all elements $z \in M$ for which there exists $c \in R^\circ$ such that

$$c\mathfrak{a}^{\lceil tq \rceil} z^q \subseteq N_M^{[q]}$$

for all large $q = p^e$, where $\lceil tq \rceil$ is the least integer which is greater than or equal to tq. The $\mathfrak{a}^t$-tight closure of an ideal $I \subseteq R$ is just defined by $I^{*\mathfrak{a}^t} = I_R^{*\mathfrak{a}^t}$.

Remark 1.2. The rational exponent t for $\mathfrak{a}^t$-tight closure is just a formal notation, but it is compatible with "actual" powers of the ideal. Namely, if $\mathfrak{b} = \mathfrak{a}^n$ for $n \in \mathbb{N}$, then $\mathfrak{a}^t$-tight closure is the same as $\mathfrak{b}^{t/n}$-tight closure. So there occurs no confusion if we call "$\mathfrak{a}^t$-tight closure for $t = 1$" just "$\mathfrak{a}$-tight closure." We have the following specialization and generalization of the concept.

(1) In the case where $\mathfrak{a} = R$ is the unit ideal, the $\mathfrak{a}$-tight closure $N_M^{*\mathfrak{a}} = N_M^{*R}$ is nothing but the (usual) tight closure N_M^* as defined in [HH1]. However, unlike the usual tight closure, it may happen that $(N_M^{*\mathfrak{a}})_M^{*\mathfrak{a}}$ is strictly larger than $N_M^{*\mathfrak{a}}$. In this sense $\mathfrak{a}$-tight closure is not an "honest" closure operation in general.

(2) Given ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_r \subseteq R$ with $\mathfrak{a}_i \cap R^\circ \neq \emptyset$ and nonnegative rational numbers $t_1, \dots, t_r$, if $t_i = tn_i$ for $t \in \mathbb{Q}$ and $n_i \in \mathbb{N}$ with $i = 1, \dots, r$, we can define $\mathfrak{a}_1^{t_1} \cdots \mathfrak{a}_r^{t_r}$-tight closure to be $(\mathfrak{a}_1^{n_1} \cdots \mathfrak{a}_r^{n_r})^t$-tight closure.

We collect some basic properties of $\mathfrak{a}^t$-tight closure in the following

Proposition 1.3. ([HY, Proposition 1.3 and Corollary 2.3]) *Let $\mathfrak{a}$, $\mathfrak{b} \subseteq R$ denote ideals not contained in any minimal prime ideal, $t \geq 0$ a rational number, and let $N \subseteq M$ be R-modules.*

1. *$N \subseteq N_M^{*\mathfrak{a}^t}$ and $N_M^{*\mathfrak{a}^t}/N \cong 0_{M/N}^{*\mathfrak{a}^t}$.*
2. *$N_M^{*\mathfrak{a}^t\mathfrak{b}} \subseteq (N_M^{*\mathfrak{a}^t} : \mathfrak{b})_M$. Moreover, if $\mathfrak{b}$ is a principal ideal, then the equality $N_M^{*\mathfrak{a}^t\mathfrak{b}} = (N_M^{*\mathfrak{a}^t} : \mathfrak{b})_M$ holds.*
3. *If $\mathfrak{b} \subseteq \mathfrak{a}$, then $N_M^{*\mathfrak{a}^t} \subseteq N_M^{*\mathfrak{b}^t}$. Moreover, if $\mathfrak{a}$ and $\mathfrak{b}$ have the same integral closure,[1] then the equality $N_M^{*\mathfrak{a}^t} = N_M^{*\mathfrak{b}^t}$ holds.*

Theorem-Definition 1.4 ([HY]). *Let R be an excellent reduced ring of characteristic $p > 0$, $\mathfrak{a} \subseteq R$ an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let $t \geq 0$ be a rational number. Let $E = \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$ be the direct sum, taken over all maximal ideals $\mathfrak{m}$ of R, of the injective envelopes of the residue field $R/\mathfrak{m}$. Then the following ideals are equal to each other and we denote it by $\tau(\mathfrak{a}^t)$.*

[1] The integral closure $\bar{\mathfrak{a}}$ of an ideal $\mathfrak{a} \subseteq R$ is defined by $\bar{\mathfrak{a}} = H^0(X, \mathfrak{a}\mathcal{O}_X) \subseteq R$, where $X \to \operatorname{Spec} R$ is a proper birational morphism from a normal scheme X such that $\mathfrak{a}\mathcal{O}_X$ is invertible.

