代数幾何学シンポジウム記録
2004年度 pp.77-91

Derived categories of coherent sheaves on algebraic surfaces

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Abstract

I give an overview of several results on derived categories of coherent sheaves on smooth algebraic surfaces. I also explain our new results in [Ue04] and [IU04].

1 Introduction

Let X be a smooth projective variety over $\mathbb{C}$. The derived category $D(X)$ of X is a triangulated category whose objects are bounded complexes of coherent sheaves on X. By an equivalence $D(Y) \simeq D(X)$, we always mean a $\mathbb{C}$-linear equivalence of triangulated categories.

If there exists an equivalence between $D(Y)$ and $D(X)$, we call X a *Fourier-Mukai (FM) partner* of Y. We denote the set of isomorphism classes of FM partners of X by $FM(X)$.

$$\mathrm{FM}(X) := \{Y \text{ smooth projective variety } \mid D(X) \simeq D(Y)\}/\cong$$

We also denote the group of isomorphism classes of autoequivalences of $D(X)$ by $\operatorname{Auteq} D(X)$.

$$\operatorname{Auteq} D(X) := \{\Phi : D(X) \simeq D(X) \text{ autoequivalence }\}/\cong$$

It is quite important to answer the following problem.

Problem 1.1. Suppose that we are given a smooth projective variety X. Then describe (i) $\mathrm{FM}(X)$ and (ii) $\operatorname{Auteq} D(X)$.

In this article, I explain some known results on Problem 1.1 for smooth algebraic surfaces. We treat Problem 1.1(i) in §2 and (ii) in §3.

Notation and conventions.

(i) All varieties are defined over $\mathbb{C}$.

(ii) Let X and Y be smooth projective varieties. We say that X and Y are *K-equivalent* if there exist a smooth projective variety Z and birational morphisms $f : Z \to X$, $g : Z \to Y$ such that $f^*K_X \sim g^*K_Y$.

(iii) For a set I, we denote by $|I|$ the cardinality of I.

(iv) Let X be an algebraic variety and Z a closed subset of X. $D_Z(X)$ denotes the full subcategory of $D(X)$ consisting of objects supported on Z.

2 Fourier-Mukai partners of smooth projective surfaces

In [BO01], Bondal and Orlov showed that a lot of information can be extracted from the objects $\{\mathcal{O}_x | x \in X\}$ when the canonical divisor K_X or the anti-canonical divisor $-K_X$ is ample. In particular, in this case they further showed that $Y \simeq X$ if $D(Y) \simeq D(X)$. Recently, Kawamata [Ka02] obtained a generalization of this result; Y is K-equivalent to X if $D(X) \simeq D(Y)$ and if K_X or $-K_X$ is big.

On the other hand, Mukai [Mu81] showed that the Poincaré bundle $\mathcal{P}$ on $A \times \hat{A}$, where A is an abelian variety and $\hat{A} := \mathrm{Pic}^0 A$, induces an equivalence $\Phi^{\mathcal{P}}_{\hat{A}\to A} : D(\hat{A}) \to D(A)$. The functor $\Phi^{\mathcal{P}}_{\hat{A}\to A}$ is the so-called Fourier-Mukai transform defined by

$$\Phi^{\mathcal{P}}_{\hat{A}\to A}(-) = \mathbf{R}\pi_{A*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathbf{L}\pi^*_{\hat{A}}(-)),$$

where $\pi_{\hat{A}} : \hat{A} \times A \to \hat{A}$ and $\pi_A : \hat{A} \times A \to A$ are the natural projections. In this equivalence, the structure sheaf $\mathcal{O}_{\hat{a}}$ of the point $\hat{a} \in \hat{A}$ is mapped to the invertible sheaf $\mathcal{P}_{\hat{a}}$ on A corresponding to $\hat{a}$. Therefore the derived category of an abelian variety does not characterize the structure sheaves of points anymore. Indeed, $\hat{A}$ is not (even birationally) isomorphic to A in general even when $D(\hat{A}) \simeq D(A)$.

This example suggests that when K_X is trivial, $D(X)$ no longer has enough information to reconstruct (the birationally equivalence class of) the variety X, and that a new and interesting relationship arises between varieties X, Y with $D(X) \simeq D(Y)$.

In connection with Problem 1.1(i) for smooth projective surfaces, we have the following.

Theorem 2.1 ([BM01], [Ka02]). *Let S be a smooth projective surface with $|\mathrm{FM}(S)| > 1$. Take $T \in \mathrm{FM}(S)$ such that $S \not\cong T$. Then one of the following holds.*

(i) S and T are K3 surfaces.

