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# Hodge theoretic approach to generalization of Abel's theorem

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The following notes are based on the lectures delivered by the author at "The Arithmetic, Geometry and Topology of Algebraic cycles", June 15-July 4 in 2002 at Morelia, Mexico. It overviews some recent works on filtrations on Chow groups and higher Abel-Jacobi maps done by the author and others. He sincerely thanks the organizers of the conference for givinig him the opportunity.

## Contents



## 1 Abel-Jacobi maps and non-representability of Chow group

We start with recalling the theorem of Abel-Jacobi: For a Riemann surface $X$ the Abel-Jacobi map

$$\rho_X \; : \; A_0(X) \to J(X)$$

is an isomorphism. Here $A_0(X)$ denotes the group of zero-cycles of degree zero on $X$ modulo rational equivalence and $J(X)$ is the Jacobian of $X$, which is defined to be the complex torus (indeed an abelian variety)

$$H^0(X, \Omega^1_X)^*/H_1(X, \mathbb{Z}),$$

the dual of the space of the holomorphic 1-forms on $X$ quotiented by the first homology group of $X$ embedded in the space by integration of forms. Fixing a base point $0 \in X$ we have the formula for $\alpha = \sum_{x \in X} n_x[x] \in A_0(X)$,

$$\rho_X(\alpha) = (\omega \to \sum_{x \in X} n_x \int_0^x \omega) \quad (\omega \in H^0(X, \Omega_X^1)),$$

where $\int_0^x$ is the integration along any chosen path from 0 to $x$.

Now let $X$ be a projective smooth variety over $\mathbb{C}$ and let $CH^r(X)$ be the group of algebraic cycles of codimension $r$ on $X$ modulo rational equivalence. A fundamental problem is to search for a reasonable (particularly Hodge theoretic) theory that provides us a good understanding of the structure of $CH^r(X)$. The first significant step toward this problem was taken by Griffiths who defined the Abel-Jacobi map

$$\rho_X^r \; : \; CH^r(X)_{hom} \to J^r(X)$$

where $CH^r(X)_{hom} \subset CH^r(X)$ is the subgroup of those cycle classes that are homologically equivalent to zero (namely whose cohomology classes are trivial) and $J^r(X)$ is the intermediate Jacobian of $X$, which is defined to be the complex torus (not an abelian variety in general)

$$F^{m+1}H^{2m+1}(X, \mathbb{C})^* / H_{2m+1}(X, \mathbb{Z}), \quad (m = \dim(X) - r)$$

where $F^pH^q(X, \mathbb{C})$ denotes the Hodge filtration on $H^q(X, \mathbb{C})$. For $\alpha \in CH^r(X)_{hom}$ we have the formula

$$\rho_X^r(\alpha) = (\omega \to \int_\Gamma \omega) \quad (\omega \in F^{m+1}H^{2m-1}(X, \mathbb{C})),$$

where $\Gamma$ is a topological $2m+1$-chain such that $\partial\Gamma = \alpha$ and $\omega \in F^{m+1}H^{2m-1}(X, \mathbb{C})$ is represented by a harmonic form on $X$. We note that in case $r = d := \dim(X)$, $CH^d(X)_{hom} = A_0(X)$, the group of zero-cycles of degree zero on $X$ modulo rational equivalence, and $J^d(X) = Alb(X)$, the Albanese variety, and $\rho_X^d$ is the so-called Albanese map.

If Griffiths Abel-Jacobi map were an isomorphism, there would not be much to explore in the world of algebraic cycles. It is not the case due to the following theorem of Mumford [M].

**Theorem 1.1** *Let $X$ be a projective smooth surface over $\mathbb{C}$ with $H^0(X, \Omega_X^2) \neq 0$. Then $CH^2(X)_{hom} = A_0(X)$ is infinite dimensional. In particular* $\mathrm{Ker}(\rho_X^2)$ *is very large.*

I will give a precise definition of infinite dimensionality later in 2.3. The above theorem implies that Chow groups in general are too large to be represented by ordinary algebro-geometric structure. Even though the result may be now considered an origin of the whole edifice of theory of mixed motives, a pessimistic feeling should have spread over the study of algebraic cycles in those times. With his ingenious insight S. Bloch shed new light by proposing the following conjecture.

**Conjecture 1.2** *Let $X$ be a projective smooth surface over $\mathbb{C}$ with $H^0(X, \Omega_X^2) = 0$. Then $\rho_X^2$ is an isomorphism.*

Note $H^0(X, \Omega_X^2) = F^2H^2(X, \mathbb{C})$. The conjecture suggests that even though the structure of Chow group seems chaotic, it is mysteriously related with Hodge structure.

## 2 Bloch-Beilinson filtration on Chow groups

An important observation is the following natural isomorphism discovered by Carlson [Ca],

$$J^r(X) \xrightarrow{\sim} \mathrm{Ext}_{MHS}(\mathbb{Z}, H^{2r-1}(X, \mathbb{Z}(r)),$$

where $MHS$ denotes the category of mixed Hodge structures introduced by Deligne [D]. This implies that an element of $CH^r(X)_{hom}/\mathrm{Ker}(\rho_X^r)$ is captured by an extension in $MHS$. One may then have a naive expectation that there may be a secondary cycle class map from $\mathrm{Ker}(\rho_X^r)$ to higher extension groups $\mathrm{Ext}^p_{MHS}$, which fails due to the fact that $\mathrm{Ext}^p_{MHS} = 0$ for $p \geq 2$. It was A.Beilinson who got an innovative idea to remedy the situation. He postulates the existence of such a suitable category (called the category of mixed motives) that all elements of Chow groups are captured by higher extensions in the category. A more precise formulation is the following conjecture.

In the rest of these notes we neglect torsion: If $M$ is an abelian group we let $M$ denote $M \otimes \mathbb{Q}$ by abuse of notation.

