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Vector Fields and Automorphism Groups

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1. Motivation.

In algebraic geometry, the basic relation between the Lie algebra of vector fields and the automorphism group schemes (see below) holds without any restriction. This suggests that there should be some parallel facts in complex geometry and/or differential geometry. So we try to set up some concepts about automorphism groups in complex geometry and in differential geometry which are quite similar to that about automorphism group schemes in algebraic geometry, and then study the corresonding problems and compare the results with that of algebraic geometry.

2. Some terminologies and notation.

Let S be a base scheme (usually noetherian) and $\tau : X \to S$ be a separated morphism of finite type. Denote by $\mathcal{D}er_S(O_X, O_X)$ the sheaf of O_S-derivations from O_X to O_X, which is isomorphic to $\mathcal{H}om_{O_X}(\Omega^1_{X/S}, O_X)$ as a coherent sheaf on X, and is a sheaf of O_X-Lie algebras. Hence $\tau_*\mathcal{D}er_S(O_X, O_X)$ is a quasi-coherent sheaf of O_S-Lie algebras, whose sections are called vector fields. In particular, if $S = \mathrm{Spec}k$ for a field k, then we denote $\Theta_{X/k} = H^0(X, \mathcal{D}er_S(O_X, O_X))$, the k-Lie algebra of vector fields on X. If $\mathrm{ch}(k) = p > 0$, $\Theta_{X/k}$ is a p-Lie algebra over k.

Let G be a separated S-group scheme of finite type. Denote by $\omega_{G/S}$ the sheaf of (left) invariant differentials of G over S, and $Lie(G/S) := \mathcal{H}om_{O_S}(\omega_{G/S}, O_S)$, the sheaf of left invariant derivations of G over S (which is a coherent sheaf of Lie algebras on S).

Let $\rho : G \times_S X \to G$ be an action of G on X over S. For any open affine subset $V \subset S$, denoting by $U = \tau^{-1}(V)$, a derivation $D \in \mathcal{D}er_S(O_X, O_X)(U)$ is called *ρ-invariant* if the following diagram is commutative:

$$\begin{array}{ccc} O_U & \xrightarrow{D} & O_U \\ \downarrow{\scriptstyle \rho^{\#}} & & \downarrow{\scriptstyle \rho^{\#}} \\ \rho_* O_{G\times_S U} & \xrightarrow{\rho_*\mathrm{pr}_2^*(D)} & \rho_* O_{G\times_S U} \end{array}$$

The sheaf of ρ-invariant derivations (as a sheaf of O_S-modules) will be denoted by $\mathcal{D}_\rho$, which is a quasi-coherent subsheaf of $\tau_*\mathcal{D}er_S(O_X, O_X)$ and is a sheaf Lie subalgebras. ρ induces a canonical homomorphism of quasi-coherent sheaves of O_S-Lie algebras

$$Lie(G/S) \to \mathcal{D}_\rho$$

For any S-scheme T, an automorphism f of $T \times_S X$ is called a *T-automorphism* if $\mathrm{pr}_1 \circ f = \mathrm{pr}_1 : T \times_S X \to T$. Denote by $Aut(T \times_S X/T)$ the set of T-automorphisms of $T \times_S X$, which has a group structure. If the following functor

$$\begin{aligned} \mathfrak{Aut}_{X/S}\colon\ ((S\text{-schemes})) &\to ((\text{sets})) \\ T &\mapsto Aut(T \times_S X/T) \end{aligned}$$

is representable, say represented by an S-scheme G and universal G-automorphism $\Phi : G \times_S X \to G \times_S X$, then G has a group scheme structure over S, called the *automorphism group scheme* of X over S, denoted by $G = Aut(X/S)$. By abstract nonsense we have

Fact. *Suppose $Aut(X/S)$ exists. Then*

i) *For any S-scheme T, $Aut(X \times_S T/T)$ exits and canonically $Aut(X \times_S T/T) \cong Aut(X/S) \times_S T$ (i.e. automorphism group schemes commute with base change).*

ii) *$\rho = \mathrm{pr}_2 \circ \Phi : Aut(X/S) \times_S X \to X$ is an action, and $(Aut(X/S), \rho)$ represents the following functor:*

$$\begin{aligned} \mathfrak{Act}_{X/S}\colon\ ((S\text{-group schemes})) &\to ((\text{sets})) \\ G &\mapsto \{\text{actions of } G \text{ on } X\} \end{aligned}$$

Explicitly, each homomorphism $h : G \to Aut(X/S)$ corresponds to the action $\rho \circ (h \times_S id_X) : G \times_S X \to X$.