1. $\bigcap_{M} \mathrm{Ann}_R(0_M^{*\mathfrak{a}^t})$, *where M runs through all finitely generated R-modules.*
2. $\bigcap_{M \subseteq E} \mathrm{Ann}_R(0_M^{*\mathfrak{a}^t})$, *where M runs through all finitely generated submodules of E.*
3. $\bigcap_{I \subseteq R} (I : I^{*\mathfrak{a}^t})$, *where I runs through all ideals of R.*

Moreover, if R is normal and $\mathbb{Q}$-Gorenstein, then

$$\tau(\mathfrak{a}^t) = \mathrm{Ann}_R(0_E^{*\mathfrak{a}^t}).$$

The following basic properties of the ideal $\tau(\mathfrak{a}^t)$ follow from Proposition 1.3.

Proposition 1.5. *Let R be a Noetherian ring R of characteristic $p > 0$, $\mathfrak{a}$, $\mathfrak{b} \subseteq R$ ideals not contained in any minimal prime ideal, and let $t \geq 0$ be a rational number.*

1. $\tau(\mathfrak{a}^t)\mathfrak{b} \subseteq \tau(\mathfrak{a}^t\mathfrak{b})$. *Moreover, if $\mathfrak{b}$ is a principal ideal of a complete local ring, then $\tau(\mathfrak{a}^t)\mathfrak{b} = \tau(\mathfrak{a}^t\mathfrak{b})$.*
2. *If $\mathfrak{b} \subseteq \mathfrak{a}$, then $\tau(\mathfrak{b}^t) \subseteq \tau(\mathfrak{a}^t)$. Moreover, if $\mathfrak{b}$ is a reduction of $\mathfrak{a}$, then the equality $\tau(\mathfrak{b}^t) = \tau(\mathfrak{a}^t)$ holds.*
3. *If R is weakly F-regular,[2] then $\mathfrak{a} \subseteq \tau(\mathfrak{a})$. Moreover, if $\mathfrak{a}$ is an ideal of pure height one, then $\mathfrak{a} = \tau(\mathfrak{a})$.*

One of the major open questions in the theory of ($\mathfrak{a}^t$-)tight closure is whether ($\mathfrak{a}^t$-)tight closure commutes with localization. Ken-ichi Yoshida shows that this is affirmative in regular local rings [Y]. The following result is a partial answer to this question in a different direction.

Proposition 1.6 ([HT]). *Let $(R, \mathfrak{m})$ be an F-finite, normal $\mathbb{Q}$-Gorenstein local ring of characteristic $p > 0$, $\mathfrak{a} \subseteq R$ an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$ and let $t \geq 0$ be a rational number. Then for any multiplicatively closed subset W of R,*

$$\tau((\mathfrak{a}R_W)^t) = \tau(\mathfrak{a}^t)R_W.$$

2. Comparison of the ideal $\tau(\mathfrak{a}^t)$ and the multiplier ideal $\mathcal{J}(\mathfrak{a}^t)$

In this section, we first see the correspondence of the ideal $\tau(\mathfrak{a}^t)$ and the multiplier ideal $\mathcal{J}(\mathfrak{a}^t)$ in a $\mathbb{Q}$-Gorenstein ring reduced from characteristic zero to characteristic $p \gg 0$. Then we show that the ideal $\tau(\mathfrak{a}^t)$ in any fixed characteristic $p > 0$ has some properties analogous to those of the multiplier ideals $\mathcal{J}(\mathfrak{a}^t)$ in characteristic zero.

Let us begin with the definition of the multiplier ideal $\mathcal{J}(\mathfrak{a}^t)$.

