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SEMI-ALGEBRAIC GEOMETRY OF BRAID GROUPS

KYOJI SAITO

0. Introduction

The braid group of n-strings is the group of homotopy types of movements of n distinct points in the 2-plane $\mathbb{R}^2$. It was introduced by E. Artin [1] in 1926 in order to study knots in $\mathbb{R}^3$. He gave a presentation of the braid group by generators and relations, which are, nowadays, called the Artin braid relations.

Since then, not only in the study of knots, the braid groups appear in several contexts in mathematics, since it is the fundamental group of the configuration space of n-points in the plane. Early in 70's the braid groups are generalized to a wider class of groups, the fundamental groups of the regular orbit spaces of finite reflection groups (Brieskorn [6]), which are called either the generalized braid group (Deligne [3]) or the Artin group (Brieskorn-Saito [2]). The regular orbit space turns out to be an Eilenberg-MacLane space (Deligne [3], c.f. Brieskorn-Saito [2]). Through the study of holonomic systems on the Eilenberg-Maclane spaces, representations of the generalized braid groups are studied (Kohno,...). Also through the braid relations, the actions of braid groups on triangulated categories are studied (Seidel-Thomas,...). Still, we are far from full understanding of their representations.

As for the study of the Eilenberg-Maclane spaces, it was from the beginning a question raised by Deligne, Brieskorn, Saito,... to find the paths in the Eilenberg-MacLane spaces which give a generator system of the Artin groups satisfying the Artin braid relations. In this note (based on [4]), we will give two answers to this question. We approach the problem by the semi-algebraic geometry of the orbit space induced from the flat structure on it [7].

1. Artin groups of finite type [2]

Definition. Let Π be a finite index set. A symmetric matrix $M = (m_{ij})_{i,h\in\Pi}$ is called a *Coxeter matrix* if

i) $m_{ii} = 1$ for $i \in \Pi$,
ii) $m_{ij} = m_{ji} \in \mathbb{Z}_{\geq 2}$ for $i \neq j \in \Pi$,
iii) indecomposability: if $\Pi = I \coprod J$ s.t. $m_{ij} = 2$ for $i \in I$, $j \in J$ $\Rightarrow I = \emptyset$ or $J = \emptyset$.

To such Coxeter matrix M, we associate two groups presented as follows.

- An Artin group

$$A(M) := \langle g_1, \ldots, g_l \mid \underbrace{g_i g_j \cdots}_{m_{ij}} = \underbrace{g_j g_i \cdots}_{m_{ij}} \text{ for } \forall i,j \in \Pi \rangle$$

- <u>A Coxeter group</u>

$$W(M) := \langle a_1, \dots, a_l \mid \underbrace{a_i a_j \dots}_{m_{ij}} = \underbrace{a_j a_i \dots}_{m_{ij}} \text{ for } \forall i, j \in \Pi,\quad a_i^2 = 1 \text{ for } i \in \Pi \rangle$$

By the definition, there is a natural surjective homomorphism:

$$A(M) \twoheadrightarrow W(M).$$

It is well known that the list of finite Coxeter groups give a complete list of finite reflection groups. They are classified by the symbols A_l $(l \geq 1)$, B_l $(l \geq 2)$, D_l $(l \geq 4)$, E_6, E_7, E_8, F_4, G_2, H_3, H_4, $I_2(p)$ $(p \geq 3)$ ([5]).

If $W(M)$ is a finite group, $A(M)$ is called of finite type. In this note, we shall consider only Artin groups of finite type.

We shall denote by A_W or by $A(W)$ the Artin group $A(M)$ for $W := W(M)$.

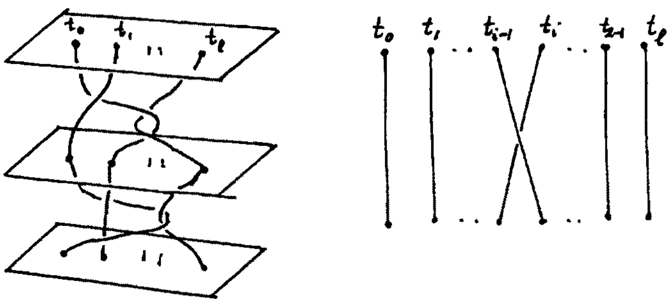
Example. Let $\Pi = \{0, 1, \dots, l\}$ and the Coxeter matrix is given by

$$m_{ij} = \begin{cases} 1 & i = j \\ 3 & |i-j| = 1 \\ 2 & |i-j| \geq 2. \end{cases}$$

Then, one has the most classical examples:

- $W(M) = \mathfrak{S}_{l+1}$= symmetric group of $l+1$ elements, where the generator a_i $(1 \leq i \leq l)$ corresponds to the transposition of $i-1$th and ith elements,
- $A(M) = B(l+1)$= braid group of $l+1$ strings, where the generator g_i $(1 \leq i \leq l)$ corresponds to the "half-braiding" of $i-1$th and ith strings (see figure below).

