

Bertini theorems and Lefschetz pencils over discrete valuation rings

Uwe Jannsen and Shuji Saito

Good hyperplane sections, whose existence is assured by Bertini's theorem, and good families of hyperplane sections, so-called Lefschetz pencils, are well-known constructions and powerful tools in classical geometry, i.e., for varieties over a field. But for arithmetic questions one is naturally led to the consideration of models over Dedekind rings and, for local questions, to schemes over discrete valuation rings. It is the aim of this note to provide extensions of the mentioned constructions to this situation. We point out some new phenomena, and refer the reader to [JS] for some arithmetic applications.

Let A be a discrete valuation ring with fraction field K , maximal ideal \mathfrak{m} and residue field $F = A/\mathfrak{m}$. Let $\eta = \text{Spec}(K)$ and $s = \text{Spec}(F)$ be the generic and closed point of $\text{Spec}(A)$, respectively. For any scheme X over A we let $X_\eta = X \times_A K$ and $X_s = X \times_A F$ be its generic and special fibre, respectively.

0. Good hyperplane sections for good reduction schemes

As a 'warm-up', we recall the classical Bertini theorem and extend it to varieties over K with good reduction. Let $X \subset \mathbb{P}_L^N$ be a smooth quasi-projective variety over a field L . Recall that another irreducible smooth subscheme $Z \subset \mathbb{P}_L^N$ is said to intersect X transversally, if the scheme-theoretic intersection $X \cdot Z = X \times_{\mathbb{P}^N} Z$ (which is just defined by the ideal generated by the equations of X and Z) is smooth and of pure codimension $\text{codim}_{\mathbb{P}^N}(Z)$ in X . Then the Bertini theorem asserts that for infinite L , there exists an L -rational hyperplane $H \subset \mathbb{P}_L^N$ intersecting X transversally (cf. [Jou, 6.11, 2], and also Theorem 3 below). In this case, one calls $Y = X \cdot H$ a smooth (or good) hyperplane section of X .

More precisely, the following holds. One has the dual projective space $(\mathbb{P}_L^N)^\vee$ parameterizing the hyperplanes in \mathbb{P}_L^N (a point $a = (a_0 : \dots : a_N)$ corresponds to the hyperplane with the equation $a_0x_0 + \dots + a_Nx_N = 0$ for the homogeneous coordinates x_i of \mathbb{P}_L^N). Then, for an arbitrary field L , there is a dense Zariski open $V_X \subset (\mathbb{P}_L^N)^\vee$ parameterizing those hyperplanes which intersect X transversally. Moreover, if L is infinite, then the set $V_X(L)$ of L -rational points is non-empty, since $\mathbb{P}_L^N(L)$ is Zariski dense in \mathbb{P}_L^N . This shows that, for an infinite field L , and finitely many smooth varieties X_1, \dots, X_n in \mathbb{P}_L^N , there also exists an L -rational hyperplane H intersecting all X_i transversally, because $V_{X_1} \cap \dots \cap V_{X_n}$ is non-empty.

If L is finite, it may happen that V_X does not have any L -rational point. But, by sieve methods, Poonen [Po] showed that in this case there always exists an L -rational point after replacing the projective embedding by the d -fold embedding for some $d > 0$, i.e., there always exists a smooth L -rational *hypersurface* section of X .

Now consider a quasi-projective A -scheme (A a discrete valuation ring as above), i.e., a subscheme X of the projective space \mathbb{P}_A^N over A .

By a hyperplane $H \subseteq \mathbb{P}_A^N$ over A we mean a closed subscheme which corresponds to an A -rational point of the dual projective space $(\mathbb{P}_A^N)^\vee$ (= Grassmannian of linear subspaces of codimension 1). Since every invertible module over A is free, H is given by a surjection $\varphi : A^{N+1} \rightarrow A^N$; or, equivalently, by an equation $\sum_{i=0}^N a_i x_i = 0$, $a_i \in A$ ($i = 0, \dots, N$), not all in the maximal ideal \mathfrak{m} , for the coordinates x_i on \mathbb{P}_A^N . The correspondence is given by

$$\ker \varphi = A \cdot \sum_{i=0}^N a_i e_i \quad ,$$

where e_0, \dots, e_N is the basis of A^{N+1} .

Theorem 0 *Let $X \subset \mathbb{P}_A^N$ be a smooth quasi-projective A -scheme. If F is infinite, then there exists a hyperplane $H \subset \mathbb{P}_A^N$ over A such that the scheme-theoretic intersection $X \cdot H = X \times_{\mathbb{P}_A^N} H$ is smooth over A and of pure codimension 1 in X . If F is finite and A is Henselian, then, for every given prime number ℓ , such a hyperplane exists after replacing A by a finite étale covering A'/A of ℓ -power-degree.*

Proof Let $H \subset \mathbb{P}_A^N$ be a hyperplane over A . Then H_η and H_s are hyperplanes in \mathbb{P}_K^N and \mathbb{P}_F^N , respectively. With the notations as above, the condition on the hyperplane is that (the K -rational point corresponding to) H_η lies in the good locus $V_{X_\eta} \subset (\mathbb{P}_K^N)^\vee$, and that H_s lies in V_{X_s} . Since H is completely determined by H_η , this means that $H_\eta \in V_{X_\eta}(K) \cap sp^{-1}(V_{X_s}(F))$, where $sp : (\mathbb{P}^N)^\vee(K) \rightarrow (\mathbb{P}^N)^\vee(F)$ is the specialization map, which sends H_η to H_s .

It remains to see when this intersection is non-empty. But for any proper scheme P over A and any open subschemes $V_1 \subset P_\eta$ and $V_2 \subset P_s$, with closed complements $Z_1 = P_\eta \setminus V_1$ and $Z_2 = P_s \setminus V_2$, respectively, one has $Z_1(K) \subset sp^{-1}(sp(Z_1)(F))$ where $sp(Z_1) = \overline{Z_1} \cap P_s$ for the Zariski closure $\overline{Z_1}$ of Z_1 in \mathbb{P}_A^N . Therefore $V_1(K) \cap sp^{-1}(V_2(F))$ contains $sp^{-1}((V_2 \setminus sp(Z_1))(F))$. The latter set has K -rational points, if $sp : P(K) \rightarrow P(F)$ is surjective and $V_2 \setminus sp(Z_1)$ has F -rational points. The latter set is open and dense in P_s , if P/S has irreducible fibres, and V_1 and V_2 are dense in their fibres.