(ii) S and T are abelian surfaces.

(iii) S and T are minimal elliptic surfaces with the non-zero Kodaira dimension $\kappa(S) = \kappa(T)$.

Combining the result above with Theorems 2.2, 2.3, we obtain a complete answer to Problem 1.1(i) for the surface case.

2.1 K3 surfaces and abelian surfaces

Let S be a K3 or an abelian surface. We shall recall the Mukai lattice [Mu87]. We define a symmetric bilinear form on $H^{ev}(S,\mathbb{Z}) := \oplus_i H^{2i}(S,\mathbb{Z})$;

$$\begin{aligned}\langle x, y\rangle &:= -\int_S (x^\vee y)\\ &= \int_S (x_1 y_1 - x_0 y_2 - x_2 y_0)\end{aligned}$$

where $x = x_0 + x_1 + x_2, x_i \in H^{2i}(S,\mathbb{Z})$ (resp. $y = y_0 + y_1 + y_2, y_i \in H^{2i}(S,\mathbb{Z})$) and $x^\vee = x_0 - x_1 + x_2$. We define a weight 2 Hodge structure by

$$\begin{cases} H^{0,2}(H^{ev}(S,\mathbb{C})) = H^{0,2}(S)\\ H^{1,1}(H^{ev}(S,\mathbb{C})) = H^{0,0}(S) \oplus H^{1,1}(S) \oplus H^{2,2}(S)\\ H^{2,0}(H^{ev}(S,\mathbb{C})) = H^{2,0}(S).\end{cases}$$

We call this lattice *Mukai lattice.* Let us denote by $T(S)$ the transcendental lattice of S, i.e. $T(S) := NS(S)^\perp$ in $H^2(S,\mathbb{Z})$. The transcendental lattice inherits a Hodge structure from $H^2(S,\mathbb{Z})$.

Now we can describe FM partners of K3 (resp. abelian) surfaces as follows:

Theorem 2.2 ([Mu87], [Or97]. See also [BM01]). *Let S and T be K3 (resp. abelian) surfaces. The following statements are equivalent.*

(i) T is is an FM partner of S.

(ii) there is a Hodge isometry of Mukai lattices $\varphi : H^{ev}(S,\mathbb{Z}) \to H^{ev}(T,\mathbb{Z})$.

(iii) there is a Hodge isometry of transcendental lattices $\varphi : T(S) \to T(T)$.

(iv) T is isomorphic to a fine, two-dimensional moduli space of stable sheaves on S.

2.2 Minimal elliptic surfaces

Let $\pi : S \to C$ be a minimal elliptic surface. For an object E of $D(S)$, we define the fiber degree of E

$$d(E) = c_1(E) \cdot f,$$

where f is a general fiber of π. Let us denote by $\lambda_{S/C}$ the highest common factor of the fiber degrees of objects of $D(S)$. Equivalently, $\lambda_{S/C}$ is the smallest number d such that there is a holomorphic d-section of π. For integers $a > 0$ and i with i coprime to $a\lambda_{S/C}$, by [Br98] there exists a smooth, 2-dimensional component $J_S(a,i)$ of the moduli space of pure dimension one stable sheaves on S, the general point of which represents a rank a, degree i stable vector bundle supported on a smooth fiber of π. There is a natural morphism $J_S(a,i) \to C$, taking a point representing a sheaf supported on the fiber $\pi^{-1}(x)$ of S to the point x. This morphism is a minimal elliptic fibration ([Br98]). Put $J^i(S) := J_S(1,i)$. Obviously, $J^0(S) \cong J(S)$, the Jacobian surface associated to S, and $J^1(S) \cong S$.

We can describe FM partners of minimal elliptic surfaces with non-zero Kodaira dimension as follows:

Theorem 2.3 ([BM01]). *Let $\pi : S \to C$ be a minimal elliptic surface and T a smooth projective variety. Assume that the Kodaira dimension $\kappa(S)$ is non-zero. Then the following are equivalent.*

(i) T is an FM partner of S.

(ii) T is isomorphic to $J^i(S)$ for some integer b with $(i, \lambda_{S/C}) = 1$.

Remark 2.4. Take a divisor D on S such that $D \cdot f = \lambda_{S/C}$. Then tensoring with $\mathcal{O}_S(D)$ gives an isomorphism

$$J^i(S) \cong J^{i+\lambda_{S/C}}(S).$$

Therefore if $\pi : S \to C$ has a section, we have

$$J^i(S) \cong J^j(S)$$

for all $i, j \in \mathbb{Z}$. Namely in this case we can conclude $FM(S) = \{S\}$.