Fig 1. <u>Composition of braids</u> , <u>ith generator</u> g_i $(i=1,\cdots,l)$



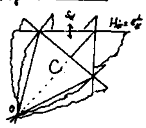
2. Topological realization of Artin groups [5], [6]

We realize the Artin groups as the fundamental group of certain complex configuration spaces [6].

First, recall the vector representation of the Coxeter group [5].

Let $V_W := \sum_{\alpha\in\Pi}\mathbb{R}e_\alpha$ be the based real vector space of rank l, is equipped with a symmetric bilinear form $B(e_\alpha, e_\beta) = \cos(\pi/m_{\alpha\beta})$ for $\alpha, \beta \in \Pi$. For each $\alpha \in \Pi$, we consider a reflection on V_Π w.r.t. e_α defined by: $s_\alpha(u) = u - 2B(u, e_\alpha)e_\alpha$

Fig 2. Chamber



whose reflection hyperplane H_α is given by the $(e_\alpha)^\perp$.

Theorem (Tits). *The correspondence $a_\alpha \mapsto s_\alpha$ $(\alpha \in \Pi)$ induces an injective homomorphism $W(M) \to GL(V)$. By this embedding, let us regard $W := W(M)$ a finite subgroup of $GL(V)$ generated by reflections. Let $R(W)$ be the set of all reflections in W (which is a union of conjugacy elements of the generators s_α $(\alpha \in \Pi)$). Then the action of W on the set of connected components of $V_W \setminus \bigcup_{s\in R} H_{s,\mathbb{R}}$ (called chambers) is simple and transitive.*

Next, we describe the configuration space as the quotient variety of V_W by the finite reflection group W-action. We give two descriptions of the quotient variety: one set-theoretic, the other categorical (the latter is also necessary, since we shall consider it over the field $\mathbb{R}$ which is not algebraically closed).

First, we recall the invariant theory for the W-action on the (real) polynomial function ring $S(V_W^*)$ (c.f. [5]). Chevalley Theorem states that the W-invariants $S(V_W^*)^W$ is generated $l := \#\Pi$ algebraically independent homogeneous elements, say $P_1, \ldots, P_l$, of degree $m_1 + 1 = 2 < \cdots < m_l + 1 = h$:

$$S(V_W^*)^W = \mathbb{R}[P_1, \ldots, P_l].$$

The set of W anti-invariant polynomial (=the polynomial which alter its sign by the action of a reflection) is a rank 1 free module over $S(V_W^*)^W$ generated by

$$\prod_{s\in R(W)} f_s = \frac{\partial(P_1, \ldots, P_l)}{\partial(X_1, \ldots, X_l)}.$$

where f_s is a linear form defining the reflection hyperplane H_s of a reflection s, and $X_1, \ldots, X_l$ is a linear coordinate system of V_W (here and in sequel in the present note, for simplicity, we disregard the positive or non-zero constant factor in such calculations). The square of the anti-invariant is an invariant:

$$\Delta_W = \left(\prod_{s\in R(W)} f_\alpha\right)^2 = \left(\frac{\partial(P_1, \ldots, P_l)}{\partial(X_1, \ldots, X_l)}\right)^2,$$

called the *discriminant*. As an element in $S(V_W^*)^W$, we develop it in a polynomial in P_l:

$$\Delta_W = A_0 P_l^l + A_1 P_l^{l-1} + \cdots + A_l \quad \text{for } A_i \in \mathbb{R}[P_1, \ldots, P_{l-1}].$$

Then, it is a highly non-trivial fact that *the leading coefficient A_0 is a non-zero constant* ([5],[7]) so that Δ_W at the origin has multiplicity l.