**Conjecture 2.1** *For every projective smooth variety $X$ over $\mathbb{C}$, there exists a canonical filtration (called Bloch-Beilinson filtration)*

$$CH^r(X) = F^0_{\mathcal{M}}CH^r(X) \supset F^1_{\mathcal{M}}CH^r(X) \supset F^2_{\mathcal{M}}CH^r(X) \supset \cdots$$

*such that the following formula holds for each integer $\nu \geq 0$:*

$$F^\nu_{\mathcal{M}}CH^r(X)/F^{\nu+1}_{\mathcal{M}}CH^r(X) \xrightarrow{\sim} \mathrm{Ext}^\nu_{\mathcal{MM}_{\mathbb{C}}}(1, h^{2r-\nu}(X)(r)).$$

*Here $\mathcal{MM}_{\mathbb{C}}$ denotes the (conjectural) category of mixed motives over $\mathbb{C}$ which contains as a full subcategory Grothendieck's category $\mathcal{M}_{\mathbb{C}}$ of (pure) motives over $\mathbb{C}$, $h^*(X)(r) \in \mathcal{M}_{\mathbb{C}}$ denotes the cohomological object with Tate twist associated to $X$ and $1 = h^0(\mathrm{Spec}(k))$.*

In [Sa2] and [J] it is proved that the Bloch-Beilinson filtration is unique if it exists under the assumption of the standard conjectures. Several candidates for Bloch-Beilinson filtrations exist. Among those we adopt the filtration

$$F^\nu_{BM}CH^r(X) \subset CH^r(X) \quad (\nu \geq 0)$$

defined in [Sa2], Def.(1-3). In this section we denote $F^\nu_{BM}CH^r(X)$ simply by $F^\nu CH^r(X)$. The definition of the filtration will be included in the appendix. It satisfies the following properties (cf. [Sa1] and [Sa2]):

($F1$) $F^0CH^r(X) = CH^r(X)$ and $F^1CH^r(X) = CH^r(X)_{hom}$.

($F2$) $F^2CH^r(X) \subset \mathrm{Ker}(\rho_X^r)$ and $F^2CH^r(X) \cap CH^r(X)_{alg} = \mathrm{Ker}(\rho_X^r) \cap CH^r(X)_{alg}$, where $CH^r(X)_{alg} \subset CH^r(X)$ is the subgroup of those cycle classes that are algebraically equivalent to zero.

($F3$) $F^\cdot CH^r(X)$ is respected by the action of algebraic correspondences.

($F4$) The induced action of algebraic correspondences on the associated graded module $Gr_F CH^r(X)$ factors through the homological equivalence.

(F5) $F^N CH^r(X) = F^{r+1}CH^r(X)$ for any $N \geq r+1$.

The existence of a Bloch-Beilinson filtration would implies the following conjecture.

**Conjecture 2.2** $F^{r+1}CH^r(X) = 0$.

If $X$ is a projective smooth surface over $\mathbb{C}$, $F^1CH^2(X) = A_0(X)$ and $F^2CH^2(X) = \mathrm{Ker}(\rho_X^2)$ by $(F1)$ and $(F2)$. One immediately sees from the definition that $H^0(X, \Omega_X^2) = 0$ implies $F^2CH^2(X) = F^3CH^2(X)$. Thus 2.2 is a generalization of Bloch's conjecture 1.2.

Now we state a result which refines 1.1 by using the above filtration. For this we need refine the notion of infinite dimensionality for Chow groups introduced in 1.1 by Mumford. Indeed we define certain hierarchy of infinite dimensionality. Let $X$ be a projective smooth variety of dimension $d$ over $\mathbb{C}$. Here we are concerned only with $CH^d(X) = CH_0(X)$, the group of zero-cycles on $X$ modulo rational equivalence. In [Sa1] more general cases are treated. Note that $F^1CH_0(X) = A_0(X)$ and $F^2CH_0(X)$ is the kernel of the Albanese map $A_0(X) \to Alb(X)$ by $(F1)$ and $(F2)$.

**Definition 2.3** *Let $X$ be a projective smooth variety of dimension $d$ over $\mathbb{C}$. Fix subgroups $F$ and $S$ of $A_0(X)$. For an integer $\mu \geq 0$, $S$ is of rank $\leq \mu$ mod $F$ if there exists $Y$, (possibly reducible) smooth projective of dimension $\leq \mu$ over $\mathbb{C}$ and a morphism $f : Y \to X$ such that*

$$S \subset F + Image(A_0(Y) \xrightarrow{f_*} A_0(X)).$$

*Let $rk(S \bmod F)$ be the minimal integer $\mu \geq 0$ for which $S$ is of rank $\leq \mu$ mod $F$. In case $F \subset S$ we also denote $rk(S/F) = rk(S \bmod F)$.*

We have the equivalence of the conditions

$$rk(A_0(X)) > 1 \Leftrightarrow A_0(X) \text{ is infinite dimensional in the sense of [M].}$$

One can prove that if $S$ is weakly representable modulo $F$ (which means that there is an abelian variety $A$ over $\mathbb{C}$ and a reasonable map $\rho : A_0(X) \to A(\mathbb{C})$ such that $\mathrm{Ker}(\rho) \cap S = F \cap S$, see [Sa1], Def.5.1), then $rk(S \bmod F) \leq 1$. We refer the readers to [Sa1] for the proof of the following theorems.

**Theorem 2.4** *Let $X$ be a projective smooth surface over $\mathbb{C}$. The following conditions are equivalent:*

(1) $F^2CH_0(X) = F^3CH_0(X)$.

(2) $rk(F^2CH_0(X)/F^3CH_0(X)) \leq 1$.

(3) $rk(A_0(X)/F^3CH_0(X)) \leq 1$.

(4) $H^0(X, \Omega_X^2) = 0$.

Note that 2.4 strengthens 1.1 which is equivalent to the implication

$$rk(F^2CH_0(X)) \leq 1 \Rightarrow H^0(X, \Omega_X^2) = 0.$$

There exists the generalization of 2.4 to higher dimensional case.

**Theorem 2.5** *Let $X$ be a projective smooth variety of dimension $d$ over $\mathbb{C}$. Assume that $B(X)$ (cf. below) holds. For an integer $\nu \geq 2$, the following conditions are equivalent:*

(1) $F^\nu CH_0(X) = F^{\nu+1}CH_0(X)$.