2.3 Nontrivial FM partners

It was well-known that there are K3 (resp. abelian) surfaces S, T such that $D(S) \simeq D(T)$ but $S \not\cong T$, namely the cases (i) and (ii) in Theorem 2.1 really occur. More strongly, Oguiso and others recently proved:

Theorem 2.5 ([Og02] and [HLOY]). *Let N be a positive integer. Then there are K3 (respectively, abelian) surfaces S such that $|\operatorname{FM}(S)| \geq N$.*

In the proof of Theorem 2.5, they actually show the following statement: *There are K3 (respectively, abelian) surfaces S_i ($1 \leq i \leq N$) such that (i) for any $1 \leq i < j \leq N$, the Néron-Severi lattices $NS(S_i)$ and $NS(S_j)$ are not isomorphic, therefore $S_i \not\simeq S_j$ as well, but such that (ii) there is a Hodge isometry $\varphi_{ij} : T(S_i) \to T(S_j)$ for any $1 \leq i, j \leq N$.*

Theorem 2.5 follows from this by Theorem 2.2. In my paper [Ue04], I show a similar result to Theorem 2.5 in the case of minimal elliptic rational surfaces. But my proof is very different from the one of Theorem 2.5 because we have no such lattice-theoretic characterization of FM partners of minimal elliptic surfaces as in Theorem 2.2.

Theorem 2.6 ([Ue04]). *(i) Let p be a positive integer. Then there is a rational elliptic surface $S(p)$ such that $S(p)$ has a singular fiber of type ${}_pI_0$ and at least three non-multiple singular fibers of different Kodaira's types.*

(ii) Let N be a positive integer and p a prime number such that $p > 6(N-1)+1$. Then we have $|\operatorname{FM}(S(p))| \geq N$.

In contrast to Theorem 2.5 and Theorem 2.6, it is predicted that the set $\operatorname{FM}(X)$ is finite for any smooth projective variety X. Actually this was shown for the 2-dimensional case ([BM01] and [Ka02]).

It was conjectured by Kawamata in [Ka02, Conjecture 1.2] that given birationally equivalent smooth projective varieties X and Y, they are FM partners each other if and only if they are K-equivalent. Theorem 2.6 produces a counterexample to his conjecture as follows:

Corollary 2.7 ([Ue04]). *As a special case in Theorem 2.6, $S = S(11)$ has an FM partner T such that $T \not\cong S$. These S and T are birational FM partners, but they are not K-equivalent.*

Proof. Note that if varieties X and Y are K-equivalent, they are isomorphic in codimension 1 ([Ka02, Lemma 4.2]). In particular, if surfaces S and T are not isomorphic, they are not K-equivalent. We can also conclude that T

is rational, since $\kappa(T) = -\infty$ and the Euler numbers $e(T)$ and $e(S)$ coincide ([BM01, Proposition 2.3]). □

Theorem 2.6(i) follows from the Persson's list [Pe90] and the logarithmic transform. We shall explain how to show Theorem 2.6(ii) below.

2.4 Sketch of the proof of Theorem 2.6(ii).

First we need some standard notation. Fix a minimal elliptic surface with a section $\pi : B \to C$. Let $\eta = \operatorname{Spec} k$ be the generic point of C, where $k = k(C)$ is the function field of C, and let $\overline{k}$ be the algebraic closure of k. Put $\overline{\eta} = \operatorname{Spec} \overline{k}$. We define the *Weil-Chatelet group* $WC(B)$ by the Galois cohomology $H^1(G, B_\eta(\overline{k}))$. Here $G = \operatorname{Gal}(\overline{k}/k)$ and $B_\eta(\overline{k})$ is the group of points of the elliptic curve B_η defined over $\overline{k}$. Suppose that we are given a pair (S, φ), where S is an elliptic surface $S \to C$ and φ is an isomorphism $J(S) \to B$ over C, fixing their 0-sections. Then we have a morphism

$$B_\eta \times S_\eta \to J(S)_\eta \times S_\eta \to S_\eta.$$

Here the first morphism is induced by $\varphi^{-1} \times id_S$ and the second is given by translation. We obtain a principal homogeneous space S_η of B_η. Since this correspondence is invertible and the group $H^1(G, B_\eta(\overline{k}))$ classifies isomorphism classes of principal homogeneous spaces of B_η, we know that $WC(B)$ consists of all isomorphism classes of pairs (S, φ). Here two pairs (S, φ) and (S', φ') are *isomorphic* if there is an isomorphism $\alpha : S \to S'$ over C, such that $\varphi' \circ \alpha_* = \varphi$, where $\alpha_* : J(S) \to J(S')$ is the isomorphism induced by α (fixing 0-sections).