1 A set theoretic description of the quotient variety: since the invariant polynomial P_i $(1 \leq i \leq l)$ defines a function on the orbit space $V_{W,\mathbb{C}}/W$ for $V_{W,\mathbb{C}} := V_W \otimes_{\mathbb{R}} \mathbb{C}$. The polynomial map $(P_1, \ldots, P_l)$ induces the homeomorphism:

$$\begin{array}{ccc} V_{W,\mathbb{C}}/W & \underset{(P_1,\ldots,P_l)}{\simeq} & \mathbb{C}^l \\ \cup & & \cup \\ (\bigcup_{s\in R(W)} H_{s,\mathbb{C}})/W & \simeq & D_{W,\mathbb{C}} := \{z \in \mathbb{C}^l \mid \Delta_W(z) = 0\} \end{array}$$

Obviously from the definition of Δ_W, the image of the reflection hyperplanes is the zero loci of Δ_W, which is denoted by $D_{W,\mathbb{C}}$ and is called the discriminant loci. A theorem of Steinberg states that a point in $V_{W,\mathbb{C}}$ is fixed by a non-trivial element of W iff it belongs to a reflection hyperplane. This means that the space of regular (=fixed point free) W-orbits in $V_{W,\mathbb{C}}$ is homeomorphic to the complement $\mathbb{C}^l \setminus D_{W,\mathbb{C}}$ of the discriminant loci.

2. A schema-theoretic description of the quotient variety: we consider the affine scheme defined over $\mathbb{R}$ and its divisor:

$$\begin{array}{ccl} S_W & := & V_W//W = \mathrm{Spec}\,\mathbb{R}[P_1, \ldots, P_l] \\ \cup & & \cup \\ D_W & := & \{\Delta_W = 0\}. \end{array}$$

• <u>An advantage of the categorical quotient.</u> Since $\mathbb{C}$ is algebraically close, the set of $\mathbb{C}$-rational points of S_W and D_W is naturally bijective to the set theoretic quotient space $\mathbb{C}^l$ and $D_{W,\mathbb{C}}$, respectively:

$$\begin{array}{lccc} S_{W,\mathbb{C}} & (= \mathrm{Hom}(S(V_W^*)^W, \mathbb{C})) & \simeq & \mathbb{C}^l \\ \cup & & & \cup \\ D_{W,\mathbb{C}} & (= \mathrm{Hom}(S(V_W^*)^W/(\Delta_W), \mathbb{C})) & \simeq & D_{W,\mathbb{C}} \end{array}$$

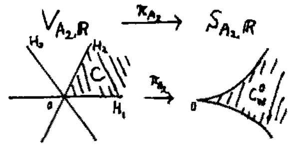
However, the set theoretic quotients $V_{W,\mathbb{R}}/W$ and $(\bigcup_{s\in R(W)} H_{\alpha,\mathbb{R}})/W$ of real vector space and real reflection hyperplanes are only a (semi-algebraic) small part of the $\mathbb{R}$-rational point sets of S_W and D_W, respectively.

$$\begin{array}{lcll} S_{W,\mathbb{R}} & (= \mathrm{Hom}(S(V^*)^W, \mathbb{R})) \simeq \mathbb{R}^l & \supset\neq & V_{W,\mathbb{R}}/W \\ \cup & & & \\ D_{W,\mathbb{R}} & (= \mathrm{Hom}(S(V_W^*)^W/(\Delta_W), \mathbb{R})) & \supset & (\bigcup_{s\in R(W)} H_{\alpha,\mathbb{R}})/W \ (\supset\neq \text{ if } l > 2). \end{array}$$

In fact, since W acts simple and transitively on the set of chambers, the set theoretical quotient $(V_{W,\mathbb{R}} \setminus \bigcup_{s\in R(W)} H_{s,\mathbb{R}})/W$ is homeomorphic to a chamber, and is bijective to a connected component of the complement of the real categorical quotient space $S_{W,\mathbb{R}} \setminus D_{W,\mathbb{R}}$, which we shall denote C_W^0 and call the *central component*. The set theoretic quotient of the real reflection hyperplanes $(\bigcup_{s\in R(W)} H_{s,\mathbb{R}})/W$ is the boundary of the central component C_W^0.

We illustrate these phenomena by the example of type A_2.

Fig 3. <u>central component</u>



Here, LHS of the figure indicates the real vector space V_W and its reflection hyperplanes, and RHS indicates the real categorical quotient variety $S_{W,\mathbb{R}}$ together with the real discriminant loci $D_{W,\mathbb{R}}$. Then shadowed aria is the central component, whose closure is the set theoretical quotient space of $V_{W,\mathbb{R}}$.

Let us state two basic theorems on the topology of the complex regular orbit space $S_{W,\mathbb{C}} \setminus D_{W,\mathbb{C}}$, where the first one is due to Brieskorn [6] and the second one is due to Deligne [3].