Definition 2.1. Let Y be a normal $\mathbb{Q}$-Gorenstein variety over a field of characteristic zero and let $\mathfrak{a} \subset \mathcal{O}_Y$ be a nonzero ideal sheaf. Let $f \colon X \to Y$ be a log resolution of the ideal $\mathfrak{a}$, that is, a resolution of singularities of Y such that the ideal sheaf $\mathfrak{a}\mathcal{O}_X$ is invertible, say, $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-Z)$ for an effective divisor Z on X, and that the union $\mathrm{Exc}(f) \cup \mathrm{Supp}(Z)$ of the f-exceptional locus and the support of Z is a simple normal crossing divisor. Given a rational number $t \geq 0$, the *multiplier ideal* $\mathcal{J}(\mathfrak{a}^t)$ associated to $\mathfrak{a}$ and t is defined to be the ideal sheaf

$$\mathcal{J}(\mathfrak{a}^t) = f_*\mathcal{O}_X(\lceil K_{X/Y} - tZ \rceil)$$

[2] We say that R is *weakly F-regular* if every ideal I of R is tightly closed, that is, $I^* = I$. It is known that regular rings of characteristic $p > 0$ are weakly F-regular.

in $\mathcal{O}_Y$, where the $\mathbb{Q}$-divisor $K_{X/Y} = K_X - f^*K_Y$ is the discrepancy of f. This definition is independent of the choice of a log resolution $f: X \to Y$ of $\mathfrak{a}$.

The following result ensures the correspondence of the ideal $\tau(\mathfrak{a}^t)$ and the multiplier ideal $\mathcal{J}(\mathfrak{a}^t)$ in $\mathbb{Q}$-Gorenstein rings.

Theorem 2.2. *Let $t \geq 0$ be a fixed rational number, R a normal $\mathbb{Q}$-Gorenstein local ring essentially of finite type over a field and let $\mathfrak{a}$ be a nonzero ideal. Assume that $\mathfrak{a} \subseteq R$ is reduced from characteristic zero to characteristic $p \gg 0$, together with a log resolution of singularities $f: X \to Y = \operatorname{Spec} R$ such that $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-Z)$ is invertible. Then*

$$\tau(\mathfrak{a}^t) = H^0(X, \mathcal{O}_X(\lceil K_{X/Y} - tZ \rceil)).$$

For the proof of the above theorem, see [HY, Section 3]. The hard part of the proof is to show the containment $\tau(\mathfrak{a}^t) \supseteq H^0(X, \mathcal{O}_X(\lceil K_{X/Y} - tZ \rceil))$, which may fail in small characteristic p, whereas the containment $\tau(\mathfrak{a}^t) \subseteq H^0(X, \mathcal{O}_X(\lceil K_{X/Y} - tZ \rceil))$ holds in arbitrary characteristic $p > 0$ as long as the right-hand side is defined.

Next we will show that some useful properties of multiplier ideals also holds true for the ideal $\tau(\mathfrak{a}^t)$ in fixed characteristic $p > 0$. In view of the above theorem, we see that the results for $\tau(\mathfrak{a}^t)$ recover those for $\mathcal{J}(\mathfrak{a}^t)$ via standard technique of reduction modulo p, but the results for $\mathcal{J}(\mathfrak{a}^t)$ do not imply those for $\tau(\mathfrak{a}^t)$, because they might disagree in small characteristic p. The following results are motivated by the corresponding statements for multiplier ideals proved in [DEL], [La], [Li].

Theorem 2.3 (Restriction theorem [HY], cf. [La]). *Let $(R, \mathfrak{m})$ be a $\mathbb{Q}$-Gorenstein normal complete local ring of characteristic $p > 0$ and let $x \in \mathfrak{m}$ be a non-zero-divisor of R. Let $S = R/xR$ and assume that S is normal. Then for any ideal $\mathfrak{a} \subseteq R$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$ and any rational number $t \geq 0$,*

$$\tau((\mathfrak{a}S)^t) \subseteq \tau(\mathfrak{a}^t)S.$$

Theorem 2.4 (Subadditivity [HY], cf. [DEL]). *Let $(R, \mathfrak{m})$ be a complete regular local ring of characteristic $p > 0$. Then for any ideals $\mathfrak{a}, \mathfrak{b} \subseteq R$ not contained in any minimal prime ideal and any rational numbers $t, t' \geq 0$,*

$$\tau(\mathfrak{a}^t\mathfrak{b}^{t'}) \subseteq \tau(\mathfrak{a}^t)\tau(\mathfrak{b}^{t'}).$$

