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KOBAYASHI-OCHIAI'S THEOREM FOR LOG SCHEMES

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1. Introduction

In 1974, S.Lang posed the following conjecture [5]:

There are only finitely many surjective meromorphic mappings from a complex space X onto a compact hyperbolic complex space Y.

One of the philosophy behind this conjecture is "The Geometric condition determines arithmetic structures on the geometry over function field ". After this conjecture was posed, Kobayashi-Ochiai proved it under the assumption that Y is of general type by using Nevanlinna theory:

Theorem 1.1 (Kobayashi-Ochiai [4]). *Let X be a compact analytic space and Y be a compact analytic spaces of general type.*

Then, there are only finitely many surjective meromorphic mappings from X to Y.

This theorem was generalized to the case Y is Kobayashi hyperbolic by J. Noguchi [7]. On the other hand, there are some algebraic variants.

Theorem 1.2 (Dechamps-Menegaux [1]). *Let X and Y be smooth projective varieties over k with any characteristic. Moreover we assume Y is of general type.*

Then, the Set of separable dominant rational maps from X to Y is finite.

Theorem1.2 was proved by purely algebraic arguments including some tricks.

The author considered the above finiteness problem in the framework of logarithmic geometry with Prof. Atsushi Moriwaki. The Motivations to study the logarithmic version of this interesting finiteness probrem are:

(1) To study moduli problems, the finiteness of the automorphism groups which we want to parametrize is very important via Deligne-Mumford criterion for the representability of log algebraic stacks. Logarithmic geometry is a new framework to cover compactifications of moduli stacks and singularities in degenerations.

(2) At the starting point, logarithmic structures in the sense of Fontaine, Illusie and K.Kato [3] are techniques to enable us to deal with degenerating objects as if these are not degenerating objects. We regard the log structures not only as such techniques but as "hooks". Through the study

on the finiteness problem, we want to understand this hybrid nature of log structures.

2. Dominant rational maps to log smooth varieties

First, we define the lead in our story.

Definition 2.1. Let k be a field. Let (X, M_X) and $(\mathrm{Spec}(k), M_K)$ be fine log schemes and $\Phi : (X, M_X) \to (\mathrm{Spec}(k), M_K)$ be a morphism of fine log schemes. We say that Φ is a *log smooth variety* if we have the following conditions

(a) X is a proper semi-stable (normal crossing) variety over k.
(b) Φ is log smooth and integral morphism of fine log schemes.

Example 2.2. If X is d-semi-stable ($\mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X) = \mathcal{O}_{X^{sing}}$), there exists a fine log structure M_X on X and a fine log structure M_k on $\mathrm{Spec}(k)$ and a morphism of fine log schemes

$$\Phi : (X, M_X) \longrightarrow (\mathrm{Spec}(k), M_k)$$

which is a log smooth variety. For example, semi-stable curves and Kulikov degenerating models of K3 surfaces are d-semi-stable.

Definition 2.3. Let $\Phi : (X, M_X) \longrightarrow (\mathrm{Spec}(k), M_k)$ be a log smooth variety. We say that Φ is *of log general type* if $det(\Omega^1_{X/k}(log(M_X/M_k))$ is a big line bundle. Here, we say line bundle H is *very big* if there is a dense open set U of X such that $H^0(X, H) \otimes \mathcal{O}_X \to H$ is surjective on U and the induced rational map $X \dashrightarrow \mathbb{P}(H^0(X, H))$ is birational to the image. Moreover, H is said to be *big* if $H^{\otimes m}$ is very big for some positive integer m.