Of course $Aut(X/S)$ may not exist. For example, if $S = \mathrm{Spec}k$ for a field k, then $Aut(\mathbb{A}^1_k/k)$ does not exist. On the other hand, if X is proper over k, then $Aut(X/k)$ exists.

When $S = \mathrm{Spec}k$ and $Aut(X/k)$ exists, we denote by $Aut^0(X/k)$ the zero component of $Aut(X/k)$, which is a normal subgroup scheme of $Aut(X/k)$.

3. The basic relation between vector fields and automorphism group schemes.

Theorem 1. *Let S be a noetherian scheme and $\tau : X \to S$ be a morphism of finite type such that $Aut(X/S)$ exists. Denote by $\rho : Aut(X/S) \times_S X \to X$ the universal action. Then there is a canonical isomorphism of coherent sheaves of O_S-Lie algebras*

$$Lie(Aut(X/S)/S) \xrightarrow{\cong} \mathcal{D}_\rho = \tau_* \mathcal{D}er_S(O_X, O_X) \tag{1}$$

Remark 1. i) For any morphism $\tau : X \to S$, one can define a "functor of vector fields":

$$\begin{aligned} \mathfrak{VF}_{X/S}\colon\ ((S\text{-schemes})) &\to ((\text{Lie algebras})) \\ T &\mapsto \Gamma(T \times_S X, \mathcal{T}_{T \times_S X/T}) \end{aligned}$$

and for any group scheme G over S, one can define a "functor of left invariant derivations":

$$\begin{aligned} \mathfrak{Lie}_{G/S}\colon\ ((S\text{-schemes})) &\to ((\text{Lie algebras})) \\ T &\mapsto \Gamma(T, Lie(G \times_S T/T)) \end{aligned}$$

Note that these two functors are usually **not** representable. Theorem 1 can be stated as: If $Aut(X/S)$ exists, then there is a natural equivalence of functors $\mathfrak{Lie}_{Aut(X/S)/S} \to \mathfrak{VF}_{X/S}$.

ii) If $S = \mathrm{Spec}k$ for a field k of characteristic $p > 0$, then (1) is an isomorphism of p-Lie algebras.

4. Some consequences and examples in algebraic geometry.

Corollary 1. *Let k be a field of characteristic 0 and X be a geometrically connected proper scheme over k. Then the following are equivalent:*

- i) *X is a homogeneous variety, i.e. $X \cong G/H$ for some connected group variety G and some subgroup scheme $H \subset G$;*
- ii) *There exists for some n a monomorphism of coherent sheaves $\Omega^1_{X/k} \hookrightarrow \mathcal{O}_X^{\oplus n}$ whose cokernel is locally free;*
- iii) *X is smooth over k and $\mathcal{T}_{X/k}$ is generated by global sections.*

Remark 2. We can define: a proper scheme X over a field k is called *homogeneous* if $\mathcal{A}ut^0(X/k)$ acts transitively on X.

In particular, if X is a geometrically connected proper scheme over a field k of characteristic 0 having a k-point such that $\Omega^1_{X/k}$ is trivial, then X is an abelian variety.

This does not hold if $\mathrm{ch}(k) \neq 0$. There is a famous counterexample of Igusa (1955):

Let E be an ordinary elliptic curve over a perfect field k of characteristic 2. Let $a \in E$ be the k-point of order 2. Let $X = E \times_k E$. Let $G = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$. Let $\bar{0}$ correspond to id_X and $\bar{1}$ correspond to the automorphism $(x, y) \mapsto (-x, y + a)$ of X. Let $Y = X/G$. Clearly Y is not an abelian variety. On the other hand, X has a closed subgroup scheme $H \cong \mu_2 \times_k E$ whose action on X (by translation) commutes with the action of G, hence induces a free action of H on Y. This shows that $\Omega^1_{Y/k} \cong \mathcal{O}_Y^2$.

In fact Y is not homogeneous, and $\mathcal{A}ut^0(Y/k) \cong H$. Furthermore, we have $\mathcal{P}ic^0(X/k) \cong H$.

Mehta and Srinivas (1987): If X is an ordinary smooth projective variety with trivial tangent bundle (in this case "ordinary" is equivalent to that the Frobenius of X induces a monomorphism on $H^1(X, \mathcal{O}_X)$), then there is an étale cover $\tilde{X} \to X$ such that $\tilde{X}$ is an abelian variety.