Theorem 2.5 (Skoda's theorem [HT], cf. [La], [Li]). *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ and assume that R is complete, or F-finite, normal and $\mathbb{Q}$-Gorenstein. Let $\mathfrak{a} \subseteq R$ be an ideal of positive height generated by r elements, $\mathfrak{b} \subseteq R$ an ideal such that $\mathfrak{b} \cap R^\circ \neq \emptyset$ and let $t \geq 0$ be a rational number. Then*

$$\tau(\mathfrak{a}^r\mathfrak{b}^t) = \tau(\mathfrak{a}^{r-1}\mathfrak{b}^t)\mathfrak{a}.$$

We shall sketch the proof of Theorem 2.5. To simplify the situation, we only consider the case where $\mathfrak{b} = R$ and R is normal, $\mathbb{Q}$-Gorenstein *and* complete. (The proof for the general case is more or less similar.) In this case, we have that $0_E^{*\mathfrak{a}^{r-1}} = \operatorname{Ann}_E(\tau(\mathfrak{a}^{r-1}))$ in $E = E_R(R/\mathfrak{m})$ by Theorem-Definition 1.4 and the Matlis duality, so that $\operatorname{Ann}_E(\tau(\mathfrak{a}^{r-1})\mathfrak{a}) = 0_E^{*\mathfrak{a}^{r-1}} : \mathfrak{a}$ in E. Then the theorem follows because

$$\tau(\mathfrak{a}^r) = \operatorname{Ann}_R(0_E^{*\mathfrak{a}^r}) = \operatorname{Ann}_R(0_E^{*\mathfrak{a}^{r-1}} : \mathfrak{a}) = \operatorname{Ann}_R(\operatorname{Ann}_E(\tau(\mathfrak{a}^{r-1})\mathfrak{a})) = \tau(\mathfrak{a}^{r-1})\mathfrak{a}$$

from the proposition below.

Proposition 2.6. *Let R be a Noetherian ring of characteristic $p > 0$ and M an R-module. Let $\mathfrak{a} \subset R$ be an ideal of positive height generated by r elements. Then*

$$0_M^{*\mathfrak{a}^r} = 0_M^{*\mathfrak{a}^{r-1}} : \mathfrak{a} \quad \text{in } M.$$

Proof. We already know that $0_M^{*\mathfrak{a}^r} \subseteq 0_M^{*\mathfrak{a}^{r-1}} : \mathfrak{a}$ by Proposition 1.3 (2), so it suffices to show the reverse inclusion $0_M^{*\mathfrak{a}^r} \supseteq 0_M^{*\mathfrak{a}^{r-1}} : \mathfrak{a}$. Let $z \in 0_M^{*\mathfrak{a}^{r-1}} : \mathfrak{a}$, i.e., $\mathfrak{a}z \subseteq 0_M^{*\mathfrak{a}^{r-1}}$. Then there exists $c \in R^\circ$ such that $c\mathfrak{a}^{[q]}(\mathfrak{a}^{r-1})^q z^q = 0$ in $\mathbb{F}^e(M)$ for all $q = p^e \gg 0$. Since $\mathfrak{a}$ is generated by r elements, one has

$$\mathfrak{a}^{qr} = \mathfrak{a}^{[q]}\mathfrak{a}^{q(r-1)},$$

so that $cz^q\mathfrak{a}^{qr} = 0$ for all $q = p^e \gg 0$, that is, $z \in 0_M^{*\mathfrak{a}^r}$. Thus we have $0_M^{*\mathfrak{a}^r} = 0_M^{*\mathfrak{a}^{r-1}} : \mathfrak{a}$, as claimed. □

Corollary 2.7 ([HT]). *Let $(R, \mathfrak{m})$ be a d-dimensional local ring of characteristic $p > 0$ with infinite residue field $R/\mathfrak{m}$, and assume either that $(R, \mathfrak{m})$ is complete, or F-finite, normal and $\mathbb{Q}$-Gorenstein. Then for any ideal $\mathfrak{a} \subseteq R$ of positive height and any $n \ge d$, one has*

$$\tau(\mathfrak{a}^n) = \tau(\mathfrak{a}^{d-1})\mathfrak{a}^{n+1-d}.$$

3. Applications

In this section, we define a few variants of the ideal $\tau(\mathfrak{a}^t)$ and give some applications. Although our method just gives alternative proofs to results obtained previously, I hope that it provides a new insight into characteristic p approach to problems in commutative algebra and algebraic geometry.