Under these preliminaries, we state the main theorem:

Theorem 2.4 (Iwanari-Moriwaki [2]). *Let k be an algebraically closed field and M_k a fine log structure of* $\mathrm{Spec}(k)$. *Let*

$$(X, M_X) \to (\mathrm{Spec}(k), M_k) \quad \textit{and} \quad (Y, M_Y) \to (\mathrm{Spec}(k), M_k)$$

be a log smooth varieties. We assume that (Y, M_Y) is of log general type, that is, $\det(\Omega^1_{Y/k}(\log(M_Y/M_k)))$ *is a big line bundle on Y . Then, the set of all log rational maps*

$$(\phi, h) : (X, M_X) \dashrightarrow (Y, M_Y)$$

over $(\mathrm{Spec}(k), M_k)$ *with the following properties* (1) *and* (2) *is finite:*

(1) *$\phi : X \dashrightarrow Y$ is a rational map defined over a dense open set U with* $\mathrm{codim}(X \backslash U) \geq 2$, *and $(\phi, h) : (U, M_X|_U) \to (Y, M_Y)$ is a log morphism over* $(\mathrm{Spec}(k), M_k)$.
(2) *For any irreducible component X' of X, there is an irreducible component Y' of Y such that $\phi(X') \subseteq Y'$ and the induced rational map $\phi' : X' \dashrightarrow Y'$ is dominant and separable.*

As an immediate corollary of the above theorem, we have the following:

Corollary 2.5. *Let X be $(X, M_X) \to (\mathrm{Spec}(k), M_k)$ is a log smooth variety. If (X, M_X) is of log general type, then the set of automorphisms of (X, M_X) over $(\mathrm{Spec}(k), M_k)$ is finite.*

Let me explain how we can regard the above corollary as the generalization of the finiteness of automorphism groups of pointed stable curves .

We can easily prove the following equivalence of categories

$$\Big(\text{pointed stable curves}\Big) \cong \Big(\text{special log smooth curves of log general type}\Big)$$

Here we omit the precise definition of *special* but *special log structure* is a fine log structure which has a certain universal property. By the equivalence, we can consider pointed stable curves to be 1-dimensional log smooth varieties and regard the corollary as the generalization to the higher dimentional cases of the finiteness of automorphism groups.

3. Proof of Theorem and log structures on semi-stable varieties

Next let us give a sketch of the proof of Theorem 2.4. For this purpose, we need to deal with the classical case and the non-classical case. In the case where $M_k = k^\times$ and X and Y are smooth over k (the classical case), we can use the similar arguments as in [1]. Actually, we prove it under the weaker conditions . However, if M_k is not trivial (the non-classical case), we have to classify local description of a fine log structure on semi-stable varieties. Indeed, we have the following theorem:

Theorem 3.1. *Let X be a semistable variety over k and M_X a fine log structure of X over M_k such that $(X, M_X) \to (\mathrm{Spec}(k), M_k)$ is log smooth and integral. Let us take a fine and sharp monoid Q with $M_k = Q \times k^\times$. For a closed point $x \in X$, there is a good chart $(Q \to M_k,\ P \to M_{X,\bar{x}},\ Q \to P)$ of $(X, M_X) \to (\mathrm{Spec}(k), M_k)$ at x, namely,*

(a) *$Q \to M_k/k^\times$ and $P \to M_{X,\bar{x}}/\mathcal{O}^\times_{X,\bar{x}}$ are bijective.*

(b) *The diagram*

$$\begin{array}{ccc} Q & \longrightarrow & P \\ \downarrow & & \downarrow \\ M_k & \longrightarrow & M_{X,\bar{x}} \end{array}$$

is commutative.

(c) *$k \otimes_{k[Q]} k[P] \to \mathcal{O}_{X,\bar{x}}$ is smooth.*

Moreover, using the good chart $(Q \to M_k, P \to M_{X,\bar{x}}, Q \to P)$, we can determine the local structure in the following ways:

(1) *If $\mathrm{mult}_x(X) = 1$, then $Q \to P$ splits and $P \simeq Q \times \mathbb{N}^r$ for some r.*

(2) *If* $\mathrm{mult}_x(X) = 2$, *then we have one of the following:*
  (2.1) *If* $Q \to P$ *does not split, then* P *is of semistable type over* Q.
  (2.2) *If* $Q \to P$ *splits, then* $\mathrm{char}(k) \neq 2$ *and* $\widehat{\mathcal{O}}_{X,x}$ *is canonically isomorphic to* $k[\![X_1, \dots, X_n]\!]/(X_1^2 - X_2^2)$.

