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EQUIVALENCES OF TWISTED K3 SURFACES

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In my talk at the Kinosaki conference in October 2004 I reported on joint work with P. Stellari published in [7]. Roughly, we showed that any Fourier–Mukai equivalence between the derived categories of two twisted K3 surfaces yields an Hodge isometry of the corresponding Hodge structures (which are introduced for this purpose). The other direction, known under the name of Căldăraru's conjecture and predicting equivalence if an Hodge isometry can be found, is proved under additional assumptions.

At the occasion of the conference K. Yoshioka informed me about his recent work [12] on moduli space of stable twisted sheaves which led, right after the conference, to a complete proof of (a modified version) of Căldăraru's conjecture (see [8]). Here, I shall give a brief outline of both results.

1. Twisted K3 surfaces

A twisted K3 surface consists of an algebraic K3 surfaces X together with a torsion class $\alpha \in H^2(X, \mathcal{O}_X^*)$. Usually we think of α as associated to a torsion B-field $B \in H^2(X, \mathbb{Q})$, i.e. $\alpha = \alpha_B := \exp(B^{0,2})$, where $B^{0,2}$ is the $(0,2)$-part of B in $H^2(X, \mathcal{O}_X)$. This allows to introduce a weight-two Hodge structure whose isomorphism type only depends on α by:

$$\widetilde{H}^{2,0}(X,B) := (\sigma + B \wedge \sigma)\mathbb{C} \subset H^*(X,\mathbb{C}),$$

where σ is a trivializing section of $H^{2,0}(X)$. The $(0,2)$-part of this newly defined Hodge structure is given by complex conjugating $\widetilde{H}^{2,0}$ and its $(1,1)$-part as the orthogonal complement of $\widetilde{H}^{2,0}$ with respect to the Mukai pairing.

We may represent α by a Čech 2-cocycle $\{\alpha_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \mathcal{O}_X^*)\}$ with $X = \bigcup_{i\in I} U_i$ an appropriate open analytic cover. An *α-twisted (coherent) sheaf E* consists of pairs $(\{E_i\}_{i\in I}, \{\varphi_{ij}\}_{i,j\in I})$ such that the E_i are (coherent) sheaves on U_i and $\varphi_{ij} : E_i|_{U_i\cap U_j} \to E_j|_{U_i\cap U_j}$ are isomorphisms satisfying the following conditions:

i) $\varphi_{ii} = \mathrm{id}$, ii) $\varphi_{ji} = \varphi_{ij}^{-1}$, and iii) $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}$.

With this definition one can introduce the abelian category $\mathbf{Coh}(X,\alpha)$ α-twisted sheaves and its K-group denoted $K(X,\alpha)$.

Note that $\mathbf{Coh}(X,\alpha)$ does not depend on the Čech 2-cocycle $\{\alpha_{ijk}\}$. More precisely, for two different choices the categories are equivalent, but unfortunately the equivalence is not canonical (see [8] for details).

We will principally be interested in the bounded derived category $\mathrm{D^b}(X,\alpha)$ of $\mathbf{Coh}(X,\alpha)$.

2. Twisted derived categories and twisted Hodge structures

In order to study the relation between the twisted bounded derived category $\mathrm{D}^{\mathrm{b}}(X,\alpha_B)$ and the twisted Hodge structure $\widetilde{H}(X,B,\mathbb{Z})$ we need to introduce a twisted Chern character. There are different ways to do this. In order to work with integral cohomology, the one introduced in [7] seems appropriate.

Let us state this as follows:

Suppose $B\in H^2(X,\mathbb{Q})$ is a rational B-field such that its $(0,2)$-part $B^{0,2}\in H^2(X,\mathcal{O}_X)$ maps to α, i.e. $\exp(B^{0,2})=\alpha$.

Then there exists an additive map

$$\mathrm{ch}^B : K(X,\alpha) \longrightarrow H^*(X,\mathbb{Z})$$

such that:

i) If $B=\mathrm{c}_1(L)\in H^2(X,\mathbb{Z})$, then $\mathrm{ch}^B(E)=\exp(\mathrm{c}_1(L))\cdot\mathrm{ch}(E)$. (Note that with this assumption α is trivial and an α-twisted sheaf is just an ordinary sheaf.)

ii) For two choices $(B_1,\alpha_1:=\exp(B_1^{0,2}))$, $(B_2,\alpha_2:=\exp(B_2^{0,2}))$ and $E_i\in K(X,\alpha_i)$ one has

$$\mathrm{ch}^{B_1}(E_1)\cdot\mathrm{ch}^{B_2}(E_2)=\mathrm{ch}^{B_1+B_2}(E_1\otimes E_2).$$

In fact, the same construction works in full generality (only that in general it is only rational) and thus in particular for the product of two K3 surfaces. As in the original argument of Mukai one can show that any complex of $\alpha^{-1}\boxtimes\alpha'$-twisted sheaves on $X\times X'$ that induces a Fourier–Mukai equivalence $\mathrm{D}^{\mathrm{b}}(X,\alpha=\alpha_B)\cong\mathrm{D}^{\mathrm{b}}(X',\alpha'=\alpha_{B'})$ induces an Hodge isometry of the corresponding Hodge structures. This yields

Proposition 2.1. [7] *Any Fourier–Mukai equivalence*

$$\mathrm{D}^{\mathrm{b}}(X,\alpha_B)\cong\mathrm{D}^{\mathrm{b}}(X',\alpha_{B'})$$

naturally induces an Hodge isometry

$$\widetilde{H}(X,B,\mathbb{Z})\cong\widetilde{H}(X',B',\mathbb{Z}).$$

Although the situation is slighly more complicated in the twisted case, we could still imitate results of Mukai and Orlov to prove the converse direction for K3 surfaces with large Picard group

Proposition 2.2. [7] *Suppose there exists an Hodge isometry $\widetilde{H}(X,B,\mathbb{Z})\cong\widetilde{H}(X',B',\mathbb{Z})$ that preserves the natural orientation of the four positive directions. If $\rho(X)\le 11$, then this Hodge isometry is induced by a Fourier–Mukai equivalence $\mathrm{D}^{\mathrm{b}}(X,\alpha_B)\cong\mathrm{D}^{\mathrm{b}}(X',\alpha_{B'})$.*

The orientation issue seems to be hard even in the untwisted case. It is not too difficult to check that any known derived equivalence (shifts, spherical twists, isomorphisms, moduli spaces) all induce orientation preserving Hodge isometry. The case of the universal family of stable sheaves being the only difficult one. This suffices to prove the above proposition.

3. Caldararu's conjecture

Relying heavily on results of Yoshioka we are now able also to prove

Proposition 3.1. [8] *Let X and X' be two projective K3 surfaces endowed with B-fields $B \in H^2(X,\mathbb{Q})$ respectively $B' \in H^2(X',\mathbb{Q})$. Suppose there exists an Hodge isometry*

$$g : \widetilde{H}(X,B,\mathbb{Z}) \cong \widetilde{H}(X',B',\mathbb{Z})$$

that preserves the natural orientation of the four positive directions. Then there exists a Fourier–Mukai equivalence $\Phi : \mathrm{D^b}(X,\alpha) \cong \mathrm{D^b}(X',\alpha')$ such that the induced action $\Phi^{B,B'}$ on cohomology equals g. Here, $\alpha := \alpha_B$ and $\alpha' := \alpha_{B'}$ are the Brauer classes induced by B respectively B'.

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