Asymptotic τ and uniform behavior of symbolic powers. The notion of multiplier ideals is defined not only for an ideal $\mathfrak{a}$ but also for a filtration of ideals $\mathfrak{a}_\bullet$ (or "graded family of ideals" in the terminology of [ELS]). This variant of multiplier ideals is called an asymptotic multiplier ideal and denoted by $\mathcal{J}(||\mathfrak{a}_\bullet||)$; see e.g. [ELS], [La]. An advantage of asymptotic multiplier ideals is that some information on infinitely many ideals in a family $\mathfrak{a}_\bullet$ is packaged in the single ideal $\mathcal{J}(||\mathfrak{a}_\bullet||)$, even in the case where the Rees algebra $\bigoplus_{n\ge 0} \mathfrak{a}_n t^n$ is not Noetherian. This idea is used successfully by Ein, Lazarsfeld and Smith [ELS] to prove the uniform behavior of symbolic powers in a regular rings of characteristic zero. Soon after that, Hochster and Huneke [HH2] established a characteristic p version of the result.

Here, we define a variant of the ideal $\tau(\mathfrak{a})$ corresponding to asymptotic multiplier ideals, and apply it to give an alternative proof to the result of [ELS] and [HH2]. Although we work in characteristic $p > 0$ as in [HH2], our strategy is essentially the same as the characteristic zero method in [ELS]. Given a filtration of ideals $\mathfrak{a}_\bullet$ in a ring of characteristic $p > 0$, we define the "asymptotic τ," which plays exactly the same role as asymptotic multiplier ideals in the proof. This answers the question raised in [ELS, Remark 3.1], which asks for a purely algebraic construction of ideals satisfying the "axiomatized" properties of asymptotic multiplier ideals.

In the following, a *filtration of ideals* on a Noethrian ring R will mean a collection $\mathfrak{a}_\bullet = \{\mathfrak{a}_n \mid n \in \mathbb{N}\}$ of ideals of R such that $\mathfrak{a}_1 \cap R^\circ \ne \emptyset$ and

$$\mathfrak{a}_m \cdot \mathfrak{a}_n \subseteq \mathfrak{a}_{m+n} \text{ for all } m, n \ge 1.$$

Given an ideal $\mathfrak{a} \subseteq R$ such that $\mathfrak{a} \cap R^\circ \ne \emptyset$, it is clear that ordinary powers $\mathfrak{a}^n$ of $\mathfrak{a}$ form a filtration. The integral closures $\overline{\mathfrak{a}^n}$ of ordinary powers also form a filtration.

Actually, given an ideal $\mathfrak{a} \subseteq R$ and a rational number $t \geq 0$, we have a filtration $\mathfrak{a}_\bullet$ defined by $\mathfrak{a}_n = \mathfrak{a}^{\lceil tn \rceil}$ or $\mathfrak{a}_n = \overline{\mathfrak{a}^{\lceil tn \rceil}}$. Another example of a filtration is given by symbolic powers: Recall that the nth *symbolic power* of an ideal $\mathfrak{a} \subseteq R$ is defined to be $\mathfrak{a}^{(n)} := \bigcap_{\mathfrak{p}} \mathfrak{a}^n R_{\mathfrak{p}} \cap R$, where the intersection ranges over all minimal prime ideals $\mathfrak{p}$ of $\mathfrak{a}$. For an integer $k \geq 0$, this gives a filtration of ideals $\mathfrak{a}^{(k\bullet)} = \{\mathfrak{a}^{(kn)} \mid n \in \mathbb{N}\}$ on R.

Definition 3.1. Let $\mathfrak{a}_\bullet$ be a filtration of ideals on a Noetherian commutative ring R of characterisitc $p > 0$ and let $N \subseteq M$ be R-modules. The $||\mathfrak{a}_\bullet||$-*tight closure* of N in M, denoted by $N_M^{*||\mathfrak{a}_\bullet||}$, is defined to be the submodule of M consisting of all elements $z \in M$ for which there exists $c \in R^\circ$ such that

$$c\mathfrak{a}_q z^q \subseteq N_M^{[q]}$$

for all large $q = p^e$.