Applying this to $P = (\mathbb{P}_A^N)^\vee$, $V_1 = V_{X_\eta}$ and $V_2 = V_{X_s}$, where all conditions are fulfilled, we see it suffices that the non-empty open subset $W = V_2 \setminus sp(Z_1)$ has F -rational points. As explained above, this is the case if F is infinite. Hence, if F is finite, it is the case over the maximal pro- ℓ -extension of F , hence over some extension F'/F of ℓ -power degree. If A is Henselian, and A'/A is the unramified extension corresponding to F'/F , then the F' -rational point lifts to an A' -rational point of P . Since the formation of the sets V_1 and V_2 is compatible with étale base change in the base, this means there is a good hyperplane section for X over A' .

Remarks 1 (i) In contrast with the classical situation, the good hyperplanes over A are not parametrized by a Zariski open in $(\mathbb{P}_K^N)^\vee$, but by a subset of the type $V_1(K) \cap sp^{-1}((V_2)(F))$ for Zariski opens $V_1 \subset (\mathbb{P}_K^N)^\vee$ and $V_2 \subset (\mathbb{P}_F^N)^\vee$.

(ii) If, with the notations as in the proof, H_s intersects the smooth variety X_s transversally, and if $X_\eta \cap H_\eta$ is non-empty, then $X \cdot H$ is a flat A -scheme of finite type, whose special fibre $(X \cdot H)_s = X_s \cdot H_s$ is smooth. Since the smooth locus of $X \cdot H$ is open, $X \cdot H$ must be smooth, if X and hence $X \cdot H$ is proper. This shows that, for smooth and proper X , one has $sp^{-1}(V_{X_s}) \subset V_{X_\eta}$, and the locus in $(\mathbb{P}_K^N)^\vee(K)$ of good hyperplanes for X/A is just $sp^{-1}(V_{X_s}(F))$. Moreover, by applying the mentioned result of Poonen, this has an K -rational point after passing to some multiple embedding.

Recall that a smooth proper variety V over K is said to have good reduction (over A) if there is a smooth proper A -scheme X with generic fibre $X_\eta = X \times_A K \cong V$.

Corollary 0 *If F is finite and V/K is a smooth projective variety with good reduction, there exists a smooth hypersurface section which again has good reduction.*

1. Good hyperplane sections for semi-stable schemes

For the applications, the case of good reduction is too restrictive, but often one can reduce questions to the case of semi-stable reduction and, by blowing up, even to the case of strictly semi-stable reduction. Therefore let $X \hookrightarrow \mathbb{P}_A^N$ be a quasi-projective strictly semi-stable A -scheme, i.e., X is flat over A , the generic fibre X_η is smooth, the reduction is semi-stable (cf. below), and the irreducible components of the special fibre X_s are smooth. The aim of this section is to prove:

Theorem 1 *If F is infinite, then there exists a hyperplane $H \subset \mathbb{P}_A^N$ over A such that the scheme-theoretic intersection $X \cdot H = X \times_{\mathbb{P}_A^N} H$ is a strictly semi-stable scheme over A . If F is finite and A is Henselian, then, for every prime ℓ there is a finite unramified extension A' of A of ℓ -power degree such that the same conclusion holds after base change with A' .*

Lemma 1 *Let $H \subset \mathbb{P}_A^N$ be a hyperplane over A , with special fibre $H_s \subset \mathbb{P}_F^N$ and generic fibre $H_\eta \subset \mathbb{P}_K^N$. Let $X_s = \bigcup_{i=1}^M Y_i$ be the union of smooth varieties intersecting transversally. Assume that*

(i) H_s intersects all p -fold intersections

$$Y_{i_1, \dots, i_p} := Y_{i_1} \cap \dots \cap Y_{i_p} \quad (i_1 < i_2 < \dots < i_p)$$

transversally, for all $p \geq 1$.

(ii) H_η intersects X_η transversally.

Then the scheme-theoretic intersection $X \cdot H$ is a strictly semi-stable scheme over A . If X is proper over A , condition (ii) is implied by condition (i).

Proof We may assume that the residue field is algebraically closed. Semi-stable reduction then means that the completion $\hat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$ at any closed point $x \in X_s$ is isomorphic to

$$B = A[[x_1, \dots, x_n]]/\langle x_1 \dots x_r - \pi \rangle \quad (1 \leq r \leq n)$$

for suitable n and r , where π is a prime element in A . Let f be the image of the local equation for H at x , and let $\mathfrak{n} \subseteq B$ be the maximal ideal. Since $B/\langle f \rangle$ is the completion of the local ring of $X \cdot H$ at x if $x \in X \cdot H$, and since the irreducible components of $(X \cdot H)_s = X_s \cdot H_s$ are the connected components of the smooth varieties $Y_i \cdot H_s$, the lemma follows from the following two claims.

Claim 1 Assumption (i) implies that

- either (a) f is a unit in B ,
or (b) $n > 1$, f is in \mathfrak{n} , and has non-zero image in $\mathfrak{n}/(\mathfrak{n}^2 + \langle x_1, \dots, x_r \rangle)$.

Claim 2 Assume condition (b) holds. Then

$$B/\langle f \rangle \cong A[[y_1, \dots, y_{n-1}]]/\langle y_1 \dots y_r - \pi \rangle$$

In fact, this gives the strictly semi-stable reduction, and by (ii) the generic fibre of $X \cdot H$ is smooth, too. If this generic fibre is non-empty, $X \cdot H$ is also flat; if it is empty, $X \cdot H$ is empty. Without (ii), claim 2 shows that every $x \in (X \cdot H)_s = X_s \cdot H_s$ has an open neighbourhood in $X \cdot H$ of the right dimension, whose generic fibre is non-empty and smooth. If X/A is proper, these neighbourhoods cover $X \cdot H$.