Theorem. 1. *The fundamental group of $S_{W,\mathbb{C}} \setminus D_{W,\mathbb{C}}$ is isomorphic to the Artin group A_W (e.g. the fundamental group of $S_{\mathfrak{S}_{l+1},\mathbb{C}} \setminus D_{\mathfrak{S}_{l+1},\mathbb{C}}$ is isomorphic to the braid group $A(\mathfrak{S}_{l+1}) = B(l+1)$).*

2. *The higher homotopy groups vanish: $\pi_i(S_{W,\mathbb{C}} \setminus D_{W,\mathbb{C}}, *) = 0$, $i = 2, 3, \ldots$. (i.e. $S_{W,\mathbb{C}} \setminus D_{W,\mathbb{C}}$ is an Eilenberg-MacLane space.)*

Remark. 1. The above theorems are proven by a use of the monoid of galleries (sequences of chambers which are adjacent successively) so that the isomorphism in 1. is not explicitly given by a path in the $S_{W,\mathbb{C}} \setminus D_{W,\mathbb{C}}$. Therefore, Brieskorn, Deligne and several other people asked the question:

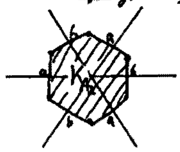
Question. Find a system of paths, say $\gamma_1, \ldots, \gamma_l$ in $S_{W,\mathbb{C}} \setminus D_{W,\mathbb{C}}$ such that their homotopy classes gives the generator system $g_1, \ldots, g_l$ in the Definition of the Artin group.

Remark. 2. We note that the concept of a polyhedron K dual to the chamber decomposition of V_W plays a crucial role in Theorems 1 and 2. Here, a dual polyhedron is a convex hull in V_W of a W-orbit of a point in a chamber C. So, the set of vertices of K is in one to one correspondence with the chambers in V_W, the set of edges of K is in one to one correspondence with the faces (of chambers) in V_W, ..., the open cell $\dot{K}$ corresponds to the origin of V_W.

Let us explain by the example.

A dual polyhedron K for the type A_2 is illustrated as the closed hexagon (with its interior). Clearly, one has: 6 vertices of $K \leftrightarrow$ 6 chambers, 6 edges of $K \leftrightarrow$ 6 faces of chambers, the open hexagon $\leftrightarrow$ the origin of V_{A_2}.

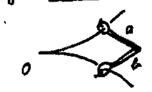
Fig 4. Dual polyhedron of type A_2



Let us explain (indicate) by this picture, relations of K and Theorems.

• Relation with Theorem 1.

Fig 5. Generators of π_1



There are two W-orbit classes of the edges of K, which correspond to two generators, say a and b, of $B(3)$. There is one W-orbit of faces of K (actually, the hexagon), which corresponds to the braid relation $aba = bab$.

• Relation with Theorem 2.

In the proof of the contractibility of $(S_{W,\mathbb{C}} \setminus D_{W,\mathbb{C}})^{\sim}$ by Deligne, the contractibility of K is essentially used.

In this note, We will give two answers to the above question by constructing the dual polyhedron by a use of certain semi-algebraic geometry on $S_{W,\mathbb{R}}$.

3. Primitive vector field D and a $\mathbb{G}_a$-action on S_W [7]

We introduce a $\mathbb{G}_a$-action on S_W which is transversal (in several strong senses, which we do not explain in the present note) by an integration of a particular vector field on it. Let $\mathrm{Der}_{\mathbb{R}}(S[V^*]^W)$ be the $S(V_W^*)^W$-module of polynomial coefficients vector fields on S_W. Since $S(V_W^*)^W$ is a graded ring, the module is also graded (e.g. $\deg(\frac{\partial}{\partial P_i}) = -\deg(P_i) = -(m_i+1)$). Then, it is easy to see that the lowest graded piece of the module is a rank 1 vector space generated by

$$D = \frac{\partial}{\partial P_l}$$

(where we recall that we have ordered as $\deg(P_1) < \cdots < \deg(P_l) = h$). In fact, D is, up to a constant factor, independent of a choice of the generator system $P_1, \ldots, P_l$ of invariants. We shall call D a primitive vector field. (*Proof.* Let us take another system of generators $S[V^*]^W = \mathbb{R}[Q_1, \ldots, Q_l]$ with $\deg Q_i = \deg P_i = m_i + 1$. Then the chain rule of the derivatives shows that

$$\begin{aligned} \frac{\partial}{\partial P_l} &= \sum_{1 \le i \le l} \frac{\partial Q_i}{\partial P_l}\frac{\partial}{\partial Q_i} \quad (\text{here}, \frac{\partial Q_i}{\partial P_l} = 0 \text{ for } i < l \text{ since it is of negative degree}) \\ &= \frac{\partial Q_l}{\partial P_l}\frac{\partial}{\partial Q_l} = \text{const.}\ \frac{\partial}{\partial Q_l} \end{aligned}$$

<u>Formal group action</u>

We, now, introduce a formal group action on $S_W := V_W//W$ by the integration of the primitive vector field D. Actually, it is globally defined as a $\mathbb{G}_a$-action (which we shall call the τ-action) as a translation of the last coordinate P_l:

$$\begin{array}{cccccc} \tau = \exp(\bullet D) : \mathbb{G}_a & \times & S_W & \to & S_W \\ \lambda & \times & (P_1, \ldots, P_l) & \mapsto & (P_1, \ldots, P_{l-1}, P_l + \lambda) \end{array}.$$

Remark. Recall that the set theoretic quotient $V_{\mathbb{R}}/W$ is the closure of the central component in $S_{W,\mathbb{R}}$. Then, as illustrated in the figure for the A_2-type, the quotient set is not invariant by the action!