(2) $rk(F^\nu CH_0(X)/F^{\nu+1}CH_0(X)) \leq \nu - 1$.

(3) $rk(A_0(X)/F^{\nu+1}CH_0(X)) \leq \nu - 1$.

(4) $H^\nu(X, \mathbb{Q}) = N^1 H^\nu(X, \mathbb{Q})$, *where* $N^p H^*(X, \mathbb{Q}) \subset H^*(X, \mathbb{Q})$ *is the coniveau filtration (cf. 6.2 for the definition).*

By definition $B(X)$ holds if the inverse of the hard Lefschetz isomorphism

$$L^{d-q} \ : \ H^q(X, \mathbb{Q}) \xrightarrow{\sim} H^{2d-q}(X, \mathbb{Q}) \quad \text{for } \forall q \leq d := \dim(X),$$

is algebraic, namely induced by an algebraic correspondence (cf. section 6), where $L \in H^2(X, \mathbb{Q})$ is the class of an ample line bundle. Recall that $B(X)$ is the so-called hard Lefschetz conjecture and is a part of the standard conjectures. The following facts are known (cf. [K]). Let $\mathcal{C}$ be the category of smooth projective varieties over $\mathbb{C}$.

($B1$) Let $X \in \mathcal{C}$. The Hodge conjecture for $X \times X$ implies $B(X)$.

($B2$) If $B(X)$ and $B(Y)$ hold for $X, Y \in \mathcal{C}$, then $B(X \times Y)$ holds.

($B3$) If $B(X)$ holds for $X \in \mathcal{C}$ and $Y \subset X$ is a smooth hypersurface section, then $B(Y)$ holds.

($B4$) If $B(X)$ holds in the following cases:

  ($i$) $\dim(X) \leq 2$.

  ($ii$) $X$ is a smooth complete intersection in a projective space.

  ($iii$) $X$ is an abelian variety.

  ($iv$) $X$ is a flag variety.

# 3 Mumford's invariants

Now the following problems arise.

($I$) Find a tractable invariant to capture cycles in $F^\nu CH^r(X)$ with $\nu \geq 2$.

($II$) Relate the invariant to higher extensions in a suitable category.

Of course the second problem arises naturally in view of Beilinson's conjecture 2.1. We will discuss it in sections 4 and 5. In this section we discuss the first problem. In his seminal work [M] Mumford used holomorphic 2-forms to capture zero-cycles on a surface. C. Voisin [V] generalized it to families of zero-cycles on surfaces. The purpose of this section is to generalize those constructions and to associate to a cycle in $F^\nu CH^r(X)$ its class in the cohomology of certain complex arising arithmetic Gauss-Manin connection, which we call Mumford invariants. Such theory of cycle classes has been formulated as a conjecture by Green and Griffiths where they use the conjectural filtration of Bloch-Beilinson (see [G]).

For that purpose we use another filtration (cf. [Sa2], Def.(1-2))

$$F_B^\nu CH^r(X) \subset F_{BM}^\nu CH^r(X) \subset CH^r(X) \quad (\nu \geq 0),$$

which is a slight modification of $F_{BM}^\nu CH^r(X)$ used in the previous section (the definition of $F_B^\nu CH^r(X)$ will be recalled in the appendix). It satisfies $(F1)$ through $(F4)$ in the previous section. We have the following fact (cf. [Sa2], Thm.(1-1)).

**Proposition 3.1** *We have $F_B^\nu CH^r(X) = F_{BM}^\nu CH^r(X)$, assuming the homological and numerical equivalences coincide.*

First we introduce some notations. In what follows $f : X \to \mathrm{Spec}(\mathbb{C})$ denotes a smooth projective variety over $\mathbb{C}$. Let $\Omega^1_{\mathbb{C}/\overline{\mathbb{Q}}}$ be the differential module of $\mathbb{C}$ over $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, and put $\Omega^p_{\mathbb{C}/\overline{\mathbb{Q}}} = \overset{p}{\wedge}\Omega^1_{\mathbb{C}/\overline{\mathbb{Q}}}$. Note that $\Omega^p_{\mathbb{C}/\overline{\mathbb{Q}}}$ is a complex vector space of infinite dimension if $p \geq 1$. Let

$$H^q_{DR}(X/\mathbb{C}) = \mathbb{H}^q(X, \Omega^{\cdot}_{X/\mathbb{C}}) \quad \text{and} \quad F^p H^q_{DR}(X/\mathbb{C}) = \mathbb{H}^q(X, \Omega^{\geq p}_{X/\mathbb{C}}) \subset H^q_{DR}(X/\mathbb{C})$$

be the de Rham cohomology of $X/\mathbb{C}$ and its Hodge filtration. Let

$$\nabla \ : \ H^q_{DR}(X/\mathbb{C}) \to \Omega^1_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes H^q_{DR}(X/\mathbb{C})$$

be the arithmetic Gauss-Manin connection, that can be defined to be the boundary map arising from the exact sequence of complexes of sheaves on $X$

$$0 \to f^*\Omega^1_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes \Omega^{\cdot}_{X/\mathbb{C}}[-1] \to \Omega^{\cdot}_{X/\overline{\mathbb{Q}}}/F^2\Omega^{\cdot}_{X/\overline{\mathbb{Q}}} \to \Omega^{\cdot}_{X/\mathbb{C}} \to 0,$$

where $\Omega^q_{X/\overline{\mathbb{Q}}}$ denotes the sheaf of differentials of $X$ over $\overline{\mathbb{Q}}$ and $F^p\Omega^{\cdot}_{X/\overline{\mathbb{Q}}}$ denotes the image of $f^*\Omega^p_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes \Omega^{\cdot-p}_{X/\overline{\mathbb{Q}}} \to \Omega^{\cdot}_{X/\overline{\mathbb{Q}}}$. We extend $\nabla$ to

$$\nabla \ : \ \Omega^\nu_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes H^q_{DR}(X/\mathbb{C}) \to \Omega^{\nu+1}_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes H^q_{DR}(X/\mathbb{C})$$

by the formula

$$\nabla(\omega \otimes \eta) = d\omega \otimes \eta + (-1)^\nu \omega \otimes \nabla\eta.$$

Two basic facts are:

(1) (flatness) $\nabla^2 = 0$

(2) (transversality) $\nabla(F^p H^q_{DR}(X/\mathbb{C})) \subset \Omega^1_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes F^{p-1} H^q_{DR}(X/\mathbb{C})$.