Proof of claim 2 The elements $x_i \pmod{\mathfrak{n}^2}$ form an F -basis of $\mathfrak{n}/\mathfrak{n}^2$ ($i = 1, \dots, n$). Hence we have

$$f = \sum_{i=1}^n a_i x_i \pmod{\mathfrak{n}^2}$$

with elements $a_i \in A$ which are determined modulo $\langle \pi \rangle$. If (b) holds, then a_i is a unit for one i with $i > r$ (in particular, we must have $n > r$), and by possibly renumbering and multiplying f by a unit we may assume $i = n$, and $a_n = 1$. But then

$$B/\langle f \rangle \cong A[[x_1, \dots, x_{n-1}]]/\langle x_1 \dots x_r - \pi \rangle.$$

Proof of claim 1 The elements x_1, \dots, x_r are the images of the local equations for Y_{i_1}, \dots, Y_{i_r} for suitable $1 \leq i_1 < \dots < i_r \leq M$. Thus the trace of $Y_{i_1} \cap \dots \cap Y_{i_r}$ in $\hat{\mathcal{O}}_{X,x} \cong B$ is given by the ideal $\langle x_1, \dots, x_r \rangle$, i.e., by the quotient

$$B' = B/\langle x_1, \dots, x_r \rangle \cong F[[x_{r+1}, \dots, x_n]]$$

This is zero-dimensional if and only if $r = n$, and in this case $Y_{i_1} \cap \dots \cap Y_{i_r}$ is zero-dimensional as well. Then, by assumption on H , H does not intersect $Y_{i_1} \cap \dots \cap Y_{i_r}$, and so f is a unit in $B/\langle x_1, \dots, x_r \rangle$ and hence in B .

If $r < n$, then H intersects $Y_{i_1} \cap \cdots \cap Y_{i_r}$ transversally at x if and only if the image of f in B' lies in $\mathfrak{n}' - (\mathfrak{n}')^2$, where \mathfrak{n}' is the maximal ideal of B' . But $\mathfrak{n}'/(\mathfrak{n}')^2 \cong \mathfrak{n}/(\mathfrak{n}^2 + \langle x_1, \dots, x_r \rangle)$ which proves claim 1.

Proof of Theorem 1 It suffices to find a hyperplane satisfying the assumption of Lemma 1, i.e., to show that, with the notations introduced earlier, the set $V_{X_\eta}(K) \cap sp^{-1}(V_2(F))$ is non-empty, where V_2 is the intersection of the sets $V_{Y_{i_1}, \dots, i_p}$, and hence open and dense in $(\mathbb{P}_F^N)^\vee$. This holds under the conditions of Theorem 1, by the arguments used in the proof of Theorem 0.

If X/A is proper, we noted that $sp^{-1}(V_2(F))$ is contained in $V_{X_\eta}(K)$. Combining this with the mentioned results of Poonen [Po] we get:

Corollary 1 *If F is finite and V/K is a smooth projective variety with strictly semi-stable reduction, there is a smooth hypersurface section which again has strictly semi-stable reduction.*

2. Lefschetz pencils for schemes with (almost) good reduction

Even if one starts with a variety V over K with good reduction, in general infinitely many fibres in a Lefschetz pencil (cf. below) for V will not have good reduction, because infinitely many hyperplanes specialize to the same hyperplane in the reduction, and usually the induced pencil for the reduction of V has a bad member. But one can arrange very mild singularities:

Definition 1 A smooth projective variety V over K is said to have almost good reduction, if there is a projective A -scheme X such that $X_\eta \cong V$, and X_s is smooth over F except for a finite number of singular points which are ordinary quadratic (cf. [SGA 7 XV, 1.2.1] and below).

In fact, one can even start with such singularities, and still get singularities which are not worse - which is useful for induction on dimension. Our aim is to prove:

Theorem 2 *Let V be projective K -variety with almost good reduction, and let $X \subset \mathbb{P}_A^N$ be a model of V as in Definition 1. let $d \geq 2$ be an integer and suppose F is infinite. Then, after possibly passing to the d -fold embedding of X , there exists a Lefschetz pencil $\{V_t\}_{t \in D}$, where D is a line in the dual projective space $(\mathbb{P}_K^N)^\vee$, satisfying the following conditions:*

- (1) *The axis of the pencil has good reduction over K .*
- (2) *There exists a finite subset $\Sigma \subset \mathbb{P}_K^1$ of closed points such that for any $t \notin \Sigma$, V_t has almost good reduction over $A_{K(t)}$, the integral closure of A in the residue field $K(t)$ of t .*

Suppose F is finite, A is Henselian, and ℓ is a fixed prime. Then the same result holds after possibly passing to a finite unramified extension K'/K of ℓ -power degree.

The proof will be achieved in four steps, numbered (2.1) ~ (2.4).

(2.1) Let $(\mathbb{P}_A^N)^\vee$ be the dual projective space over A . Its fibres over K and F coincide with the dual projective spaces $(\mathbb{P}_K^N)^\vee$ and $(\mathbb{P}_F^N)^\vee$, respectively. Furthermore, let $\mathcal{G} = \text{Gr}(1, (\mathbb{P}_A^N)^\vee)$ be the Grassmannian of lines in $(\mathbb{P}_A^N)^\vee$; again its fibres over K and F are the corresponding Grassmannians for $(\mathbb{P}_K^N)^\vee$ and $(\mathbb{P}_F^N)^\vee$, respectively. According to [SGA 7, XVII, 2.5], after possibly passing to the d -fold projective embedding, there is a dense open subscheme $W_{X_\eta} \subset \mathcal{G}_K$ such that the lines in W_{X_η} give Lefschetz pencils for $X_\eta \subset \mathbb{P}_K^N$.

(2.2) Since X_F is possibly singular, we need a slight extension of the results in [SGA 7, XVII]. First we extend the results to smooth, but only quasi-projective varieties.

Theorem 3 *Let L be any field, let $U \subset \mathbb{P}_L^N$ be a smooth irreducible quasi-projective variety, and let $d \geq 2$ be an integer. After possibly passing to the d -fold embedding, there is a non-empty open subscheme W_U in the Grassmannian $\text{Gr}(1, (\mathbb{P}_L^N)^\vee)$ of lines in the dual projective space, such that the lines D in W_U satisfy all properties of Lefschetz pencils with respect to U , i.e.:*

- (1) *The axis of D (i.e., the intersection of any two different and hence all hyperplanes parametrized by D) intersects U transversally.*
- (2) *There is a finite subscheme $\Sigma \subset D$ such that for $t \in D \setminus \Sigma$ the hyperplane H_t corresponding to t intersects U transversally.*
- (3) *For $t \in \Sigma$ the scheme-theoretic intersection $U \cdot H_t = U \times_{\mathbb{P}_L^N} H_t$ is smooth except for one singular point which is ordinary quadratic.*