Fig 6. Primitive vector field

Remark. There is a one to one correspondence between the set of connected components of $S_{W,\mathbb{R}} \setminus D_{W,\mathbb{R}}$ and the set of conjugacy class of involutive elements of W. E.g. $\mathfrak{S}_3$: $\{1\}, \{\sigma_{ij}\}$

Remark. A Coxeter element of W is, by definition, a product (in any order) of a system of reflections whose reflection hyperplanes give a system of walls of a chamber in V_W. The conjugacy class of the Coxeter elements is uniquely defined independent of the ambiguities in the above definition. The order of a Coxeter element, say c, is denoted by h. Then, the primitive hth root of unity is an eigenvalue of c of multiplicity 1, and the eigenspace belonging the eigenvalue is regular (in the sense that it is not contained in any reflection hyperplane) (see [5]). More precisely, we can show that the inverse image in $V_{W,\mathbb{C}}$ of the one dimensional τ-orbit $\tau(\mathbb{G}_a)\cdot O$ of the origin in S_W decomposes into a union of lines, each of which is the eigenspace of a Coxeter element for the eigenvalue of hth primitive root of unity.

4. Main theorems [4, Theorem A, B]

We formulate two theorems on certain semi-algebraic geometry in $S_{W,\mathbb{R}}$ and in $V_{W,\mathbb{R}}$, respectively. In order to formulate the result, let us prepare a notation

$$C_W^{\{\pm\}} := \begin{array}{l}\text{the connected component in } S_{W,\mathbb{R}} \setminus D_{W,\mathbb{R}} \text{ containing} \\ \text{the half line } \tau(\pm\mathbb{R}_{>0})\dot{O}.\end{array}$$

Theorem. 1. *For $\lambda \in \mathbb{R}_{>0}$, consider the intersection of three components*

$$J_W(\lambda) := C_W^0 \ \cap\ \tau(-\lambda)\cdot C_W^{\{+\}} \ \cap\ \tau(\lambda)\cdot C_W^{\{-\}}.$$

Then, $J_W(\lambda)$ is a connected component of the complement of three discriminants: $S_{W,\mathbb{R}} \setminus \left(D_{W,\mathbb{R}} \cup \tau(\lambda)D_{W,\mathbb{R}} \cup \tau(-\lambda)D_{W,\mathbb{R}}\right)$, and is homeomorphic to a parallelotope $[0,\lambda]^l$. The origin O is a vertex of $J_W(\lambda)$ and let $ao = ao(\lambda)$ be the vertex of $J_W(\lambda)$ anti-podal to the origin. The edges of $J_W(\lambda)$ adjacent to $ao(\lambda)$ are indexed by Π and are transversal to the discriminant $D_{W,\mathbb{R}}$.

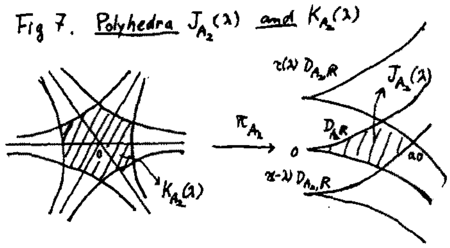
2. *The inverse image $\bar{K}_W(\lambda)$ in $V_{W,\mathbb{R}}$ of the parallelotope $\bar{J}_W(\lambda)$ by the quotient map $\pi_W : V_{W,\mathbb{R}} \to S_{W,\mathbb{R}}$ is a semi-algebraic polyhedron which is dual to the chamber decomposition of V_W.*

Remark. The set of vertices of the polyhedron $K_W(\lambda)$ are mapped to the vertex $ao(\lambda)$ of $J_W(\lambda)$. The trace of the vertices $AO^+ := \{ao(\lambda) \mid \lambda \in \mathbb{R}_{>0}\}$ is a half line, called the *half vertex orbit axis* (in fact, one has an a priori description of the vertex orbit axis by a use of Coxeter element [4], playing a crucial role in the whole theory).