Corollary 2. *Let X be a proper variety of dimension n over a field k of characteristic $\neq 2$, having a k-point. If X is ordinary and $\Omega^1_{X/k} \cong \mathcal{O}_X^n$, then X is an abelian variety.*

If X is not ordinary, we have a counterexample when $\mathrm{ch}(k) = 3$: Let E be the supersingular elliptic curve $y^2 = x^3 - x$ over $\mathbb{F}_3$. It has an automorphism σ of order 3 given by $(x, y) \mapsto (x + 1, y)$. Let E' be an ordinary elliptic curve over a finite field K of characteristic 3 and $a \in E$ be a K-point of order 3. Let $X = E \times E'$. Then X has an automorphism of order 3 given by $(g, g') \mapsto (\sigma(g), g' + a)$. Let G be the group generated by this automorphism. We see that $Y = X/G$ has trivial cotengent sheaf, and is not an abelian variety (not homogeneous). But till now I have not seen any counterexample for $\mathrm{ch}(k) > 3$. We can conjecture that when $\mathrm{ch}(k) > 3$, any proper variety over k having a

k-point with trivial cotengent sheaf is an abelian variety.

Corollary 3. *Let $f : \tilde{X} \to X$ be a finite étale covering of proper varieties over a field k. If X is homogeneous, so is $\tilde{X}$.*

This can be viewed as a generalization of Serre-Lang Theorem.

5. Parallel facts in complex geometry.

For a real or complex analytic space X, we denote by Θ_X its Lie algebra of vector fields, and denote by O_X the sheaf of analytic functions.

For complex analytic spaces, the following fact is similar to the corresponding fact in algebraic geometry.

Theorem 2. *Let X be a compact complex analytic space. Then $Aut(X)$ has a complex Lie group structure, denoted by $\mathcal{A}ut(X)$, which has the following properties.*

i) *The canonical action of $\mathcal{A}ut(X)$ on X is analytic.*
ii) *$\mathcal{A}ut(X)$ represents the following functor*

$$\begin{aligned}((\text{complex analytic manifolds})) &\to ((\text{sets}))\\ T &\mapsto Aut(T \times X/T)\end{aligned}$$

iii) *The Lie algebra of $\mathcal{A}ut(X)$ is canonically isomorphic to Θ_X.*

The following corollaries are also similar to the corresponding fact in algebraic geometry.

Corollary 4. *Let X be a connected compact complex analytic space. Then the following are equivalent:*

i) *X is homogeneous, i.e. $X \cong G/H$ for a complex Lie group G and a Lie subgroup $H \subset G$;*
ii) *There exists for some n a monomorphism of coherent sheaves $\Omega^1_X \hookrightarrow O_X^{\oplus n}$ whose cokernel is locally free;*
iii) *X is a complex manifold and its tangent bundle is generated by vector fields.*

Corollary 5. *Let $f : Y \to X$ be a smooth analytic map of compact connected complex manifolds of the same dimension. If X is homogeneous, so is Y. (In particular, if X is a complex torus, so is Y.)*

However, the statement parallel to Corollary 2 does not hold, i.e. a compact complex manifold with trivial tangent bundle need not be a complex torus. We have the following counterexample given by Nakamura (1975):

Let $G \subset GL_3(\mathbb{C})$ be the Lie subgroup of upper-triangular matrices with diagonal entries 1. Let $K \subset \mathbb{C}$ be a quadratic imaginary field, and $O_K \subset K$ be the ring of integers. Let $H = GL_3(O_K) \cap G$, then H is a discrete subgroup of G. It is easy to check that the homogeneous space $X = G/H$ is compact, and X has a trivial tangent bundle. However X is not a Lie group, because a compact Lie group is a complex torus, but $\Theta_X \cong Lie(G)$ is not a commutative Lie algebra. In fact $Aut^0(X) \cong G$.

This shows that X is not algebraic. By GAGA, $SL_3(\mathbb{C})/SL_3(O_K)$ is not quasi-projective.

For trivial tangent bundles, we have the following result of Wang (1954):

Corollary 6. *Let X be a connected compact complex analytic manifold. Then X has trivial tangent bundle iff $X \cong G/H$ for a connected Lie group G and a discrete subgroup $H \subset G$. Furthermore, in this case the following are equivalent:*

i) *X is a Lie group;*
ii) *Θ_X is a commutative Lie algebra;*
iii) *X is a complex torus.*

6. Parallel facts in differential geometry.

For a real analytic space X, it is usually hard to set up a geometric structure of $Aut(X)$, this is simply because Θ_X is usually infinite (uncountably large) dimensional, and we have no good set-up of infinite dimensional Lie groups.