Proof Let X be the closure of U in $\mathbb{P} = \mathbb{P}_L^N$, and let $Z = X \setminus U$ (both endowed with the reduced subscheme structure). For $Q = \mathbb{P}_L^N \setminus Z$, let \mathcal{J} be the ideal sheaf of the closed immersion $U \subset Q$, and denote by $\mathcal{N} = \mathcal{J}/\mathcal{J}^2$ the conormal sheaf, regarded as a locally free sheaf on U , and by \mathcal{N}^\vee its dual. As in [SGA 7, XVII] consider the closed immersion of projective bundles on U

$$\mathbb{P}_U(\mathcal{N}^\vee) \hookrightarrow \mathbb{P}_U(\mathcal{O}_U(1) \otimes_L \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))^\vee) \cong U \times (\mathbb{P}_L^N)^\vee$$

induced by the canonical monomorphism of bundles

$$\mathcal{J}/\mathcal{J}^2 \hookrightarrow \Omega_{Q/U}^1 \hookrightarrow \mathcal{O}_U(-1)^{N+1} = \mathcal{O}_U(-1) \otimes_L \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)).$$

(Here we adopt the convention that, for a vector bundle \mathcal{F} on U , the projective bundle $\mathbb{P}(\mathcal{F}) = \text{Proj}(\text{Sym}(\mathcal{F}))$ parametrizes line bundle quotients of \mathcal{F} .) The above immersion identifies $\mathbb{P}_U(\mathcal{N}^\vee)$ with the subvariety of points (x, H) in $U \times (\mathbb{P}_L^N)^\vee$ for which H touches U in x . Let U^\vee be the closure of the image of $\mathbb{P}_U(\mathcal{N}^\vee)$ in $(\mathbb{P}_L^N)^\vee$. It is the dual variety to U and contains all hyperplanes in \mathbb{P}_L^N which touch U in some point. One has $\dim U^\vee \leq \dim \mathbb{P}_U(\mathcal{N}^\vee) = N - 1$. Hence $(\mathbb{P}_L^N)^\vee \setminus U^\vee$ is non-empty,

and the set $M''_U \subseteq \mathcal{G}_L = Gr(1, (\mathbb{P}_L^N)^\vee)$ of lines in $(\mathbb{P}_L^N)^\vee$ contained in U^\vee is closed and different from \mathcal{G}_L .

Moreover, let $(U^\vee)^0$ be the set of hyperplanes which touch U in exactly one point which is an ordinary quadratic singularity. Then $(U^\vee)^0$ is open in U^\vee by results of Elkik and Deligne ([SGA 7, XVII, 3.2], [SGA 7, XV, 1.3.4]). (If $\text{char } L \neq 2$ or if $n = \dim U$ is even, then it is the locus where $\mathbb{P}_U(\mathcal{N}^\vee) \rightarrow (\mathbb{P}_L^N)^\vee$ is unramified.) It is non-empty after replacing the given embedding by its d -multiple ($d \geq 2$), by the argument in [SGA 7, XVII, 3.7, 4.2]. Since U^\vee is irreducible (by irreducibility of $\mathbb{P}_U(\mathcal{N}^\vee)$), the closed subscheme $F''' = U^\vee \setminus (U^\vee)^0$ has codimension ≥ 2 in $(\mathbb{P}_L^N)^\vee$ in this case. Then the set $M'''_U \subset \mathcal{G}_L$ of lines in \mathbb{P}_L^N which meet F''' is closed and different from \mathcal{G}_L .

Finally, the set $W'_U \subseteq Gr(N-2, \mathbb{P}_L^N)$ of codimension 2 linear subspaces in \mathbb{P}_L^N which intersect U transversally is open [Jou, 6.11, 2]). It is also non-empty: Since $U^\vee \neq (\mathbb{P}_L^N)^\vee$, over the algebraic closure there is a hyperplane H_1 intersecting U transversally, and similarly, there is a hyperplane H_2 intersecting $U \cdot H_1$ transversally. This means that the codimension 2 linear subspace $H_1 \cdot H_2$ intersect U transversally. Recall the isomorphism

$$\mathcal{G}_L = Gr(1, (\mathbb{P}_L^N)^\vee) \xrightarrow{\sim} Gr(N-2, \mathbb{P}_L^N)$$

sending a pencil to its axis. We denote the preimage of W'_U in \mathcal{G}_L by W'_U again.

The conclusion is that there is a non-empty open subscheme $W_U = W'_U \cap (\mathcal{G}_L \setminus M''_U) \cap (\mathcal{G}_L \setminus M'''_U) \subseteq \mathcal{G}_L$ such that the lines in W_U satisfy all properties of Lefschetz pencils with respect to U , and thus Theorem 3 is proved.

(2.3) Now we deal with the singular points of the special fibre X_F of X in Theorem 2.

Theorem 4 *Let L be any field, and let $X \subset \mathbb{P}_L^N$ be a projective variety which is smooth except for finitely many singular points x_1, \dots, x_r which are ordinary quadratic. After possibly passing to the d -fold embedding (any $d \geq 2$), there is a non-empty open subscheme $W_X \subset Gr(1, (\mathbb{P}_L^N)^\vee)$ such that for the lines D in W_X the following holds:*

- (i) *The axis of D does not meet the singular points of X and intersects the regular locus X^{reg} transversally.*
- (ii) *There is a finite subscheme $\Sigma \subseteq D$ such that for $t \in D \setminus \Sigma$ the hyperplane H_t does not meet the singular points of X and intersects X^{reg} transversally.*
- (iii) *For $t \in \Sigma$ the scheme-theoretic intersection $X^{\text{reg}} \cdot H_t$ is smooth except for possibly one singular point which is an ordinary quadratic singularity.*
- (iv) *If $t \in \Sigma$ and $x_i \in H_t$, then x_i is an ordinary quadratic singularity of $X \cdot H_t$.*

Proof Applying Theorem 3 to $X^{reg} = X \setminus \{x_1, \dots, x_r\}$ we find a non-empty open subset $W' = W_{X^{reg}} \subseteq \mathcal{G}_L$ such that the lines D in V satisfies the properties (i) to (iii) for X^{reg} instead of X .