(3) *If* $\mathrm{mult}_x(X) \geq 3$, *then* $Q \to P$ *does not split and* P *is of semistable type over* Q.

Here we define a monoid of semistable type:

Let $f : Q \to P$ be an integral homomorphism of fine and sharp monoids with $Q \neq \{0\}$. We say P is *of semi-stable type*

$$(r, l, p_1, \dots, p_r, q_0, b_{l+1}, \dots, b_r)$$

over Q if the following conditions are satisfied:

(1) r and l are positive integers with $r \geq l$, $p_1, \dots, p_r \in P$, $q_0 \in Q \setminus \{0\}$, and $b_{l+1}, \dots, b_r$ are non-negative integers.

(2) P is generated by $f(Q)$ and $p_1, \dots, p_r$. The submonoid of P generated by $p_1, \dots, p_r$ in P, which is denoted by N, is canonically isomorphic to $\mathbb{N}^r$, namely, a homomorphism $\mathbb{N}^r \to N$ given by $(t_1, \dots, t_r) \mapsto \sum_i t_i p_i$ is an isomorphism.

(3) We set $\Delta_l, B \in \mathbb{N}^r$ as follows:

$$\Delta_l = (\underbrace{1, \dots, 1}_{l}, \underbrace{0, \dots, 0}_{r-l}) \quad \text{and} \quad B = (\underbrace{0, \dots, 0}_{l}, b_{l+1}, \dots, b_r).$$

Then, $\Delta_l \cdot p = f(q_0) + B \cdot p$, i.e., $p_1 + \cdots + p_l = f(q_0) + \sum_{i>l} b_i p_i$.

(4) If we have a relation

$$I \cdot p = f(q) + J \cdot p \quad (I, J \in \mathbb{N}^r)$$

with $q \neq 0$, then $I(i) > 0$ for all $i = 1, \dots, l$.

The classification of fine log structure on semi-stable varieties was studied by M.Olsson under a certain condition in [6]. Theorem 1.8 gives the complete classification under no conditions. When study the moduli problems such as compactifications, this result makes us be able to equip semi-stable varieties with fine log structure which is suitable for each context.

By using the above local structure result, we can see the uniqueness of a log morphism over the fixed scheme morphism, namely, we have the following:

Theorem 3.2. *Let* (X, M_X) *and* (Y, M_Y) *be log smooth varieties over* $(\mathrm{Spec}(k), M_k)$. *Let* $\mathrm{Supp}(M_Y/M_k)$ *be the union of* $\mathrm{Sing}(Y)$ *and the boundaries of the log structure of* M_Y *over* M_k. *Let* $\phi : X \to Y$ *be a morphism over* k *such that* $\phi(X') \not\subseteq \mathrm{Supp}(M_Y/M_k)$ *for any irreducible component* X' *of* X. *If* $(\phi, h) : (X, M_X) \to (Y, M_Y)$ *and* $(\phi, h') : (X, M_X) \to (Y, M_Y)$ *are morphisms of log schemes over* $(\mathrm{Spec}(k), M_k)$, *then* $h = h'$.

By virtue of this theorem, the non-classical case can be reduced to the classical case, so that we complete the proof of the theorem.

REFERENCES

[1] M. M. Deschamps and R. L. Menegaux, Applications rationnelles séparables dominantes sur une variété de type général, Bull. Soc. math. France, 106 (1978), 279–287.

[2] I. Iwanari and A. Moriwaki, Dominant rational maps in the category of log schemes, preprint

[3] K. Kato, Logarithmic structure of Fontaine-Illusie, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 191–224, Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[4] S. Kobayashi and T. Ochiai, Meromorphic mappings onto complex spaces of general type, Invent. math., 31 (1975) 7–16.

[5] S.Lang, Higher dimensional diophantine problems, Bull. Amer. Math. Soc. 80, No.5 (1974) 779–787

[6] M. Olsson, Universal log structures on semi-stable varieties, Tohoku Math. J., 55 (2003), 397–438.

[7] J.Noguchi, Meromorphic mappings into compact hyperbolic complex spaces and geometric diophantine problems, Internat. J. of Math. 3. No.2 (1992) 277–289.

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