**Definition 3.2** *For integers $r, \nu \geq 0$ the space $\nabla J^{r,\nu}(X)$ of Mumford invariants is defined to be the cohomology of the following complex:*

$$\Omega^{\nu-1}_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes F^{r-\nu+1} H^{2r-\nu}_{DR}(X/\mathbb{C}) \xrightarrow{\nabla} \Omega^{\nu}_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes F^{r-\nu} H^{2r-\nu}_{DR}(X/\mathbb{C}) \xrightarrow{\nabla} \Omega^{\nu+1}_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes F^{r-\nu-1} H^{2r-\nu}_{DR}(X/\mathbb{C})$$

**Proposition 3.3** *There exists a natural map*

$$\phi_X^{r,\nu} \;:\; F^\nu CH^r(X) \to \nabla J^{r,\nu}(X)$$

*which is functorial for a morphism $Y \to X$ of projective smooth varieties over $\mathbb{C}$ and satisfies $\phi_X^{r,\nu}(F^{\nu+1}CH^r(X)) = 0$.*

In case $\nu = 1$ $\phi_X^{r,\nu}$ is related to the Griffiths Abel-Jacobi map $\rho_X^r$ as follows. It is the Griffiths construction of infinitesimal invariants of normal functions. We note

$$J^r(X) = H^{2r-1}(X,\mathbb{C})/F^{r-1}H^{2r-1}(X,\mathbb{C}) + H^{2r-1}(X,\mathbb{Z}(r)).$$

We have the comparison isomorphism $H^q(X,\mathbb{C}) \xrightarrow{\sim} H^q_{DR}(X/\mathbb{C})$ preserving the Hodge filtrations and the arithmetic Gauss-Manin connection $\nabla$ annihilates the image of the subspace $H^q(X,\mathbb{Q}(r)) \subset H^q(X,\mathbb{C})$. Hence $\nabla$ induces

$$\tau \;:\; J^r(X) \to \nabla J^{r,1}(X)$$

and one can check that $\phi_X^{r,1} = \tau \circ \rho_X^r$.

The map $\phi_X^{r,\nu}$ can be defined by the same idea of the definition of the infinitesimal invariants of higher normal functions introduced in [Sa3], which generalizes Griffiths normal functions and infinitesimal invariants. It is based on the theory of cycle classes in Deligne cohomology. In the next section we will explain a refinement of the above cycle class map by using the theory of cycle classes in higher extension groups in the category of arithmetic Hodge structures developed by Green-Griffiths, M.Asakura and M.Saito (cf. [A1], 3.2 and 4.2 and [MSa2]).

Here we give a brief explanation of the construction of $\phi_X^{r,\nu}$. We start with the cycle class map defined by El-Zein [EZ]

$$c_{EZ}^r \;:\; CH^r(X) \to \mathbb{H}^r(X, \Omega^{\geq r}_{X/\overline{\mathbb{Q}}}),$$

where the important point is to use $\Omega^*_{X/\overline{\mathbb{Q}}}$ instead of $\Omega^*_{X/\mathbb{C}}$. There is a filtration

$$\Omega^{\geq r}_{X/\overline{\mathbb{Q}}} \supset F^p\Omega^{\geq r}_{X/\overline{\mathbb{Q}}} := Image(f^*\Omega^p_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes \Omega^{\geq r-p}_{X/\overline{\mathbb{Q}}}[-p]). \quad (f : X \to \mathrm{Spec}(\mathbb{C}))$$

Its graded quotients are given by

$$Gr_F^p\Omega^{\geq r}_{X/\overline{\mathbb{Q}}} = f^*\Omega^p_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes \Omega^{\geq r-p}_{X/\mathbb{C}}[p].$$

In view of

$$H^q(X, f^*\Omega^p_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes \Omega^{\geq r-p}_{X/\mathbb{C}}) \xrightarrow{\sim} \Omega^p_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes F^{r-p}H^q_{DR}(X/\mathbb{C}),$$

we get the spectral sequence

$$E_1^{p,q} = H^{p+q}(X, Gr_F^p\Omega^{\geq r}_{X/\overline{\mathbb{Q}}}) = \Omega^p_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes F^{r-p}H^q_{DR}(X/\mathbb{C}) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^{\geq r}_{X/\overline{\mathbb{Q}}})$$

whose differential $d_1^{p,q}$ is identified with $\nabla$ by [KO]. Thus the construction of $\phi_X^{r,\nu}$ is reduced to show the following facts which are shown by the same argument as [Sa3], Lem.(1-1) and Prop.(2-1):

(1) The above spectral sequence degenerates at $E_2$.

(2) $\phi_X^{r,\nu}(F^\nu CH^r(X)) \subset F^\nu \mathbb{H}^r(X, \Omega^{\geq r}_{X/\overline{\mathbb{Q}}})$, where the last group denotes the filtration on $\mathbb{H}^r(X, \Omega^{\geq r}_{X/\overline{\mathbb{Q}}})$ associated to the spectral sequence.

Now we present a theorem implying that $\phi_X^{r,\nu}$ with $\nu \geq 1$ is able to detect non-trivial cycles in the kernel of Griffiths Abel-Jacobi map.

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d$ and of dimension $n$. By the Lefschetz theory the restriction map $H^i(\mathbb{P}^{n+1}, \mathbb{Q}) \to H^i(X, \mathbb{Q})$ is an isomorphism unless $i = n$. By the definition of $F^\nu CH^r(X)$ it implies that if $2r \geq n+1$,

(1) $CH^r(X)_{hom} = F^1 CH^r(X) = F^{2r-n} CH^r(X)$.