It remains to consider the singular points x_1, \dots, x_r . For each x_i , the hyperplanes in \mathbb{P}_L^N which pass through x_i form a hyperplane $\tilde{H}_i \subseteq (\mathbb{P}_L^N)^\vee$. By the following lemma there is a non-empty open subset $U_i \subseteq \tilde{H}_i$ such that for any hyperplane H in U_i the intersection $Y \cdot H$ has an ordinary quadratic singularity at x_i . Then $F_i = \tilde{H}_i \setminus U_i$ is closed and of codimension ≥ 2 in $(\mathbb{P}_L^N)^\vee$, and so is $F = \cup_{i=1}^r F_i$. The set $W'' \subseteq \mathcal{G}_L$ of lines in $(\mathbb{P}_L^N)^\vee$ which do not meet F and are not contained in any \tilde{H}_i is thus open and non-empty, and the properties (i) to (iv) above hold for the lines in $W_X = W' \cap W'' \subseteq \mathcal{G}_L$.

Lemma 2 *Let L be any field, let $X \subset \mathbb{P}_L^N$ be a projective variety of positive dimension, and let x be an isolated singularity which is an ordinary quadratic point. If $\tilde{H}_x \subset (\mathbb{P}_L^N)^\vee$ denotes the locus of hyperplanes passing through x , then there is an open dense subset $U \subset \tilde{H}_x$ such that for all hyperplanes H in U the point x is an ordinary quadratic singularity of $X \cdot H$.*

Proof We may assume that L is algebraically closed. Let $A = \hat{\mathcal{O}}_{X,x}$ be the completion of the local ring at x . Then x is called an ordinary quadratic singularity, if A is isomorphic to the quotient

$$L[[x_1, \dots, x_{n+1}]]/\langle f \rangle,$$

where f starts in degree 2, and where f_2 , the homogeneous part of degree 2 of f , is non-zero, and defines a non-singular quadric in \mathbb{P}_L^n (where $n \geq 1$ by assumption). We shall call A the ring of an ordinary quadratic singularity in this case.

Lemma 3 *Let $\mathfrak{m} \subset A$ be the maximal ideal, and let $g \in \mathfrak{m} \setminus \{0\}$ be an element. Then $A' := A/\langle g \rangle$ is the ring of an ordinary quadratic singularity if the following two conditions hold*

- (i) *The image \bar{g} of g in $\mathfrak{m}/\mathfrak{m}^2$ is non-zero.*
- (ii) *The non-singular projective quadric $\text{Proj}(\text{Sym}(\mathfrak{m}/\mathfrak{m}^2)/\langle Q \rangle)$ and the hypersurface $\text{Proj}(\text{Sym}(\mathfrak{m}/\mathfrak{m}^2)/\langle \bar{g} \rangle)$ intersect transversally in $\text{Proj}(\text{Sym}(\mathfrak{m}/\mathfrak{m}^2)) \cong \mathbb{P}_L^n$. Here Q corresponds to f_2 under the isomorphism*

$$L[[x_1, \dots, x_{n+1}]] \xrightarrow{\sim} \text{Sym}(\mathfrak{m}/\mathfrak{m}^2).$$

(More intrinsically, Q is determined up to a scalar factor as the generator of the 1-dimensional kernel of the surjection $\text{Sym}^2(\mathfrak{m}/\mathfrak{m}^2) \rightarrow \mathfrak{m}^2/\mathfrak{m}^3$).

Proof Lift g to an element $\bar{g} \in B := L[[x_1, \dots, x_{n+1}]]$, and let \mathfrak{n} be the maximal ideal of B . By (i) and a substitution we may assume $\bar{g} = x_{n+1}$. Then $B' := B/\langle \bar{g} \rangle \cong L[[x_1, \dots, x_n]]$, and

$$A' = B'/\langle f' \rangle$$

where f' is the image of f in B' . Then f' starts in degree 2 as well, and f'_2 , its degree 2 part with respect to the variables x_1, \dots, x_n , is just the image of f_2 . If f'_2 is zero, then $\langle Q \rangle \subseteq \langle \bar{g} \rangle$ in $\text{Sym}(\mathfrak{m}/\mathfrak{m}^2)$, in contradiction to (ii). Hence $f'_2 \neq 0$, and by (ii) it gives rise to a non-singular quadric in

$$\text{Proj}(\text{Sym}(\mathfrak{m}/\mathfrak{m}^2)/\langle \bar{g} \rangle) \cong \text{Proj}(\text{Sym}(\mathfrak{m}/(\mathfrak{m}^2 + \langle g \rangle))) \cong \mathbb{P}_L^{n-1}$$

by (ii) for $n \geq 2$, i.e., A' is the ring of an ordinary quadric singularity.

We proceed with the proof of Lemma 2. Choose coordinates X_0, \dots, X_N on \mathbb{P}_L^N such that $x = (1 : 0 : \dots : 0)$. The hyperplanes in \mathbb{P}_L^N are given by points $b = (b_0 : \dots : b_N)$ in the dual projective space $(\mathbb{P}_L^N)^\vee$, corresponding to the hyperplanes

$$\sum_{i=0}^N b_i X_i = 0 \quad .$$

The hyperplanes through x are given by those b with $b_0 = 0$ and are parametrized by $(b_1 : \dots : b_N) \subseteq (\mathbb{P}_L^{N-1})^\vee$. If $x_i = \frac{X_i}{X_0}$, $i = 1, \dots, N$, are the affine coordinates on the open affine neighbourhood $\{x_0 \neq 0\} \cong \mathbb{A}_L^N$, x corresponds to the zero point, and the hyperplane associated to $(b_1 : \dots : b_N)$ is determined by the element $\sum_{i=1}^N b_i x_i \in L[x_1, \dots, x_N]$.

Let \mathfrak{n} be the maximal ideal $\langle x_1, \dots, x_N \rangle$. Then one has an isomorphism

$$\begin{aligned} L^N &\xrightarrow{\sim} \mathfrak{n}/\mathfrak{n}^2 \\ (b_1, \dots, b_N) &\longmapsto \sum_{i=1}^N b_i x_i \pmod{\mathfrak{n}^2} \quad . \end{aligned}$$