We illustrate the results of Theorem 1. and 2. by the example of type A_2. A more precise figures for the types A_2 and B_2 are given in Appendix 2 and 3.

The figure in RHS draw the real discriminant $D_{A_2,\mathbb{R}}$ and its translations to a positive direction λ and to a negative direction $-\lambda$. The shadowed aria is the 2-dimensional parallelotope $J_{A_2}(\lambda)$. The figure in LHS draw the union of the reflection hyperplanes $\cup H_{s,\mathbb{R}} = \pi_{A_2}^{-1}(D_{A_2,\mathbb{R}})$ and the inverse images of the shifted discriminants. The shadowed aria is the two dimensional dual polyhedron $K_{A_2}(\lambda)$.

Fig 7. Polyhedra $J_{A_2}(\lambda)$ and $K_{A_2}(\lambda)$



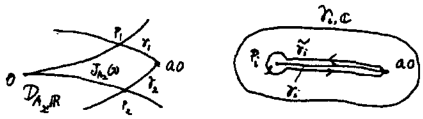
The proof of Theorems is based on some more basic result on the semi-algebraic description of the τ action, and is indicated in 6. To obtain a comprehensive description and understanding of the polyhedron, we should study not only the polyhedron $K_W^{+1}(\lambda) := K_W(\lambda)$ in the real vector space $V_{W,\mathbb{R}}$, but also its twin polyhedron $K_W^{-1}(\lambda)$ in the imaginary vector space $\sqrt{-1}V_{W,\mathbb{R}}$. For the details, one is referred to [4] and its complete version, which is in preparation.

5. APPLICATIONS [4, §4]
(THE DESCRIPTION OF A GENERATOR SYSTEM OF $\pi_1(S_{W,\mathbb{C}} \setminus D_{W,\mathbb{C}}, *)$)

As the applications to the Theorems 1. and 2. in previous section, we give two answers to the question posed at the end of section 2. (and in the introduction).

Theorem. 3. *Let $\bar{J}_W(\lambda)$ be the parallelogram in Theorem 1. Let γ_i $(i \in \Pi)$ be the edges of $\bar{J}_W(\lambda)$ adjacent to ao. For $(i \in \Pi)$, choose a path $\tilde{\gamma}_i$ in the complexification $\gamma_{i,\mathbb{C}}$ of γ_i (i.e. an open Riemann surface in $S_{W,\mathbb{C}}$ containing γ_i) which is based at ao and turning around the point $D_{W,\mathbb{R}} \cap \gamma_{i,\mathbb{C}}$ once counter-clockwisely (see Fig.).*

Fig 8. The generators $\tilde{\gamma}_i$ of π_1



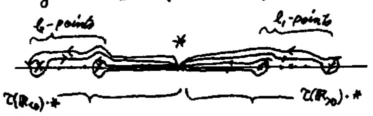
Then the correspondence $g_i \in A(W) \mapsto \tilde{\gamma}_i$ $(i \in \Pi)$ induces the isomorphism:

$$\begin{array}{ccc} A_W & \xrightarrow{\sim} & \pi_1(S_{W,\mathbb{C}} \setminus D_{W,\mathbb{C}}, ao). \\ g_i & \mapsto & \tilde{\gamma}_i \end{array}$$

The fundamental group of a complement of a divisor has another presentation by so called Zariski-van Kampen method (see Le and Cheniot [ ]). We give a comparison of the Zariski-van Kampen type generator system and our generators system.

Theorem. 4 *(Zariski-van Kampen type generators) Take any general point $*$ in the central component C_W^0, Then the real orbit $\tau(\mathbb{R}) \cdot *$ intersects with the discriminant $D_{W,\mathbb{R}}$ at $l = \#\Pi$ points. Let $l^{\pm} = (\#\tau(\pm\mathbb{R}_{>0}) \cdot *) \cap D_{W,\mathbb{R}}$ s.t. $l = l^+ + l^-$. Consider the paths $\delta_1^+, \ldots, \delta_{l^+}^+$ (resp. $\delta_1^-, \ldots, \delta_{l^-}^-$) in the half complexification $\tau(\mathbb{H}) \cdot *$ (resp. $\tau(-\mathbb{H}) \cdot *$ based at the point $*$ and turning once around each point at the intersection $\tau(\mathbb{R}_{>0}) \cdot * \cap D_{W,\mathbb{R}}$ (resp. $\tau(-\mathbb{R}_{>0}) \cdot * \cap D_{W,\mathbb{R}}$), where $\mathbb{H} := \{\lambda \in \mathbb{C} \mid Im(\lambda) > 0\}$.*