(2) $F^N CH^r(X) = F^{2r-n+1} CH^r(X) \quad$ for $\forall N \geq 2r - n + 1$.

Thus $F^\nu CH^r(X)/F^{\nu+1} CH^r(X)$ is interesting only if $\nu = 2r - n$. Now we consider the case $r = n-1$, namely the case of one cycles on $X$.

Let $\ell \subset X$ be a line on $X$ and consider

$$\beta_\ell := d \cdot [\ell] - [X \cap L] \in CH^{n-1}(X)_{hom} = F^{n-2} CH^{n-1}(X),$$

where $L \subset \mathbb{P}^{n+1}$ is a linear subspace of dimension 2 intersecting properly with $X$. We are interested in the image of $\beta_\ell$ under

$$\phi := \phi_X^{n-1,n-2} \; : \; F^{n-2} CH^{n-1}(X) \to \nabla J^{n-1,n-2}(X).$$

**Theorem 3.4** *([Sa3], Thm.(0-1))*

(1) *If $d \geq n \geq 3$ and $X$ is general with respect to $\ell$, then $\phi(\beta_\ell) \neq 0$ and hence $\beta_\ell \notin F^{n-1} CH^{n-1}(X)$.*

(2) *Assume $d \geq 2n-1$ and $X$ is general. Then $\phi(CH^{n-1}(X)_{alg}) = 0$.*

By [Bo], Prop.2.1 there exists always a line $\ell \subset X$ if $d \leq 2n-1$. Thus we get the following.

**Corollary 3.5** *For every line $\ell$ on a general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d = 2n-1$,*

$$\beta_\ell \notin F^{n-1} CH^{n-1}(X) + CH^{n-1}(X)_{alg}.$$

**Remark 3.6** (1) *In case $n = 3$ and $d = 5$ (quintic 3-folds) 3.5 is due to Griffiths, who found the first example where the homological and algebraic equivalence do not coincide even modulo torsion.*

(2) *If $n \geq 4$, the intermediate Jacobian $J^{n-1}(X)$ vanishes so that it is not possible to capture $\beta_\ell$ by using the Griffiths Abel-Jacobi map.*

(3) *In [N] Nori captured a nontrivial cycle in the kernel of the Griffiths Abel-Jacobi map by using the de Rham cohomology of the total space of a family of projective smooth varieties over a parameter sapce $S$.*

Here we explain the idea of the proof of 3.4 very briefly. We refer the readers to [Sa3] for the details. As for (1) we use the induction on $n$. We take a hyperplane $H \subset \mathbb{P}^{n+1}$ such that $\ell \subset H$ and that $H$ intersects transversally $X$. Put $Y = X \cap H$. The first key step is the construction of the commutative diagram

$$\begin{array}{ccc} F^{n-3}CH^{n-2}(Y) & \xrightarrow{\phi_Y^{n-2,n-3}} & \nabla J^{n-2,n-3}(Y) \\ \downarrow i_* & & \downarrow i_* \\ F^{n-2}CH^{n-1}(X) & \xrightarrow{\phi_X^{n-1,n-2}} & \nabla J^{n-1,n-2}(X) \end{array}$$

where the vertical maps are the Gysin maps for the immersion $i : Y \to X$. By definition $\beta_\ell \in F^{n-2}CH^{n-1}(X)$ lies in the image of $i_*$ on the left hand side. The key point is to show the fact: if $H$ is a general hyperplane containing $\ell$, $i_*$ on the right hand side is injective. Then 3.4(1) is reduced to the same assertion for $Y$ and the induction proceeds. By the definition of $i_*$ on the right hand side, the injectivity follows from the exactness at the middle of the complex:

$$\Omega^{n-4}_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes F^3H^n_{DR}(U/\mathbb{C}) \xrightarrow{\nabla} \Omega^{n-3}_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes F^2H^n_{DR}(U/\mathbb{C}) \xrightarrow{\nabla} \Omega^{n-2}_{\mathbb{C}/\overline{\mathbb{Q}}} \otimes F^1H^n_{DR}(U/\mathbb{C})$$

where $U = X - Y$ and $F^pH^q_{DR}(U/\mathbb{C}) = \mathbb{H}^q(X, \Omega^{\geq p}_{X/\mathbb{C}}(\log Y))$ with $\Omega^{\cdot}_{X/\mathbb{C}}(\log Y)$, the complex of the sheaves of differentials of $X$ over $\mathbb{C}$ with logarithmic poles along $Y$. It can be controlled by using (generalized) Jacobian rings that provide an algebraic description of the infinitesimal part of variation of mixed Hodge structures associated to the cohomology of the universal family of the moduli space of $U$.

The key idea to show 3.4(2) is to define a subspace

$$\nabla J^{r,\nu}_{alg}(X) \subset \nabla J^{r,\nu}(X),$$

which we would call the algebraic part of $\nabla J^{r,\nu}(X)$. The idea originates from the theory of formal tangent spaces of Chow groups which was developed by Bloch and Stienstra. It is also inspired by a Hodge theoretic study of this subject done by Green-Griffiths (cf. [G]). The desired assertion then follows from the following:

(1) $\phi^{r,\nu}_X(F^\nu CH^r(X) \cap CH^r(X)_{alg}) \subset \nabla J^{r,\nu}_{alg}(X)$.

(2) If $X$ is general and $d(n-r) \geq 2r+1$, then $\nabla J^{r,\nu}_{alg}(X) = 0$.

These are shown by the same argument as [Sa3], section 4.