$$\begin{array}{ccc} J(S) & \xrightarrow{\alpha_*} & J(S') \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi'} \\ B & = & B \end{array}$$

Now let $\pi : S(p) \to C$ be the elliptic fibration obtained in (i). Put $S := S(p)$ and $B := J(S)$. Because every (-1)-curve on S is a p-section of π, we know that $\lambda_{S/C} = p$. For $i \in \mathbb{Z}$, there is an isomorphism $\varphi_i : J(J^i(S)) \to B$ such that $(J^i(S), \varphi_i)$ corresponds to $i\xi \in WC(B)$ ([Fr95, page 38]). By Theorem 2.3, each $J^i(S)$ is an FM partner of S for $1 \le i < p$. We can also check $J^i(S)$ is rational (see the proof of Corollary 2.7). Put

$$I = \{1, \ldots, p-1\}, \qquad I(a) = \{ i \in I \mid J^i(S) \cong J^a(S) \}$$

for $a \in I$. Then there are $i_1, \ldots, i_M \in I$ such that $I = \coprod_{k=1}^{M} I(i_k)$ (disjoint union). Fix $a \in I$. We construct an injection (between sets) $\Phi : I(a) \to G$, where

$$G := \{\gamma \in \mathrm{Aut}(B/C) \mid \gamma \text{ fixes the zero section of } B \to C\}$$

To construct Φ, we use the information of singular fibers of $S \to C$ (see [Ue04] for the details). Then since the order of the group G is at most 6, we have $6M \geq |I| = p - 1$. By the assumption $p > 6(N-1)+1$, we have $M \geq N$, which completes the proof of Theorem 2.6(ii). □

3 Autoequivalences of derived categories

3.1 Twist functors

Let X be a smooth projective variety. We note that $\mathrm{Auteq}\, D(X)$ always contains the group $A(X) := (\mathrm{Aut}\, X \ltimes \mathrm{Pic}\, X) \times \mathbb{Z}$, generated by functors of tensoring with invertible sheaves, automorphisms of X and the shift functor. When K_X or $-K_X$ is ample, it is shown that $\mathrm{Auteq}\, D(X) \cong A(X)$ in [BO01]. When X is an abelian variety, Orlov solves Problem 1.1 in [Or02]. In this case, $\mathrm{Auteq}\, D(X)$ is strictly larger than $A(X)$.

The twist functors along spherical objects are autoequivalences of another kind that are not in $A(X)$. Seidel and Thomas [ST01] introduced them, expecting that they should correspond via Kontsevich's homological mirror conjecture to the generalized Dehn twists along Lagrangian spheres. These functors play an essential role in our paper [IU04] and we recall the definition.

For an object $\mathcal{P} \in D(X \times Y)$, an *integral functor*

$$\Phi^{\mathcal{P}}_{X \to Y} : D(X) \to D(Y)$$

is defined by

$$\Phi^{\mathcal{P}}_{X \to Y}(-) = \mathbf{R}\pi_{Y*}(\mathcal{P} \overset{\mathbf{L}}{\otimes} \mathbf{L}\pi_X^*(-)),$$

where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the projections.

Definition 3.1 ([ST01]). (i) We say that an object $\alpha \in D(X)$ is *spherical* if we have $\alpha \otimes \omega_X \cong \alpha$ and

$$\mathrm{Hom}^k_{D(X)}(\alpha, \alpha) \cong \begin{cases} 0 & k \neq 0, \dim X \\ \mathbb{C} & k = 0, \dim X. \end{cases}$$

(ii) Let $\alpha \in D(X)$ be a spherical object. We consider the mapping cone

$$\mathcal{C} = Cone(\pi_1^*\alpha^\vee \overset{\mathbf{L}}{\otimes} \pi_2^*\alpha \to \mathcal{O}_\Delta)$$

of the natural evaluation $\pi_1^*\alpha^\vee \overset{\mathbf{L}}{\otimes} \pi_2^*\alpha \to \mathcal{O}_\Delta$, where $\Delta \subset X \times X$ is the diagonal, and π_i is the projection of $X \times X$ to the i-th factor. Then the integral functor $T_\alpha := \Phi^{\mathcal{C}}_{X \to X}$ defines an autoequivalence of $D(X)$, called the *twist functor* along the spherical object α.