Fig 9. The generators δ_i of π_1



Since the base points ao and $*$ lie in the same contractible set C_W^0, one has a canonical isomorphism: $\pi_1(S_{W,\mathbb{C}} \setminus D_{W,\mathbb{C}}, ao) \simeq \pi_1(S_{W,\mathbb{C}} \setminus D_{W,\mathbb{C}}, *)$. Then one has:
1. The isomorphism induces a bijection between the generator systems:

$$\{\tilde{\gamma}_i \mid i \in \Pi\} \simeq \{\tilde{\delta^+}_1, \ldots, \tilde{\delta^+}_{l^+}, \tilde{\delta^-}_1, \ldots, \tilde{\delta^-}_{l^-}\}.$$

2. The generators $\delta_1^+, \ldots, \delta_{l^+}^+$ and $\delta_1^-, \ldots, \delta_{l^-}^-$ are mutually commutative among themselves, respectively.

6. Proof ([4, Theorem C])

Theorems 3 and 4 are consequences of Theorems 1 and 2 (in a stronger form, proof is omitted, see [4]). Theorems 1 and 2 are consequences of the linearization of the real discriminant loci, roughly formulated in a Theorem in this section (to be exact, the formulation is not sufficient for the application). Interested reader is referred to [4] for a complete formulation and a proof.

We prepare some notations.

1. The $\mathbb{G}_a$-quotient space T_W and the bifurcation divisor B_W

Let us introduce an affine scheme over $\mathbb{R}$:

$$T_W := S_W//\tau\mathbb{G}_a = \text{Spec}\ \mathbb{R}[P_1, \dots, P_{l-1}]$$

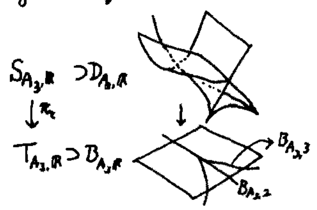
and define the quotient map $\pi_\tau : S_W \to T_W$. The restriction $\pi_\tau|_{D_W}$ is a finite cover over T_W, which is branching along a divisor $B_W \subset T_W$ where B_W is defined by the resultant (discriminant) of the discriminant $\Delta_W = A_0 P_l^l + A_1 P_l^{l-1} + \cdots + A_l$ as a polynomial in one variable P_l:

$$\delta\left(\Delta_W, \frac{\partial \Delta_W}{\partial P_l}\right) = \omega_2^2 \cdot \omega_3^3 \cdots \quad \in \mathbb{R}[P_1, \dots, P_{l-1}],$$

where RHS is the decomposition of LHS according to the multiplicities: 2, 3,... of the factor ω_2, $\omega_3, \dots$ (there is no reduced factor due to the transversality of the τ-action to the discriminant).

The divisor $B_{W,p} := \{\omega_p = 0\}$ is called the pth bifurcation loci, and we define the bifurcation divisor $B_W := \bigcup_{p\geq 3} B_{W,p}$. The sub-divisor $B_{W,\geq 3} := \bigcup_{p\geq 3} B_{W,p}$ is called the caustics.

Fig 10. τ-quotient morphism



Recall the vertex orbit half axis AO^+, discussed in Remark of section 4. Let us denote by $O^+ := \pi_\tau(AO^+)$ its projection image in the real form $T_{W,\mathbb{R}}$ of T_W and call it the vertex orbit half line. It is a highly non-trivial fact that $O^+ \subset B_{W,2,\mathbb{R}}$ but $O^+ \cap B_{W,\geq 3,\mathbb{R}} = \emptyset$. Therefore, we can define

Key concept: The central region:

$$E_W \ := \ \text{the connected component of } T_{W,\mathbb{R}} \setminus B_{W,\geq 3,\mathbb{R}} \text{ containing the vertex orbit half line } O^+.$$

We shall see that *E_W is a simplicial cone of dimension $l-1$, whose faces are indexed by the edges of the Dynkin-Coxeter graph $\Gamma(W)$ for the type of the Coxeter group W* (in fact, this is the back ground for the fact that the generator system $\tilde{\gamma}_i$ ($i \in \Pi$) in Theorem 3 satisfy the Artin-braid relations. In order to formulate the result, we prepare some more notation.