Now let $\mathfrak{m} \subset \mathcal{O}_{X,x}$ be the maximal ideal. Then we get a surjection

$$\varphi : \mathfrak{n}/\mathfrak{n}^2 \rightarrow \mathfrak{m}/\mathfrak{m}^2 \quad ,$$

and for a point $b = (b_1, \dots, b_N) \in L^N$ and the associated hyperplane H_b , the local ring of x in $X \cdot H_b$ is

$$\mathcal{O}_{X,x}/\langle \sum_{i=1}^N b_i x_i \rangle.$$

By the above lemma, x is an ordinary quadratic singularity if the image $\overline{\sum b_i x_i}$ of $\sum b_i x_i$ in $\mathfrak{m}/\mathfrak{m}^2$ is non-zero, and if the associated hyperplane in $\mathbb{P}_L(\mathfrak{m}/\mathfrak{m}^2)$ intersects the quadric in $\mathbb{P}_L(\mathfrak{m}/\mathfrak{m}^2)$ associated to the singularity transversally. The latter condition defines an open subset U' in the dual projective space $\mathbb{P}_L((\mathfrak{m}/\mathfrak{m}^2)^\vee)$ parametrizing the hyperplanes in $\mathbb{P}_L(\mathfrak{m}/\mathfrak{m}^2)$. Consider the non-empty open subset $U'' \subseteq \mathbb{P}_L((\mathfrak{n}/\mathfrak{n}^2)^\vee)$ on which the projection

$$p : \mathbb{P}_L((\mathfrak{n}/\mathfrak{n}^2)^\vee) \dashrightarrow \mathbb{P}_L((\mathfrak{m}/\mathfrak{m}^2)^\vee)$$

associated to φ^\vee is defined. (To wit: U'' is the complement of $\mathbb{P}_L((\ker \varphi)^\vee) \subseteq \mathbb{P}_L((\mathfrak{n}/\mathfrak{n}^2)^\vee)$. Letting $U = p^{-1}(U')$, we see that for the hyperplanes H in U the intersection $X \cdot H$ has an ordinary quadratic singularity at x .

(2.4) We can now finish the proof of Theorem 2. Applying Theorem 4 to X_F and combining it with the result on X_K , we obtain the wanted Lefschetz pencil over $\text{Spec}(A)$ provided there is an A -rational point in \mathcal{G} , corresponding to a line L over A , such that L_η lies in the open $W_{X_\eta} \subset \mathcal{G}_\eta$ (constructed in (2.1)) and L_s lies in the open $U_{X_s} \subset \mathcal{G}_s$ (constructed in Theorem 4). This existence, under the conditions of Theorem 2, follows now by applying the arguments in the proof of Theorem 0 to $P = \mathcal{G}$, $V_1 = W_{X_\eta}$ and $V_2 = U_{X_s}$. Note that the specialization map $\mathcal{G}(K) = \mathcal{G}(A) \rightarrow \mathcal{G}(F)$ is surjective, and that \mathcal{G}_L , over a field L , has a cellular decomposition, so that $\mathcal{G}(L)$ is dense in \mathcal{G} for infinite L .

3. Desingularization of ordinary quadratic singularities

For the applications, it is important to have a good description of varieties with almost good reduction, and also a description of their desingularization, because such schemes may be non-regular. We recall the following description of local rings around an ordinary quadratic singularity [SGA 7, XV, 1.32].

Lemma 4 *Let X be a flat scheme of finite type over A , and assume that X is smooth over A except for one singular point $x \in X_s$ which is an ordinary quadratic singularity (in X_s). Assume that X_s is of dimension n at x . Then, after possibly passing to a finite étale extension of A , the Henselization of $\mathcal{O}_{X,x}$ is isomorphic to the Henselization of the following ring B at the maximal ideal $(x_1, \dots, x_{n+1}, \pi)$.*

(i) *If x is non-degenerate:*

$$B = A[x_1, \dots, x_{n+1}] / \langle Q(x_1, \dots, x_{n+1}) - c \rangle,$$

where Q is a non-degenerate quadratic form over A and $c \in \mathfrak{m} \setminus \{0\}$.

(ii) *If x is degenerate (which can only happen if $\text{char}(F) = 2$ and $n = 2m$ is even):*

$$B = A[x_1, \dots, x_{n+1}] / \langle P(x_1, \dots, x_{2m}) + x_{n+1}^2 + bx_{n+1} + c \rangle,$$

where P is a non-degenerate quadratic form over A and $b, c \in A$ with $b^2 - 4c \in \mathfrak{m} \setminus \{0\}$.

In the situation of Lemma 4 (i), let $r = v(c)$, where v is the normalized valuation of K , so that $c = \eta\pi^r$, where π is a prime element in A and η is a unit in A . Then we say that X has an ordinary quadratic singularity of order r . By possibly passing to a ramified extension of degree 2 (extracting a square root of π), we may assume that r is even.

In the situation of Lemma 4 (ii), by possibly passing to a ramified extension of degree 2 (the splitting field of $x^2 + bx + c$), and by a coordinate transformation, we

may assume that $c = 0$. In this case we let $q = v(b)$, so that $b = \epsilon\pi^q$ with a unit ϵ , and say that X has an ordinary quadratic singularity of order q .

Theorem 5 *Let X be as in Lemma 4, and let $\varphi : \tilde{X} \rightarrow X$ be the blowing up of X at the singular point x . Assume r is even in case 2, and $c = 0$ in case (ii). Then the strict transform \tilde{Y} of $Y = X_s$ is smooth, and the exceptional fibre $F_x = \varphi^{-1}(x)$ contains a point $\tilde{x} \notin \tilde{Y}$ such that the following holds:*

(a) $F_x \setminus \{\tilde{x}\}$ is smooth, and \tilde{Y} and F_x intersect transversally, i.e., the scheme-theoretic intersection of these inside \tilde{X} is smooth.

(b) $\tilde{X} \setminus \{\tilde{x}\}$ is regular and has strict semi-stable reduction.

(c) In case (i), if x is of order r , then the behavior of \tilde{X} at \tilde{x} is as follows: If $r > 2$, then \tilde{x} is ordinary quadratic of order $r - 2$. If $r = 2$, then \tilde{X} is also smooth at \tilde{x} , and hence has strict semi-stable reduction.

(d) In case (ii), if x is of order q , then the behavior of \tilde{X} at \tilde{x} is as follows: If $q > 1$, then \tilde{x} is ordinary quadratic of order $q - 1$. If $q = 1$, then \tilde{X} is also smooth at \tilde{x} , and hence has strict semi-stable reduction.

Proof Since blowing-ups are compatible with flat base change, and since smoothness and type of the quadratic singularity just depend on the Henselization of the local ring, we may consider the rings B in Lemma 4.