To get rid of this problem, we need to find a natural way to restrict Θ_X to a finite dimensional Lie subalgebra. There are at least two methods for this.

One method is to use some differential equations. Denote by $Diff(O_X, O_X)$ the sheaf of linear differential operators from O_X to O_X. Let $\mathcal{H} \subset Diff(O_X, O_X)$ be a coherent subsheaf. For any open subset $U \subset X$ such that $\mathcal{H}|_U$ is generated by $\mathcal{H}(U)$, let

$$\mathcal{K}(U) = \bigcap_{D \in \mathcal{H}(U)} \ker(D : O_X(U) \to O_X(U))$$

Then the $\mathcal{K}(U)$'s glue together to give an $\mathbb{R}$-linear subsheaf $\mathcal{K} \subset O_X$. We may denote $\mathcal{K} = \ker(\mathcal{H})$. A vector field $\theta \in \Theta_X$ is called *$\mathcal{H}$-harmonic* if $\theta(\mathcal{K}) \subset \mathcal{K}$. The $\mathcal{H}$-harmonic vector fields form a Lie subalgebra of Θ_X, denoted by $\Theta_{\mathcal{H}}$. An automorphism $g : X \to X$ is called *$\mathcal{H}$-harmonic* if $g^*(\mathcal{K}) = \mathcal{K}$. The $\mathcal{H}$-harmonic automorphisms form a subgroup of $Aut(X)$, denoted by $Aut(X, \mathcal{H})$. Furthermore, for a real analytic manifold T, a T-automorphism f of $T \times X$ is called *$\mathcal{H}$-harmonic* if $f^*\mathrm{pr}_2^*\mathcal{K} = \mathrm{pr}_2^*\mathcal{K}$. this is equivalent to that each fiber f_t (over $t \in T$) is $\mathcal{H}$-harmonic.

Theorem 3. *Let X be a compact real analytic space, and $\mathcal{H}$ be a coherent subsheaf of $Diff(O_X, O_X)$ such that $\dim_{\mathbb{R}} \Theta_{\mathcal{H}} < \infty$. Then $Aut(X, \mathcal{H})$ has a real Lie group structure, denoted by $Aut(X, \mathcal{H})$, which has the following properties.*

i) *The canonical action of $Aut(X, \mathcal{H})$ on X is analytic.*
ii) *$Aut(X, \mathcal{H})$ represents the following functor*

$$((\text{real analytic manifolds})) \to ((\text{sets}))$$
$$T \mapsto Aut(T \times X/T, \mathcal{H})$$

iii) *The Lie algebra of $Aut(X, \mathcal{H})$ is canonically isomorphic to $\Theta_{\mathcal{H}}$.*

Another method is to use a metric. Let μ be a metric on a real analytic manifold X, i.e. a symmetric positive definite bilinear map

$$\langle , \rangle_\mu : \mathcal{T}_X \times_X \mathcal{T}_X \to \mathbb{R} \times X \tag{1}$$

of analytic vector bundles over X. This can be understood as an O_X-linear map $\mathcal{T}_X \otimes_{O_X} \mathcal{T}_X \to O_X$ (where $\mathcal{T}_X$ is viewed as the tangent sheaf of X), or a bilinear form on $T_{X,x}$ for each $x \in X$ which varies analytically with respect to x. An automorphism g of X is called *μ-orthogonal* if for any sections $\tilde{D}, \tilde{D}'$ of $\mathcal{T}_X$ (over some open subset $U \subset X$),

$$\langle g_* \tilde{D}, g_* \tilde{D}' \rangle_\mu = g^{-1*} \langle \tilde{D}, \tilde{D}' \rangle_\mu$$

where $g_* \tilde{D} = g^{-1*} \circ \tilde{D} \circ g^* : O_{gU} \to O_{gU}$. This can be understood as: for any $x \in X$ and any $D, D' \in T_{X,x}$, $\langle g_* D, g_* D' \rangle_\mu = \langle D, D' \rangle_\mu$. All of the μ-orthogonal automorphisms of X form a group, denoted by $Aut(X, \mu)$.

More generally, for an analytic manifold T, a T-automorphism $f \in Aut(T \times X/T)$ is called *μ-orthogonal* if its fibers over T are all μ-orthogonal. Denote by $Aut(T \times X/T, \mu)$ the set of μ-orthogonal T-automorphisms of $T \times X$, which is a subgroup of $Aut(T \times X/T)$.