Example 3.2. (i) Let X be a K3 surface and $\mathcal{L}$ a line bundle on Z. Then $\mathcal{L}$ is a spherical object of $D(X)$.

(ii) Let Z be the fundamental cycle of -2-curves in ADE configurations on a smooth surface X and $\mathcal{L}$ a line bundle on Z. Then $\mathcal{L}$ is a spherical object of $D_Z(X)$.

In Example 3.12 below, I give a highly non-trivial example of a spherical object.

Consider the derived category $D(X)$ for a smooth surface X. It is natural to ask how large the subgroup of $\operatorname{Auteq} D(X)$ generated by $A(X)$ and the twists along spherical objects is.

In our paper [IU04], we consider a chain Z of -2-curves on a smooth surface X and study the autoequivalences of the derived category $D_Z(X)$ of coherent sheaves on X supported by Z.

Note that the twist functor T_α can be defined as long as the support of α is projective, even if X is not projective. Moreover, the category $D_Z(X)$ depends only on the formal neighborhood of Z in X. Thus we can assume as follows:

$$Y = \operatorname{Spec} \mathbb{C}[[x,y,z]]/(x^2 + y^2 + z^{n+1})$$

is the A_n-singularity,

$$f : X \to Y$$

its minimal resolution and

$$Z = f^{-1}(P) = C_1 \cup \cdots \cup C_n$$

where $P \in Y$ is the closed point.

For an autoequivalence $\Phi \in \operatorname{Auteq} D_Z(X)$, we don't know if it is always isomorphic to an integral functor. Here, an integral functor from $D_Z(X)$ to $D_Z(X)$ is defined by an object $\mathcal{P} \in D(X \times X)$ whose support is projective over X with respect to each projection. If an autoequivalence is given as an integral functor, we call it a *Fourier-Mukai transform* (*FM transform*). Let

$$\operatorname{Auteq}^{\mathrm{FM}} D_Z(X) \subset \operatorname{Auteq} D_Z(X)$$

be the subgroup consisting of FM transforms. Remark that $\operatorname{Aut} X \cong \operatorname{Aut} Y$ and $\operatorname{Pic} X \cong \operatorname{Pic}(X/Y)$ act faithfully on $D_Z(X)$ in our setting; therefore we see $A(X) \subset \operatorname{Auteq}^{\mathrm{FM}} D_Z(X)$.

We denote the dualizing sheaf on Z by ω_Z and put

$$B = \left\langle T_{\mathcal{O}_{C_l}(-1)}, T_{\omega_Z} \;\middle|\; 1 \le l \le n \right\rangle \subset \operatorname{Auteq}^{\mathrm{FM}} D_Z(X).$$

The following is a main result of [IU04].

Theorem 3.3 (**[IU04]**). *We have*

$$\operatorname{Auteq}^{FM} D_Z(X) = (\langle B, \operatorname{Pic} X\rangle \rtimes \operatorname{Aut} X) \times \mathbb{Z}.$$

Here $\mathbb{Z}$ is the group generated by the shift [1].

Remark 3.4. We know more about subgroups of $\operatorname{Auteq}^{\mathrm{FM}} D_Z(X)$, that is, we have the following:

- $B \cap \operatorname{Pic} X = \langle \otimes\mathcal{O}_X(C_1), \ldots, \otimes\mathcal{O}_X(C_n)\rangle$.
- $\langle B, \operatorname{Pic} X\rangle \cong B \rtimes \mathbb{Z}/(n+1)\mathbb{Z}$.
- $B = \langle T_\alpha \mid \alpha \in D_Z(X),\ \textit{spherical}\ \rangle$.

Put $\alpha_i := \mathcal{O}_{C_i}(-1)$ $(1 \le i \le n)$ and $\alpha_0 := \alpha_{n+1} := \omega_Z$, where we consider the suffix i of α_i modulo $n+1$ (that is, $\alpha_i = \alpha_{n+1+i}$ for all $i \in \mathbb{Z}$). B is generated by all T_{α_i}'s by definition. We denote by B_k the subgroup of B generated by all T_{α_i}'s except T_{α_k}. The result in [ST01] implies that the defining relations of the group B_k is as follows:

$$\begin{cases} T_{\alpha_i}T_{\alpha_{i+1}}T_{\alpha_i} \cong T_{\alpha_{i+1}}T_{\alpha_i}T_{\alpha_{i+1}} & \textit{if}\ 0 \le i \le n, \quad i \ne k-1, k \\ T_{\alpha_i}T_{\alpha_j} \cong T_{\alpha_j}T_{\alpha_i} & \textit{if}\ i-j \ne \pm 1, 0. \end{cases}$$