Remark 3.2 (cf. [La]). Let $\mathfrak{a}_\bullet$ be a filtration of ideals on a Noetherian ring R of characterisitic $p > 0$. Then by the ascending chain condition of ideals in R, the set of ideals $\{\tau((\mathfrak{a}_k)^{1/k}) \mid k \in \mathbb{N}\}$ has a maximal element with respect to inclusion, which is easily seen to be unique. Indeed, for any integers $k, l > 0$ and any R-module M, we have $0_M^{*(\mathfrak{a}_{kl})^{1/kl}} \subseteq 0_M^{*(\mathfrak{a}_k)^{1/k}}$, because $(\mathfrak{a}_k)^{\lceil q/k \rceil} \subseteq (\mathfrak{a}_{kl})^{\lceil q/kl \rceil}$ for every $q = p^e$, so that

$$\tau((\mathfrak{a}_k)^{1/k}) \subseteq \tau((\mathfrak{a}_{kl})^{1/kl}).$$

Hence, if $\tau((\mathfrak{a}_k)^{1/k})$ and $\tau((\mathfrak{a}_l)^{1/l})$ are both maximal, then they must coincide with $\tau((\mathfrak{a}_{kl})^{1/kl})$.

Proposition-Definition 3.3. *Let $\mathfrak{a}_\bullet$ be a filtration of ideals on an excellent local ring $(R, \mathfrak{m})$ of characterisitc $p > 0$ and let $E = E_R(R/\mathfrak{m})$ be the injective envelope of the R-module $R/\mathfrak{m}$. We define*

$$\tau(||\mathfrak{a}_\bullet||) = \bigcap_{M \subset E} \mathrm{Ann}_R(0_M^{*||\mathfrak{a}_\bullet||}),$$

where the intersection on the right is taken over all finitely generated R-submodules M of E. Then $\tau(||\mathfrak{a}_\bullet||)$ is equal to the unique maximal element of the set of ideals $\{\tau((\mathfrak{a}_k)^{1/k}) \mid k \in \mathbb{N}\}$ with respect to inclusion.

Proposition 3.4 (Subadditivity). *Let $(R, \mathfrak{m})$ be a complete regular local ring of characteristic $p > 0$ and let $\mathfrak{a}_\bullet$ be a filtration of ideals R. Then for all $n \geq 0$,*

$$\tau(||\mathfrak{a}_{n\bullet}||) \subseteq \tau(||\mathfrak{a}_\bullet||)^n.$$

Proof. We can choose sufficiently large $k > 0$ so that $\tau(||\mathfrak{a}_{n\bullet}||) = \tau((\mathfrak{a}_{nk})^{1/k})$ and $\tau(||\mathfrak{a}_\bullet||) = \tau((\mathfrak{a}_{nk})^{1/nk})$. On the other hand, by the subaddtivity (Theorem 2.4), we have

$$\tau((\mathfrak{a}_{nk})^{1/k}) = \tau(((\mathfrak{a}_{nk})^{1/nk})^n) \subseteq \tau((\mathfrak{a}_{nk})^{1/nk})^n.$$

Therefore $\tau(||\mathfrak{a}_{n\bullet}||) \subseteq \tau(||\mathfrak{a}_\bullet||)^n$ as required. □

Theorem 3.5 ([ELS], [HH2]). *Let $(R, \mathfrak{m})$ be a complete regular local ring of characteristic $p > 0$ and let $\mathfrak{a}$ be an unmixed ideal of R of height h. Then*

$$\mathfrak{a}^{(hn)} \subseteq \mathfrak{a}^n$$

for all integers $n \geq 0$.