A vector field $\theta \in \Theta_X$ is called *μ-orthogonal* if for any sections $\tilde{D}, \tilde{D}'$ of $\mathcal{T}_X$ (over some open subset $U \subset X$),

$$\theta \langle \tilde{D}, \tilde{D}' \rangle_\mu = \langle [\theta, \tilde{D}], \tilde{D}' \rangle_\mu + \langle \tilde{D}, [\theta, \tilde{D}'] \rangle_\mu$$

It is easy to see that the μ-orthogonal vector fields in Θ_X form a Lie subalgebra, denoted by Θ_μ.

Theorem 4. *Let X be a real compact analytic manifold, and μ be a metric on X. If $\dim \Theta_\mu < \infty$, then $Aut(X, \mu)$ has a real Lie group structure, denoted by $Aut(X, \mu)$, which has the following properties.*

i) *The canonical action of $Aut(X, \mu)$ on X is analytic.*

ii) *$Aut(X, \mu)$ represents the following functor*

$$\begin{aligned} ((\text{real analytic manifolds})) &\to ((\text{sets})) \\ T &\mapsto Aut(T \times X/T, \mu) \end{aligned}$$

iii) *The Lie algebra of $Aut(X, \mu)$ is canonically isomorphic to Θ_μ.*

Remark 3. The above formulation works for any bilinear form as in (1), not necessarily symmetric or positive definite. Furthermore, we can use more than one bilinear forms.

We can also combine the above two methods, i.e. use both differential equations and metrics. As long as we can get a finite dimensional Lie subalgebra of Θ_X, we have a similar theorem. In particular, this can be set up for complex symplectic manifolds.

Remark 4. We see that the concept of fine moduli can be adopted to complex geometry and differential geometry, for automorphism groups. This concept can also be adopted for Picard group, at least for manifolds. We hope there be more examples on this.

4. The idea of the proof.

In the proof of the above results, a key method is the so called "calculus of actions" (by "calculus" we mean differential operators, de Rham complexes, Lie algebra of vector fields, connections, etc.).

For simplicity we denote $k = \mathbb{R}$ or $\mathbb{C}$. For a Lie group G over k, denote by ω_G the k-linear space of (left) invariant differentials. Similar to algebraic geometry, for an action ρ of G on an analytic space X over k, we can define ρ-invariant vector fields on X, and the ρ-invariant vector fields form a Lie subalgebra $\Theta_\rho \subset \Theta_X$.

The following are some facts in calculus of actions.

Lemma 1. *Let G be a Lie group over k, X be an analytic space over k and $\rho : G \times X \to X$ be an analytic action. Then*

i) *There is a canonical complex induced by ρ:*

$$O_X \to \omega_G \otimes_k O_X \to \bigwedge_k^2 \omega_G \otimes_k O_X \to \bigwedge_k^3 \omega_G \otimes_k O_X \dots \tag{2}$$

ii) *The identity map of O_X induces a (unique) canonical O_X-linear map from $\Omega^{\cdot}_X$ to (2), in particular ρ induces a caninical O_X-linear map*

$$\Omega^1_X \to \omega_G \otimes_k O_X \tag{3}$$

which is surjective when ρ is free.

iii) *ρ induces a canonical homomorphism*

$$\rho_* : Lie(G) \to \Theta_\rho \tag{4}$$

of Lie algebras over k.

Lemma 2. *Let X be an analytic space over k. Let T be an analytic manifold and f be a T-automorphism of $T \times X$. Then f induces a canonical homomorphism $f_* : \mathcal{T}_T \to \Theta_X \times T$ of vector bundles over T. Furthermore, if $T = G$ is a Lie group and $f = \Phi_\rho = (\mathrm{pr}_1, \rho)$ for an analytic action $\rho : G \times X \to X$, then f_* coincides with the canonical homomorphism induced by (4):*

$$f_* = \rho_* \times \mathrm{id}_G : \mathcal{T}_G \cong Lie(G) \times G \to \Theta_X \times G \tag{5}$$

The above facts are compatible with respect to a harmonic structure and/or a metric. These are all parallel to some facts in algebraic geometry.

In algebraic geometry, we combine calculus with deformation theory, and we use calculus to prove the basic relation between the Lie algebra of vector fields and the automorphism group scheme. But in differential geometry, we combine calculus with solutions of differential equations, and we bear in mind the basic relation between the Lie algebra of vector fields and the automorphism group to set up the Lie group structure of the automorphism group.

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