In other words, B_k is the Artin group of type A_n (the braid group on $n+1$ strands). Therefore the generators, $T_{\mathcal{O}_{C_l}(-1)}$ $(1 \le l \le n)$ and T_{ω_Z}, of B satisfy the defining relations of the Artin group of type $\tilde{A}_n$, which is denoted by $\mathcal{A}(\tilde{A}_n)$. Consequently, we know that the group $\mathcal{A}(\tilde{A}_n)$ acts on $\operatorname{Auteq}^{\mathrm{FM}} D_Z(X)$.

Conjecture 3.5. *The action of the group $\mathcal{A}(\tilde{A}_n)$ on* $\operatorname{Auteq}^{FM} D_Z(X)$ *is faithful. Therefore the group B is the Artin group of type $\tilde{A}_n$.*

According to Orlov's theorem [Or97], any autoequivalence $\Phi \in \operatorname{Auteq} D(S)$ for a smooth *projective* variety S is isomorphic to an integral functor $\Phi^{\mathcal{P}}_{S\to S}$ for some $\mathcal{P} \in D(S \times S)$. Using the Orlov's theorem, we obtain Theorem 3.6 below. The main part of the proof of Theorem 3.6 is parallel to the one of Theorem 3.3.

Theorem 3.6 ([IU04]). *Let S be a smooth projective surface of general type whose canonical model has A_n-singularities at worst. Then we have*

$$\operatorname{Auteq} D(S) = \left\langle T_{\mathcal{O}_C(a)}, A(S) \mid C : -2\text{-curve},\ a \in \mathbb{Z} \right\rangle.$$

3.2 Sketch of the proof of Theorem 3.3.

We explain how to show Theorem 3.3.

Let η_i be the generic point of C_i. For a coherent sheaf $\mathcal{E} \in \operatorname{Coh}_Z(X)$, we define

$$l(\mathcal{E}) := \sum_i \operatorname{length}_{\mathcal{O}_{X,\eta_i}} \mathcal{E}_{\eta_i}$$

and for $\alpha \in D_Z(X)$, define

$$l(\alpha) := \sum_p l(\mathcal{H}^p(\alpha)).$$

When α is spherical, we can see that every cohomology sheaf $\mathcal{H}^p(\alpha)$ is a pure one-dimensional $\mathcal{O}_Z$-module (Lemma 3.7). Hence if $l(\alpha) = 1$, we get $\alpha \cong \mathcal{O}_{C_b}(a)[i]$ for some $a, b, i \in \mathbb{Z}$.

We divide the proof of Theorem 3.3 into two steps.

3.2.1 Step 1

First of all, we prove the following.

Key Proposition. *For a spherical α with $l(\alpha) > 1$, there is an autoequivalence $\Psi \in B$ such that $l(\alpha) > l(\Psi(\alpha))$.*

In the proof of the theorem, showing Key Proposition is the most essential. Our proof of Key Proposition is rather technical and difficult, so I give a very rough idea of it here. In §3.3, I find $\Psi \in B$ as in the proposition for the concrete example.

By using a spectral sequence, we can show

$$l(\Psi(\alpha)) \leq \sum_q l(\Psi(\mathcal{H}^q(\alpha))) \tag{1}$$

for any $\Psi \in \operatorname{Auteq} D_Z(X)$ ([IU04, Lemma 3.11]). Therefore to prove Key Claim it suffices to find $\Psi \in B$ such that

$$\sum_p l(\Psi(\mathcal{H}^p(\alpha))) < \sum_p l(\mathcal{H}^p(\alpha))(= l(\alpha)).$$

Actually we can find such Ψ when[1] $n > 1$.

To carry out this, we need more information of cohomology sheaves of spherical objects in $D_Z(X)$. The following lemma is a key tool to enable computations.

Recall that a coherent sheaf $\mathcal{F}$ on a variety X is *rigid* if $\operatorname{Ext}^1_X(\mathcal{F},\mathcal{F}) = 0$.

Lemma 3.7 ([IU04]). *(i) If $\alpha \in D_Z(X)$ is a spherical object, then the sheaf $\bigoplus_p \mathcal{H}^p(\alpha)$ is a rigid $\mathcal{O}_Z$-module, pure of dimension* 1.