Proof. By the weak F-regularity of R and Proposition 3.4,

$$\mathfrak{a}^{(hn)} \subseteq \tau(||\mathfrak{a}^{(hn\bullet)}||) \subseteq \tau(||\mathfrak{a}^{(h\bullet)}||)^n.$$

Hence it is sufficient to show that $\tau(||\mathfrak{a}^{(h\bullet)}||) \subseteq \mathfrak{a}$. Let $\mathfrak{p}$ be any minimal prime ideal of $\mathfrak{a}$. Then $\mathfrak{a}^{(hn)}R_{\mathfrak{p}} = (\mathfrak{a}R_{\mathfrak{p}})^{hn}$ is an $\mathfrak{p}R_{\mathfrak{p}}$-primary ideal for all $n > 0$, so that

$$\tau(||\mathfrak{a}^{(h\bullet)}||) \subseteq \tau(||\mathfrak{a}^{(h\bullet)}R_{\mathfrak{p}}||) = \tau((\mathfrak{a}R_{\mathfrak{p}})^h) \subseteq \mathfrak{a}R_{\mathfrak{p}}.$$

by Theorem 2.5 and Proposition 1.3 (3). Thus $\tau(||\mathfrak{a}^{(h\bullet)}||) \subseteq \mathfrak{a}^{(1)} = \mathfrak{a}$, as required. □

The second application is related to a Skoda-type theorem like Corollary 2.7.

Global generation of adjoint line bundles in characteristic p. We give an alternative proof to the following result due to Karen Smith [S].

Theorem 3.6 (K. E. Smith [S]). *Let X be a projective variety over an algebraically closed field k of characteristic $p > 0$ such that the local ring $\mathcal{O}_{X,x}$ is F-rational[3] for every $x \in X$ and let $\mathcal{L}$ be an ample invertible sheaf on X which is generated by global sections. Then $\omega_X \otimes \mathcal{L}^{\otimes \dim X+1}$ is generated by global sections.*

This is a spacial case of Fujita's freeness conjecture [F], which turns out to be much easier to prove under the extra assumption that $\mathcal{L}$ itself is globally generated. Actually, if we assume that $\mathcal{L}$ is globally generated in characteristic zero, then the global generation of $\omega_X \otimes \mathcal{L}^{\otimes \dim X+1}$ is easily deduced from the Kodaira vanishing theorem. Smith succeeded in bypassing the use of vanishing theorems that fail in characteristic p, by translating the problem into the language of commutative algebra and applying the theory of tight closure, especially, the "colon-capturing" property of tight closure [HH1].

3.7. *Proof of Theorem* 3.6 *via $\mathfrak{a}$-tight closure.* We will briefly overview an alternative proof of Theorem 3.6 given in [H]. Let

$$R = R(X, \mathcal{L}) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})T^n$$

be the graded ring associated to $(X, \mathcal{L})$ and put $d = \dim R = \dim X + 1 \geq 2$. Then R is F-rational off the irrelevant maximal ideal $\mathfrak{m} = R_+$ if X has only F-rational singularities. Also, for any $n \geq 0$, the integral closure $\overline{\mathfrak{m}^n}$ of $\mathfrak{m}^n$ is equal to the graded part $R_{\geq n}$ of R of degree $\geq n$ if $\mathcal{L}$ is generated by global sections.

Now the top local cohomology module $H^d_{\mathfrak{m}}(R)$ of R with the support at $\mathfrak{m}$ is k-dual to the graded canonical module $\omega_R = \bigoplus_{n \in \mathbb{Z}} [\omega_R]_n$ of R with the graded pieces

$$[\omega_R]_n = H^0(X, \omega_X \otimes \mathcal{L}^{\otimes n}).$$

It is easy to see that $\omega_X \otimes \mathcal{L}^{\otimes d}$ is globally generated if and only if $[\omega_R]_n = R_{n-d}[\omega_R]_d$ for all $n \gg 0$. We will reduce this equality to a Skoda-type theorem in ω_R.