# 4 Mumford invariants as higher extensions

The purpose of this section is to relate the Mumford invariants to higher extensions in a suitable category following the works of Green-Griffiths [G] and Asakura [A1], 3.2. It is motivated by the Beilinson's formula in 2.1. Let $R$ be a smooth algebra over $\overline{\mathbb{Q}}$ and let $D_R \subset \mathrm{End}_{\overline{\mathbb{Q}}}(R)$ be the ring of differential operators generated as a subring by $\Theta_R = \mathrm{Der}_{\overline{\mathbb{Q}}}(R, R)$ (derivations) and $R$ (scalars). If $R$ has a local coordinate $\{x_i\}_{1 \leq \leq n}$ and $\{\partial_i\}_{1\leq i \leq n}$ is the dual basis of $\{dx_i\}_{1 \leq i \leq n}$, we have

$$D_R = \bigoplus_{\alpha \in \mathbf{N}} R \cdot \partial^\alpha \quad (\partial^\alpha := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}).$$

It is endowed with the filtration of differential order

$$F_p D_{\mathbb{C}} = \bigoplus_{|\alpha| \le p} R \cdot \partial^\alpha \quad (|\alpha| = \sum_{i=1}^m \alpha_i).$$

A filtered $D_R$-module is a pair $(M, F)$ of a $D_R$-module $M$ and an increasing filtration of *finite* $R$-modules $F_p M \subset M$ $(p \in \mathbb{Z})$ satisfying

(1) $M = \cup_{p \in \mathbb{Z}} F_p M \quad (F_p M = 0\, p << 0)$,

(2) $F_p D_R \cdot F_q M \subset F_{p+q} M$,

(3) $F$ is a *good filtration,* namely there exists $q_0 \in \mathbb{Z}$ such that

$$F_p D_R \cdot F_q M \subset F_{p+q} M \quad \text{for } \forall p \text{ and } \forall q \ge q_0.$$

We remark that we may drop the last condition (3) to get the same result in this section. Let $\mathcal{MF}_R$ be the category of filtered $D_R$-modules. For an object $M = (M, F)$ of $\mathcal{MF}_R$ its Tate twist $M(r)$ is defined to be $(M, F(r))$ with $F(r)_p = F_{p-r}$. Every morphism in $\mathcal{MF}_R$ has a kernel and a cokernel but its image and coimage are not isomorphic in general. Thus $\mathcal{MF}_R$ is not an abelian category but it becomes an exact category by defining a complex in $\mathcal{MF}_R$

$$(M_1, F_1) \to (M_2, F_2) \to (M_2, F_2)$$

to be exact if and only if

$$Gr^{F_1} M_1 \to Gr^{F_2} M_2 \to Gr^{F_3} M_3$$

is an exact sequence of $R$-modules. Thus we can consider higher extension groups in the sense of Yoneda in $\mathcal{MF}_R$.

Let $f : X \to \mathrm{Spec}(R)$ be a smooth projective morphism. The algebraic de Rham cohomology $M = H^q_{DR}(X/R) = R^q f_* \Omega^\cdot_{X/R}$ with the filtration $F_p M := F^{-p} H^q_{DR}(X/R)$ gives rise to an object of $\mathcal{MF}_R$: We let $\theta \in \Theta_R$ act on $M$ via $\nabla_\theta$, the covariant derivative of $\theta$ with respect to the algebraic Gauss-Manin connection

$$\nabla : H^q_{DR}(X/R) \to \Omega^1_{R/\overline{\mathbb{Q}}} \otimes H^q_{DR}(X/R).$$

**Proposition 4.1** *(cf. [G] and [A1]) For integers $p, q \ge 0$, $\mathrm{Ext}^p_{\mathcal{MF}_R}(R, H^q_{DR}(X/R)(r))$ ($R$ is considered as a $D_R$-module via the augmentation $D_R \to R$ and endowed with the filtration $F_p R = 0$ if $p < 0$ and $F_p R = R$ if $p \ge 0$) is isomorphic to the cohomology of the following complex*

$$\Omega^{p-1}_{R/\overline{\mathbb{Q}}} \otimes F^{r-p+1} H^q_{DR}(X/R) \xrightarrow{\nabla} \Omega^p_{R/\overline{\mathbb{Q}}} \otimes F^{r-p} H^q_{DR}(X/\mathbb{C}) \xrightarrow{\nabla} \Omega^{p+1}_{R/\overline{\mathbb{Q}}} \otimes F^{r-p-1} H^q_{DR}(X/R)$$

For a morphism $\phi : R \to S$ of smooth $\overline{\mathbb{Q}}$-algebras we have the functor

$$\mathcal{MF}_R \to \mathcal{MF}_S \; ; \; (M, F) \to (S \otimes_R M, S \otimes_R F),$$

Here $S \otimes_R M$ is endowed with structure of $D_S$-module in such a way that if $\{x_i\}_{1 \le \le n}$ is a local coordinate on $R$ and $\{\partial_i\}_{1 \le i \le n}$ is the dual basis of $\{dx_i\}_{1 \le i \le n}$,

$$\theta(a \otimes m) = \theta(a) \otimes m + \sum_{i=1}^{n} \theta(\phi(x_i)) \otimes \partial_i m \quad (a \in S,\ m \in M, \theta \in \Theta_S).$$

Now we put

$$\mathcal{MF}_{\mathbf{C}} = \varinjlim_{R \subset \mathbf{C}} \mathcal{MF}_R.$$

where $R$ ranges over the subalgebras of $\mathbb{C}$ which are smooth over $\overline{\mathbb{Q}}$. Let $X$ be a smooth projective variety over $\mathbb{C}$. By the above construction $H^*_{DR}(X/\mathbb{C})$ gives rise to an object of $\mathcal{MF}_{\mathbf{C}}$. 4.1 immediately implies the following.