(ii) Let $\mathcal{E}$ be a coherent $\mathcal{O}_Z$-module, pure of dimension 1. Then $\mathcal{E}$ decomposes into a direct sum of sheaves in $\Sigma(Z)$, where $\Sigma(Z)$ is the set of line bundles $\mathcal{L}$ on $C_s \cup \cdots \cup C_t$ where $1 \leq s \leq t \leq n$.

3.2.2 Step 2

For $\Phi \in \operatorname{Auteq} D_Z(X)$, put $\alpha = \Phi(\mathcal{O}_{C_1})$ and $\beta = \Phi(\mathcal{O}_{C_1}(-1))$. By Key Proposition, we may assume that $l(\alpha) = 1$. Next we show the existence of $\Psi \in B$ such that $l(\Psi(\alpha)) = 1$ and $l(\beta) > l(\Psi(\beta))$. The induction on $l(\beta)$ induces:

Claim 3.8. *For $\Phi \in \operatorname{Auteq} D_Z(X)$, there is an autoequivalence $\Psi \in B$ such that $\Psi \circ \Phi(\alpha) = \Psi \circ \Phi(\beta) = 1$.*

Consequently, we can assume that there are integers a, b $(1 \leq b \leq n)$ and i, and there is an autoequivalence $\Psi \in B$ such that

$$\Psi \circ \Phi(\mathcal{O}_{C_1}) \cong \mathcal{O}_{C_b}(a)[i]$$

[1]When $n = 1$, we cannot find such Ψ. Instead of it, we can find $\Psi_1 \in B$ such that

$$l(\Psi_1(\alpha)) < \sum_q l(\Psi_1(\mathcal{H}^q(\alpha))) = \sum_q l(\mathcal{H}^q(\alpha))(= l(\alpha)).$$

and

$$\Psi \circ \Phi(\mathcal{O}_{C_1}(-1)) \cong \mathcal{O}_{C_b}(a-1)[i].$$

In particular, for any point $x \in C_1$, we can find a point $y \in C_b$ with $\Psi \circ \Phi(\mathcal{O}_x) \cong \mathcal{O}_y[i]$. Then we can rather easily show the following claim by induction on n.

Claim 3.9. *For any $\Phi \in \operatorname{Auteq} D_Z(X)$, there exists an integer i and $\Psi \in B$ such that $\Psi \circ \Phi$ sends every skyscraper sheaf $\mathcal{O}_x$ with $x \in Z$ to $\mathcal{O}_y[i]$ for some $y \in Z$.*

Note that up to here we do not assume that Φ is an FM transform. Finally Lemma 3.10 below completes the proof of Theorem 3.3.

Lemma 3.10 ([BM98]). *Suppose an autoequivalence $\Phi \in \operatorname{Auteq}^{FM} D(X)$ for an algebraic variety X satisfies the following: for any point $x \in X$, there is a point $y \in X$ such that $\Phi(\mathcal{O}_x) \cong \mathcal{O}_y$. Then $\Phi \in \operatorname{Pic} X \rtimes \operatorname{Aut} X$.*

□

3.3 Example of spherical object

The aim of this subsection is to find $\Psi \in B$ as in Key Proposition for a non-trivial spherical object.

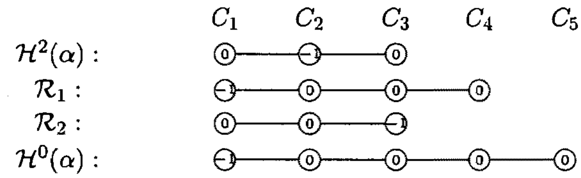
Let S be a smooth projective surface. The following proposition shows that an object α of $D(S)$ is determined by its cohomology sheaves $\mathcal{H}^i(\alpha)$ and the classes $e^i(\alpha)$, up to (non-canonical) isomorphisms.

Proposition 3.11 ([IU04]). *Suppose we are given coherent sheaves $\mathcal{G}^i$ on S and elements*

$$e^i \in \operatorname{Ext}^2_X(\mathcal{G}^i, \mathcal{G}^{i-1})$$

for all $i \in \mathbb{Z}$ such that $\mathcal{G}^i$'s are zero except for finitely many i's. Then there is an object $\alpha \in D(S)$ and isomorphisms $\mu_i : \mathcal{H}^i(\alpha) \cong \mathcal{G}^i$ such that $\mu_{i-1}[2] \circ e^i(\alpha) = e^i \circ \mu_i$. This α is uniquely determined up to isomorphisms.