Case (i): 1) Here the blowing-up of B at the ideal $\mathfrak{n} = \langle x_1, \dots, x_{n+1}, \pi \rangle$ is $\text{Proj}(C)$, for the B -algebra

$$C = B[U_1, \dots, U_{n+1}, T]/I$$

$$I = \langle x_i U_j - x_j U_i, x_i T - \pi U_i, Q(U_1, \dots, U_{n+1}) - \eta\pi^{r-2}T^2 \rangle,$$

which is graded as quotient of the polynomial ring over B . In fact, the coordinate ring of the affine chart $\{U_{n+1} \neq 0\}$ is

$$A[u_1, \dots, u_n, x_{n+1}, t]/\langle Q(u_1, \dots, u_n, 1) - \eta\pi^{r-2}t^2, x_{n+1}t - \pi \rangle,$$

with $x_{n+1}u_i = x_i$ ($i = 1, \dots, n$). A similar description holds for the other charts $\{U_i \neq 0\}$. The coordinate ring for the chart $\{T \neq 0\}$ is

$$A[u_1, \dots, u_{n+1}]/\langle Q(u_1, \dots, u_{n+1}) - \eta\pi^{r-2} \rangle,$$

with $\pi u_i = x_i$ ($i = 1, \dots, n + 1$). This shows that the inverse image of \mathfrak{n} is an invertible ideal: it is generated by one element (by x_{n+1} , x_i , and π , respectively), which is not a zero divisor. Moreover, the morphism $\text{Proj}(C) \rightarrow \text{Spec}(B)$ becomes an isomorphism after inverting any of the elements x_1, \dots, x_{n+1}, π . Finally there is a surjection of graded B -algebras

$$C \longrightarrow \bigoplus_{n \geq 0} \mathfrak{n}^n,$$

by sending U_i and T to x_i and π in \mathfrak{n} , respectively. Thus, by lemma 5 below, $\text{Proj}(C)$ is isomorphic to $\text{Proj}(\bigoplus_{n \geq 0} \mathfrak{n}^n)$, the blowing-up of B in \mathfrak{n} .

2) Assume $r > 2$. We consider the special fiber of the blowing-up, obtained by setting $\pi = 0$. Thus its chart $\{U_{n+1} \neq 0\}$ is

$$\begin{aligned} & \text{Spec}(k[u_1, \dots, u_n, x_{n+1}, t] / \langle Q(u_1, \dots, u_n, 1), x_{n+1}t \rangle) \\ &= \text{Spec}(R[x_{n+1}, t] / \langle x_{n+1}t \rangle) \end{aligned}$$

where $R = k[u_1, \dots, u_n] / \langle Q(u_1, \dots, u_n, 1) \rangle$. It is reduced, with two smooth irreducible components intersecting transversally - the first one being the locus $\{t = 0\}$, the second one being the locus $\{x_{n+1} = 0\}$. A similar result holds for the other charts $\{U_i \neq 0\}$. The chart $\{T \neq 0\}$ is

$$\text{Spec}(k[u_1, \dots, u_{n+1}] / \langle Q(u_1, \dots, u_{n+1}) \rangle),$$

which is smooth except for one ordinary quadratic singularity at $u = (0, \dots, 0)$.

We may identify the irreducible components as follows. The strict transform \bar{Y} of the special fiber of $\text{Spec}(B)$ is obtained by blowing up

$$\bar{B} = B / \langle \pi \rangle = F[x_1, \dots, x_{n+1}] / \langle Q(x_1, \dots, x_{n+1}) \rangle$$

in the ideal $\bar{n} = \langle x_1, \dots, x_{n+1} \rangle$. This is $\text{Proj}(\bar{C})$, for

$$\bar{C} = \bar{B}[U_1, \dots, U_{n+1}] / \langle x_i U_j - x_j U_i, Q(U_1, \dots, U_{n+1}) \rangle.$$

The affine ring of the chart $\{U_i \neq 0\}$ is

$$F[x_i, u_1, \dots, \check{u}_i, \dots, u_{n+1}] / \langle Q(u_1, \dots, 1, \dots, u_{n+1}) \rangle,$$

where $x_i u_j = x_j u_i$ ($j \neq i$), \check{u}_i means omission of u_i , and the 1 in $Q(u_1, \dots, 1, \dots, u_{n+1})$ is at position i . This is smooth over F , and corresponds to the locus $T = 0$ in \bar{X} .

The exceptional fibre F_x is obtained by letting $x_1 = \dots = x_{n+1} = 0 = \pi$ in C . For $r > 2$ we get

$$\text{Proj}(F[U_1, \dots, U_{n+1}, T] / \langle Q(U_1, \dots, U_{n+1}) \rangle).$$

In the chart $\{U_i \neq 0\}$ this corresponds to the locus $x_i = 0 = \pi$ which is

$$\text{Spec}(F[u_1, \dots, \check{u}_i, \dots, u_{n+1}, t] / \langle Q(u_1, \dots, 1, \dots, u_{n+1}) \rangle)$$

and thus smooth. In the chart $\{T \neq 0\}$ we get

$$\text{Spec}(F[u_1, \dots, u_{n+1}] / \langle Q(u_1, \dots, u_{n+1}) \rangle).$$

This shows that the exceptional fiber has one ordinary quadratic singular point which does not lie on \bar{Y} . From the previous description of the chart $\{T \neq 0\}$ for the whole blowing-up we see that the order of the quadratic singularity is $r - 2$.

3) Now let $r = 2$. Then the chart $\{U_{n+1} \neq 0\}$ of the whole blowing-up is

$$\text{Spec}(S / \langle x_{n+1} t - \pi \rangle),$$

where $S = A[u_1, \dots, u_n, x_{n+1}, t] / \langle Q(u_1, \dots, u_n, 1) - \eta t^2 \rangle$ is smooth over A and x_{n+1}, t are part of a local parameter system where they vanish. Thus we get a regular scheme with semi-stable reduction over A . The same holds in the other charts $\{U_i \neq 0\}$. In the chart $\{T \neq 0\}$ we get the smooth A -scheme

$$\text{Spec}(A[u_1, \dots, u_{n+1}] / \langle Q(u_1, \dots, u_{n+1}) - \eta \rangle).$$

The strict transform \tilde{Y} of Y has exactly the same description as before; it is smooth, and it is again the locus where $T = 0$. The exceptional fiber is

$$\text{Proj}(F[U_1, \dots, U_{n+1}, T] / \langle Q(U_1, \dots, U_{n+1}) - \eta T^2 \rangle)$$

which is smooth as well. Therefore \tilde{X} has strict semistable reduction.