**Corollary 4.2** *For integers $r, \nu \ge 0$ there is a canonical isomorphism*

$$\nabla J^{r,\nu}(X) \xrightarrow{\sim} \mathrm{Ext}^{\nu}_{\mathcal{MF}_{\mathbf{C}}}(\mathbb{C}, H^{2r-\nu}_{DR}(X/\mathbb{C})(r)).$$

*Proof of 4.1.* For simplicity we assume that $R$ has a local coordinate $\{x_i\}_{1 \le \le n}$. Let $\{\partial_i\}_{1 \le i \le n}$ be the dual basis of $\{dx_i\}_{1 \le i \le n}$. We have the Koszul resolution of $R$ as a filtered $D_R$-module:

$$0 \to D_R(-n) \otimes \overset{n}{\wedge} V \to \cdots \to D_R(-1) \otimes V \to D_R \to R \to 0,$$

where $V$ is a $\mathbb{Q}$-vector space with a basis $\{e_i\}_{1 \le i \le n}$ and the boundary maps are given by

$$D_R(-p) \otimes \overset{p}{\wedge} V \to D_R(-p+1) \otimes \overset{p-1}{\wedge} V \ ; \ \xi \otimes e_{i_1} \wedge \cdots \wedge e_{i_p} \to \sum_{j=1}^{p} (-1)^j \xi \cdot \partial_{i_j} \otimes e_{i_1} \wedge \cdots \widehat{\wedge e_{i_j} \wedge} \cdots \wedge e_{i_p}.$$

It is an exact sequence in $\mathcal{MF}_R$ and 4.1 follows easily from the following standard fact.

**Lemma 4.3** *For $(M, F) \in \mathcal{MF}_R$ we have*

$$\mathrm{Ext}^{\nu}_{\mathcal{MF}_R}(D_R(p), (M, F)) = \begin{cases} F_p M & \text{if } \nu = 0 \\ 0 & \text{otherwise} \end{cases}$$

# 5 Arithmetic Hodge structures

In this section we give a brief explanation of cycle classes in higher extension group in the category of arithmetic Hodge structures due to M. Asakura and M. Saito (cf. [A1] and [MSa2]). In the construction of the previous section, the $\mathbb{Q}$-structure is not taken into account. The theory of arithmetic Hodge structures remedies the defect and gives a refinement of the Mumford invariants.

Let $S$ be a smooth scheme over $\overline{\mathbb{Q}}$. An admissible variation of mixed Hodge structures on $S$ is defined to be a datum $(H_{\mathbf{Q}}, H_{\mathcal{O}}, W_., F^{\cdot}, \nabla, i)$, where

- $H_{\mathbf{Q}}$ is a local system of finite dimensional $\mathbb{Q}$-vector space on $S(\mathbb{C})^{an}$,
- $H_{\mathcal{O}}$ is a locally free Zariski sheaf of $\mathcal{O}_S$-module of finite rank,

- $W_\cdot \subset H_{\mathbb{Q}}$ and $W_\cdot \subset H_{\mathcal{O}}$ are increasing filtrations, called the weight filtration,
- $F^\cdot \subset H_{\mathcal{O}}$ is a decreasing filtration, called the Hodge filtration,
- $\nabla : H_{\mathcal{O}} \to H_{\mathcal{O}} \otimes \Omega^1_S$ is an algebraic connection such that $\nabla^2 = 0$,
- $i : H_{\mathbb{Q}} \otimes \mathcal{O}^{an}_S \xrightarrow{\sim} H_{\mathcal{O}} \otimes \mathcal{O}^{an}_S$ which induces $H_{\mathbb{Q}} \otimes \mathbb{C} \xrightarrow{\sim} \mathrm{Ker}(\nabla^{an})$, called the comparison isomorphism,

and they satisfy the following conditions:

(1) Two weight filtrations are compatible with respect to $i$,

(2) For every point $s \in S(\mathbb{C})$, the fibers $H_{\mathbb{Q},s} \overset{i}{\hookrightarrow} H_{\mathcal{O},s}$ with the induced weight filtration and Hodge filtration define a mixed Hodge structure, namely an object of $MHS$.

(3) (transversality) $\nabla(W_\ell) \subset W_\ell \otimes \Omega^1_S, \quad \nabla(F^p) \subset F^{p-1} \otimes \Omega^1_S \quad$ for $\forall \ell, p$.

(4) (polarrizability) omitted,

(5) (admissibility) omitted.

We let $VMHS(S)$ denote the category of admissible variation of mixed Hodge structures on $S$. Some important remarks are the following:

($i$) $VMHS$ is an abelian category.

($ii$) For a projective smooth morphism $f : X \to S$, we have the associated cohomological object

$$\underline{H}^q(X/S)(r) = (R^q f^{an}_* \mathbb{Q}(r), H^q_{DR}(X/S), \nabla, W_{\cdot+2r}, F^{\cdot+r}),$$

where $f^{an} : X(\mathbb{C}) \to S(\mathbb{C})$, $H^q_{DR}(X/S) = \mathbb{H}^q(X, \Omega^\cdot_{X/S})$, $\nabla$ is the algebraic Gauss-Manin connection, $W_{\cdot+2r}$ (resp. $F^{\cdot+r}$) is the degree shift of the usual weight (resp. Hodge filtrations). In particular it gives rise to the Tate object $\underline{\mathbb{Q}}(r)$ in case $X = S$.

($iii$) For $S = \mathrm{Spec}(R)$ we have the functor

$$r_{MF} \ : \ VMHS(S) \to \mathcal{MF}_R \ ; \ (H_{\mathbb{Q}}, H_{\mathcal{O}}, W_\cdot, F^\cdot, \nabla, i) \to (H_{\mathcal{O}}, F_\cdot),$$

where $F_p = F^{-p}$ and the action of $D_R$ on $H_{\mathcal{O}}$ is given by the action of $\Theta_R$ induced by covariant derivatives with respect to $\nabla$.