Example 3.12. Now we go back to the situation of Theorem 3.3. We give a rather non-trivial example of a spherical object $\alpha \in D(X)$, supported on $C_1 \cup \cdots \cup C_5$, a union of -2-curves in A_5-configuration on X. First we define the cohomology sheaves of α as follows:



with $\mathcal{H}^1(\alpha) = \mathcal{R}_1 \oplus \mathcal{R}_2$. Here by the figure above for $\mathcal{H}^2(\alpha)$ we mean

$$\mathcal{H}^2(\alpha) = \mathcal{O}_{C_1 \cup C_2 \cup C_3}(0, -1, 0),$$

the line bundle on $C_1 \cup C_2 \cup C_3$ such that the degrees of it on C_1, C_2, C_3 are $0, -1, 0$ respectively. The rest are similar. Notice that

$$\operatorname{Ext}^2_X(\mathcal{H}^2(\alpha), \mathcal{H}^1(\alpha)) \cong \operatorname{Ext}^2_X(\mathcal{H}^2(\alpha), \mathcal{R}_1) \oplus \operatorname{Ext}^2_X(\mathcal{H}^2(\alpha), \mathcal{R}_2) \cong \mathbb{C} \oplus \mathbb{C}$$

and

$$\operatorname{Ext}^2_X(\mathcal{H}^1(\alpha), \mathcal{H}^0(\alpha)) \cong \operatorname{Ext}^2_X(\mathcal{R}_1, \mathcal{H}^0(\alpha)) \oplus \operatorname{Ext}^2_X(\mathcal{R}_2, \mathcal{H}^0(\alpha)) \cong \mathbb{C} \oplus \mathbb{C}.$$

Keep these isomorphisms in mind, and take

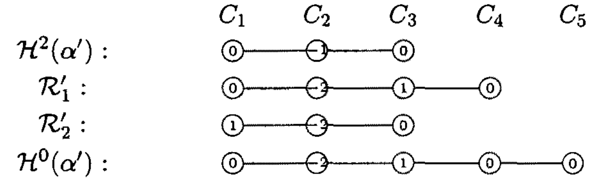
$$e^2(\alpha) = (e^2_1, e^2_2) \in \operatorname{Ext}^2_X(\mathcal{H}^2(\alpha), \mathcal{H}^1(\alpha))$$

and

$$e^1(\alpha) = (0, e^1_2) \in \operatorname{Ext}^2_X(\mathcal{H}^1(\alpha), \mathcal{H}^0(\alpha))$$

with $e^2_1, e^2_2, e^1_2 \in \mathbb{C}^*$. The data $\mathcal{H}^i(\alpha)$ and $e^i(\alpha) \in \operatorname{Ext}^i_X(\mathcal{H}^i(\alpha), \mathcal{H}^{i-1}(\alpha))$ determine an object $\alpha \in D(X)$ by Proposition 3.11. We can see that α is spherical (see [IU04]).

For all of computations below, see [IU04, Lemma 3.15]. Now let us put $\alpha' := T_{\mathcal{O}_{C_2}(-2)}(\alpha)$. First we have



with $\mathcal{H}^1(\alpha') = \mathcal{R}'_1 \oplus \mathcal{R}'_2$. In particular,

$$l(T_{\mathcal{O}_{C_2}(-2)}(\alpha)) = l(\alpha).$$

Then we can see from computations that

$$\begin{aligned} l(T_{\mathcal{O}_{C_1}(-1)}(\mathcal{H}^2(\alpha'))) &= l(\mathcal{H}^2(\alpha')) - 1 \\ l(T_{\mathcal{O}_{C_1}(-1)}(\mathcal{R}'_1)) &= l(\mathcal{R}'_1) - 1 \\ l(T_{\mathcal{O}_{C_1}(-1)}(\mathcal{R}'_2)) &= l(\mathcal{R}'_2) + 1 \\ l(T_{\mathcal{O}_{C_1}(-1)}(\mathcal{H}^0(\alpha'))) &= l(\mathcal{H}^0(\alpha')) - 1. \end{aligned}$$

Consequently, we have

$$\sum_p l(T_{\mathcal{O}_{C_1}(-1)} \circ T_{\mathcal{O}_{C_2}(-2)}(\mathcal{H}^p(\alpha))) = 15 + 1 - 1 - 1 - 1 = 13 < 15 = l(\alpha).$$

Then (1) implies that

$$l(T_{\mathcal{O}_{C_1}(-1)} \circ T_{\mathcal{O}_{C_2}(-2)}(\alpha)) < l(\alpha)$$

as desired.

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