For an integer $n \geq 0$, we define

$$\tau(\mathfrak{m}^n, \omega_R) := \mathrm{Ann}_{\omega_R}(0^{*\mathfrak{m}^n}_{H^d_{\mathfrak{m}}(R)}) \subseteq \omega_R,$$

[3]A local ring $(R, \mathfrak{m})$ of characteristic $p > 0$ is said to be *F-rational* if every ideal generated by a system of parameters of R is tightly closed. The F-rationality assumption on singularities of X is relaxed to the F-injectivity as we will see in 3.8.

the annihilator of $0^{*\mathfrak{m}^n}_{H^d_{\mathfrak{m}}(R)}$ in ω_R with respect to the duality pairing $\omega_R \times H^d_{\mathfrak{m}}(R) \to k$. Then it follows from Proposition 2.6, together with Proposition 1.3 (3), that

$$\tau(\mathfrak{m}^{n+d-1}, \omega_R) = \overline{\mathfrak{m}^n} \cdot \tau(\mathfrak{m}^{d-1}, \omega_R)$$

for all $n \geq 0$. On the other hand, the F-rationality of Spec $R \setminus \{\mathfrak{m}\}$ implies that $\tau(\mathfrak{m}^n, \omega_R) = [\omega_R]_{>n}$ for all $n \gg 0$ ([H, Lemma 2.6]). Thus we have the inclusions

$$R_{\geq n-d}[\omega_R]_{>d-1} \subseteq [\omega_R]_{>n-1} = \tau(\mathfrak{m}^{n-1}, \omega_R) = R_{\geq n-d} \cdot \tau(\mathfrak{m}^{d-1}, \omega_R) \subseteq R_{\geq n-d}[\omega_R]_{>d-1},$$

so that $[\omega_R]_{>n-1} = R_{\geq n-d}[\omega_R]_{>d-1}$. Comparing the graded pieces of the least degree of the both sides, we obtain the required equality $[\omega_R]_n = R_{n-d} \cdot [\omega_R]_d$ for $n \gg 0$. □

3.8. *Geometric reinterpretation.* Recently, the author found a geometric simplification (or even a trivialization!) of the above proof. To begin with, let us recall

Lemma 3.9 (Mumford [M]). *Let $\mathcal{L}$ be a globally generated ample line bundle on an projective variety X and $\mathcal{F}$ a coherent sheaf on X satisfying the vanishing*

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes -i}) = 0 \text{ for } i > 0.$$

Then $\mathcal{F}$ is generated by global sections.

In characteristic zero, the assertion of Theorem 3.6 immediately follows from the Kodaira vanishing theorem applied to Mumford's lemma for $\mathcal{F} = \omega_X \otimes \mathcal{L}^{\otimes \dim X+1}$. In characteristic $p > 0$, we can use the Serre vanishing theorem instead by virtue of the following

Lemma 3.10. *Let X be a projective variety of characteristic $p > 0$ with the Frobenius endomorphism $F \colon X \to X$, and let $\mathcal{L}$ be an invertible sheaf on X. Assume that X has only F-injective singularities.*[4] *If the $\mathcal{O}_X$-module $\mathcal{L} \otimes F^e_*\omega_X \cong F^e_*(\omega_X \otimes \mathcal{L}^{p^e})$ is generated by global sections for some $e \in \mathbb{N}$, then so is $\omega_X \otimes \mathcal{L}$.*

The above lemma follows because the F-injectivity guarantees the surjectivity of the dual Frobenius map $\mathcal{L} \otimes F^e_*\omega_X \cong \mathcal{H}om_X(F^e_*\mathcal{O}_X, \omega_X \otimes \mathcal{L}) \to \omega_X \otimes \mathcal{L}$.

Now, let $\mathcal{L}$ be a globally generated ample line bundle on a projective variety X with only F-injective singularities, and put $\mathcal{F} = \mathcal{L}^{\otimes \dim X+1} \otimes F^e_*\omega_X$ for $e \gg 0$. Then $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes -i}) = H^i(X, \omega_X \otimes \mathcal{L}^{\otimes p^e(\dim X+1-i)}) = 0$ for $i > 0$ by the vanishing theorems of Serre and Grothendieck. Thus we see that $\omega_X \otimes \mathcal{L}^{\otimes \dim X+1}$ is globally generated by Lemmas 3.9 and 3.10.

The author does not know the answer to the following question unless $\dim X = 1$.

Question. Let X be a smooth projective variety of characteristic $p > 0$ and let $\mathcal{L}$ be an ample line bundle on X. Is the rank $p^{e \dim X}$ vector bundle $\mathcal{L}^{\otimes \dim X+1} \otimes F^e_*\omega_X$ generated by global sections for some $e \in \mathbb{N}$?

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