For a morphism $T \to S$ of smooth varieties over $\overline{\mathbb{Q}}$, the inverse image functor $VMHS(S) \to VMHS(T)$ is defined. The category of arithmetic Hodge structures is now defined by:

$$\mathcal{MM}^{AH}_{\mathbb{C}} = \varinjlim_{(S,\eta)} VMHS(S),$$

where the limit is taken over the set of smooth schemes $S$ over $\overline{\mathbb{Q}}$ with embeddings $\eta : \overline{\mathbb{Q}}(S) \hookrightarrow \mathbb{C}$. By the above construction $\mathcal{MM}^{AH}_{\mathbb{C}}$ is an abelian category with the Tate object $\underline{\mathbb{Q}}(r)$ and the cohomological object $\underline{H}^q(X/\mathbb{C})(r)$ associated to a projective smooth variety $X$ over $\mathbb{C}$. It is endowed with the functors

$$r_{MF} \ : \ \mathcal{MM}^{AH}_{\mathbb{C}} \to \mathcal{MF}_{\mathbb{C}} \quad \text{and} \quad r_H \ : \ \mathcal{MM}^{AH}_{\mathbb{C}} \to MHS,$$

where the first one is induced by the functor in (*iii*) and the second is obtained by taking the fibers over the point of $S(\mathbb{C})$ corresponding to the given $\eta$ in view of (2). The proof of the following theorem due to M. Asakura and M. Saito requires the theory of mixed Hodge modules [MSa1], which is beyond scope of these notes.

**Theorem 5.1** *There exists a cycle map*

$$\rho_X^{r,\nu} : F^\nu CH^r(X) \to \mathrm{Ext}^\nu_{\mathcal{MM}^{AH}_{\mathbb{C}}}(\underline{\mathbb{Q}}(0), \underline{H}^{2r-\nu}(X/\mathbb{C})(r))$$

*satisfying the following:*

(1) *The composite of $\rho_X^{r,1}$ with the map*

$$\mathrm{Ext}^1_{\mathcal{MM}^{AH}_{\mathbb{C}}}(\underline{\mathbb{Q}}(0), \underline{H}^{2r-1}(X/\mathbb{C})(r)) \to \mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), H^{2r-1}(X, \mathbb{Q}(r))) \simeq J^r(X)$$

*coincides with the Griffiths Abel-Jacobi map $\rho_X$, where the first map is induced by $r_H$ and the second is Carlson's isomorphism.*

(2) *The composite of $\rho_X^{r,\nu}$ with the map*

$$\mathrm{Ext}^\nu_{\mathcal{MM}^{AH}_{\mathbb{C}}}(\underline{\mathbb{Q}}(0), \underline{H}^{2r-\nu}(X/\mathbb{C})(r)) \to \mathrm{Ext}^\nu_{\mathcal{MF}_{\mathbb{C}}}(\mathbb{C}, H^{2r-\nu}_{DR}(X/\mathbb{C})(r)) \simeq \nabla J^{r,\nu}(X)$$

*coincides with $\phi_X^{r,\nu}$, where the first map is induced by $r_{MF}$ and the second is the isomorphism of 4.2.*

Finally we remark that there exist cycles whose Mumford invariants vanish but whose images under the refined cycle maps do not vanish (cf. [A2], Thm.1.3, [RS]).

# 6 Appendix

In this section we recall the definitions of two filtrations on Chow groups used in these notes. They are excerpted from [Sa2], §1. First we introduce notations.

Let $\mathcal{C}$ be the category of smooth projective varieties over $\mathbb{C}$. For $X \in \mathcal{C}$ we denote $H^i(X) = H^i(X, \mathbb{Q})$ for simplicity. For $V, X \in \mathcal{C}$ and for $\Gamma \in CH^q(V \times X)$ let

$$\Gamma : CH^s(V) \to CH^r(X) \quad \text{and} \quad \varphi^i_\Gamma : H^{i-2(r-s)}(V) \to H^i(X) \quad (s = r - q + \dim(V))$$

be the homomorphisms induced by $\Gamma$ as an algebraic correspondence. They are given by the formulae

$$\Gamma_*(\alpha) = (\pi_X)_*((\pi_V)^*(\alpha) \cdot \Gamma) \quad \text{for } \alpha \in CH^s(V),$$

$$\varphi^i_\Gamma(\beta) = (\pi_X)_*((\pi_V)^*(\beta) \cup [\Gamma]) \quad \text{for } \beta \in H^{i-2(r-s)}(V),$$

where $\pi_X : V \times X \to X$ and $\pi_V : V \times X \to V$ are the projections and $[\Gamma] \in H^{2q}(V \times X)$ is the cohomology class of $\Gamma$.

**Definition 6.1** *For $\nu \geq 0$ we define $F^\nu_B CH^r(X)$ for $\forall X \in \mathcal{C}$ and for $\forall r \geq 0$ in the following inductive way:*

(1) *$F^0_B CH^s(V) = CH^s(V)$ for $\forall V \in \mathcal{C}$ and for $\forall s \geq 0$.*

(2) *Assume that we have defined* $F_B^\nu CH^s(V) = CH^s(V)$ *for* $\forall V \in \mathcal{C}$ *and for* $\forall s \geq 0$. *Then we define*

$$F_B^{\nu+1}CH^r(X) = \sum_{V,q,\Gamma} Image(\Gamma_* : F_B^\nu CH^{r+d_V-q}(V) \to CH^r(X)),$$

*where* $V, q, \Gamma$ *range over the following data:*

(a) $V \in \mathcal{C}$ *of dimension* $d_V$,

(b) $r \leq q \leq r + d_V$,

(c) $\Gamma \in CH^q(V \times X)$ *satisfying the condition* $\varphi_\Gamma^{2r-\nu} = 0$, *where*

$$\varphi_\Gamma^{2r-\nu} : H^{2s-\nu}(V) \to H^{2r-\nu}(X) \quad (s = r - q + d_V).$$

**Definition 6.2** *We define* $F_{BM}^\nu CH^r(X)$ *in the same inducitve way as 6.1 except that we replace* (2)(c) *by the following condition:*

(c)′ $\Gamma \in CH^q(V \times X)$ *satisfying the condition*

$$\varphi_\Gamma^{2r-\nu}(H^{2s-\nu}(V)) \subset N^{r-\nu+1}H^{2r-\nu}(X).$$

Here $N^pH^*(X) \subset H^*(X)$ denotes the $p$-th coniveau filtration:

$$N^pH^*(X) = \varinjlim_{\substack{Y \hookrightarrow X \\ codim_X(Y) \geq p}} \mathrm{Ker}(H^*(X) \to H^*(X - Y)),$$

where $Y$ ranges over all closed subschemes of $X$ of codimension$\geq p$.

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