Case (ii): Here the blowing-up of

$$B = A[x_1, \dots, x_{n+1}] / \langle P(x_1, \dots, x_{2m}) + x_{n+1}^2 + bx_{n+1} \rangle$$

($b \in \mathfrak{m} \setminus \{0\}$) in the ideal $\mathfrak{n} = \langle x_1, \dots, x_{n+1}, \pi \rangle$ is $\text{Proj}(C)$, for

$$C = B[U_1, \dots, U_{n+1}, T] / I,$$

where the ideal I is generated by the elements

$$\begin{aligned} x_i U_j - x_j U_i & \text{ for } i, j \in \{1, \dots, n+1\} \\ x_i T = \pi U_i & \text{ for } i \in \{1, \dots, n+1\} \\ P(U_1, \dots, U_n) + U_{n+1}^2 + \epsilon \pi^{q-1} T U_{n+1}. & \end{aligned}$$

The coordinate ring of the chart $\{U_{n+1} \neq 0\}$ is

$$A[u_1, \dots, u_n, x_{n+1}, t] / \langle x_{n+1} t - \pi, P(u_1, \dots, u_n) + 1 + \epsilon \pi^{q-1} t \rangle.$$

For $i \in \{1, \dots, n\}$, the chart $\{U_i \neq 0\}$ is

$$\text{Spec}(A[u_1, \dots, u_{i-1}, x_i, u_{i+1}, \dots, u_{n+1}, t] / J),$$

where

$$J = \langle x_i t - \pi, P(u_1, \dots, u_{i-1}, 1, u_{i+1}, \dots, u_n) + u_{n+1}^2 + \epsilon \pi^{q-1} t u_{n+1} \rangle,$$

The affine ring for the chart $\{T \neq 0\}$ is

$$A[u_1, \dots, u_{n+1}] / \langle P(u_1, \dots, u_n) + u_{n+1}^2 + \epsilon \pi^{q-1} u_{n+1} \rangle.$$

The strict transform of the special fiber X_s is the locus $T = 0$, and it has exactly the same description as in case (i)2), except that the quadratic form is now $Q(U_1, \dots, U_{n+1}) = P(U_1, \dots, U_n) + U_{n+1}^2$. Thus it is smooth.

The exceptional fibre corresponds to the locus $\{x_i = 0 = \pi\}$. In the chart $\{U_i \neq 0\}$, for $i \neq n+1$, we get the subscheme

$$\text{Spec}(F[u_1, \dots, \check{u}_i, \dots, u_{n+1}, t] / \langle f_i \rangle)$$

with

$$f_i = P(u_1, \dots, 1, \dots, u_n) + u_{n+1}^2 + \overline{\epsilon\pi^{q-1}}tu_{n+1}$$

where $\bar{a} = a \bmod \pi$. In the chart $\{U_{n+1} \neq 0\}$ we get the subscheme

$$\text{Spec}(F[u_1, \dots, u_n, t] / \langle f_{n+1} \rangle),$$

$$f_{n+1} = P(u_1, \dots, u_n) + 1 + \overline{\epsilon\pi^{q-1}}t.$$

These are smooth. In the chart $\{T \neq 0\}$ we get the scheme

$$\text{Spec}(F[u_1, \dots, u_{n+1}] / \langle g \rangle),$$

$$g = P(u_1, \dots, u_n) + u_{n+1}^2 + \overline{\epsilon\pi^{q-1}}u_{n+1}.$$

If $q > 1$, this has one quadratic singularity of order $q - 1$. If $q = 1$, the scheme is smooth, since $\partial g / \partial u_{n+1} = \bar{\epsilon} \neq 0$. It is also clear that the strict transform of the special fibre and the exceptional fibre intersect transversally (in their smooth loci). Hence the claim follows.

Lemma 5 *Let B be a noetherian ring, let $I \subset B$ be an ideal, and let $\tilde{X} = \text{Proj}(\bigoplus I^n)$ be the blowing-up of $X = \text{Spec}(B)$ in the closed subscheme $Y = \text{Spec}(B/I)$ corresponding to I . Let*

$$\varphi: C \longrightarrow \bigoplus_{n \geq 0} I^n$$

be a surjection of graded B -algebras. Then the X -morphism

$$f = \varphi^*: \text{Proj}(\bigoplus I^n) = \tilde{X} \longrightarrow Z = \text{Proj}(C)$$

induced by φ is an isomorphism if and only if the following two conditions hold.

(i) *I generates an invertible ideal in $Z = \text{Proj}(C)$.*

(ii) *g induces an isomorphism $g^{-1}(X \setminus Y) \xrightarrow{\sim} X \setminus Y$.*

Proof: The two conditions are known to hold for $Z = \tilde{X}$, and by the surjectivity of φ , the morphism f is a closed immersion. In particular, f is affine. By (i), each point in Z has an open affine neighbourhood $V = \text{Spec}(R) \subset Z$ over which the image of I is generated by one element $a \in R$ which is not a zero divisor. Hence $f^{-1}(V) \rightarrow V$ corresponds to a surjection of rings $R \rightarrow R/J$, which induces an isomorphism after inverting a , by condition (ii) (for Z and \tilde{X}). This means that $J_a = 0$ for the localization of the ideal J with respect to a . It follows that $J = 0$, because a is not a zero divisor. Therefore f is an isomorphism.

References

- [Ha] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math. **52**, Springer, New York 1977.
- [JS] U. Jannsen, S. Saito, *Kato Homology of Arithmetic Schemes and Higher Class Field Theory over Local Fields*, Documenta Math. Extra Volume: Kazuya Kato's Fiftieth Birthday (2003) 479–538.
- [Jou] J.-P. Jouanolou, *Théorèmes de Bertini et Applications*, Progress in Math. **42**, Birkhäuser, Basel 1983.
- [Po] B. Poonen, *Bertini theorems over finite fields*, preprint, 23 p., <http://math.berkeley.edu/~poonen/>
- [SGA 7] P. Deligne, N. Katz *Groupes de Monodromie en Géométrie Algébrique*, I: Lect. Notes in Math. **288**, Springer, Berlin 1972, II: Lect. Notes in Math. **340**, Springer, Berlin 1973.

Uwe Jannsen, Fakultät für Mathematik, Universität Regensburg, Universitätsstr. 31, 93040 Regensburg, GERMANY, uwe.jannsen@mathematik.uni-regensburg.de

Shuji Saito, Graduate School of Mathematics, Nagoya University, Chikusa-ku, NAGOYA 464-8602, JAPAN, sshuji@msb.biglobe.ne.jp