2. Linear model space

Let us introduce a based vector space, where the basis are indexed by the set Π:

$$\hat{V}_\Pi := \oplus_{\alpha\in\Pi}\mathbb{G}_a \cdot v_\alpha.$$

Let us define the diagonal action of $\mathbb{G}_a$ on $\hat{V}_\Pi$ by letting $\lambda \in \mathbb{G}_a$ acts on $\tilde{t} \in \tilde{V}_\Pi$ by $\tilde{t} \mapsto \tilde{t} + \lambda \sum_{\alpha\in\Pi} v_\alpha$. Let us introduce the quotient space:

$$V_\Pi := \hat{V}_\Pi / \mathbb{G}_a \cdot \sum_{\alpha\in\Pi} v_\alpha.$$

Let λ_α ($\alpha \in \Pi$) be the dual basis of the basis v_α so that any element $\tilde{t} \in \hat{V}_\Pi$ is expressed as $\sum_{\alpha\in\Pi} \lambda_\alpha(\tilde{v})v_\alpha$ (i.e. λ_α ($\alpha \in \Pi$) are coordinates for $\hat{V}_\Pi$). Note that $\lambda_{\alpha\beta} := \lambda_\alpha - \lambda_\beta$ ($\alpha \neq \beta \in \Pi$) form a root system of type A_{l-1} on V_Π.

Solving the algebraic equation $\Delta_W = 0$ in the indeterminate P_l, we obtain l number of (multivalued) algebroid functions on $T_{W,\mathbb{C}}$ branching along $B_{W,odd,\mathbb{C}}$. In fact, by choosing the half vertex orbit line O^+, as for the base point of the multivalued functions, we can naturally index the algebroid functions by the set Π [4]. That is: we have the "decomposition" of the discriminant polynomial:

$$\begin{aligned} \Delta_W &= A_0 P_l^l + A_1 P_l^{l-1} + \cdots + A_l \\ &= A_0 \textstyle\prod_{\alpha\in\Pi}(P_l - \underbrace{\varphi_\alpha(P_1,\ldots,P_{l-1})}_{\text{algebraic functions}}) \end{aligned}$$

Theorem. *Consider the multivalued algebroid maps*) c_W and b_W defined by the correspondences*

$$\begin{array}{lcll} c_W & : & \lambda_\alpha = P_l - \varphi_\alpha, & \alpha \in \Pi \\ b_W & : & \lambda_{\alpha\beta} = \varphi_\beta - \varphi_\alpha, & \alpha, \beta \in \Pi \end{array}$$

which makes the following diagram) commutative:*

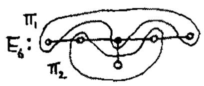
$$\begin{array}{ccc} S_W & \xrightarrow{c_W} & \tilde{V}_\Pi. \\ \downarrow{\scriptstyle \pi_\tau} & & \downarrow{\scriptstyle \mathbb{G}_a \sum_{\alpha\in P_i} v_\alpha} \\ T_W & \xrightarrow{b_W} & V_\Pi. \end{array}$$

The restriction of the maps to their real forms $c_{W,\mathbb{R}} : S_{W,\mathbb{R}} \to \hat{V}_{\Pi,\mathbb{R}}$ and $b_{W,\mathbb{R}}$: $T_{W,\mathbb{R}} \to V_{\Pi,\mathbb{R}}$ induce the following semi-algebraic isomorphisms of the central component and the central region to certain linear simplicial cones:

$$\begin{array}{lcll} c_{W,\mathbb{R}} & : & C_W^0 & \simeq & \sigma \cdot \{\tilde{t} \in \tilde{V}_{\Pi,\mathbb{R}} : \lambda_\alpha < 0 \ \ for\ \alpha \in \Pi_1, \lambda_\alpha > 0 \ \ \underline{for}\ \alpha \in \Pi_2\} \\ b_{W,\mathbb{R}} & : & E_W & \simeq & \sigma \cdot \{t \in V_{\Pi,\mathbb{R}} : \lambda_{\alpha\beta} > 0 \ for\ \alpha \in \Pi_1, \beta \in \Pi_2, \overline{\alpha\beta} \in Edge(\Gamma(W))\} \end{array}$$

where Π_i ($i = 1, 2$) is a decomposition $\Pi = \Pi_1 \coprod \Pi_2$ such that each Π_i is totally disconnected subset of the vertices of the Coxeter-Dynkin graph $\Gamma(W)$ (see the figure for the example of type E_6), and $\sigma \in \{\pm 1\}$. The sign factor σ can be determined exactly [4]. The linearization maps c_{A_3} and b_{A_3} are illustrated in Appendix 1 (taken from [4]).

Fig 11. Decomposition $\Pi = \Pi_1 \amalg \Pi_2$



*) To be exact, the maps c_W and b_W should be defined on a suitable covering spaces of S_W and T_W, and the branch of the maps in consideration should be specified. In [4] (and in the forthcoming paper in preparation), we proceed this by two means either by a use of suitable topological covering spaces with some careful consideration of base points, or by a use of suitable finite algebraic covering. Both are technically complicated and we do not go into any details in the present note.

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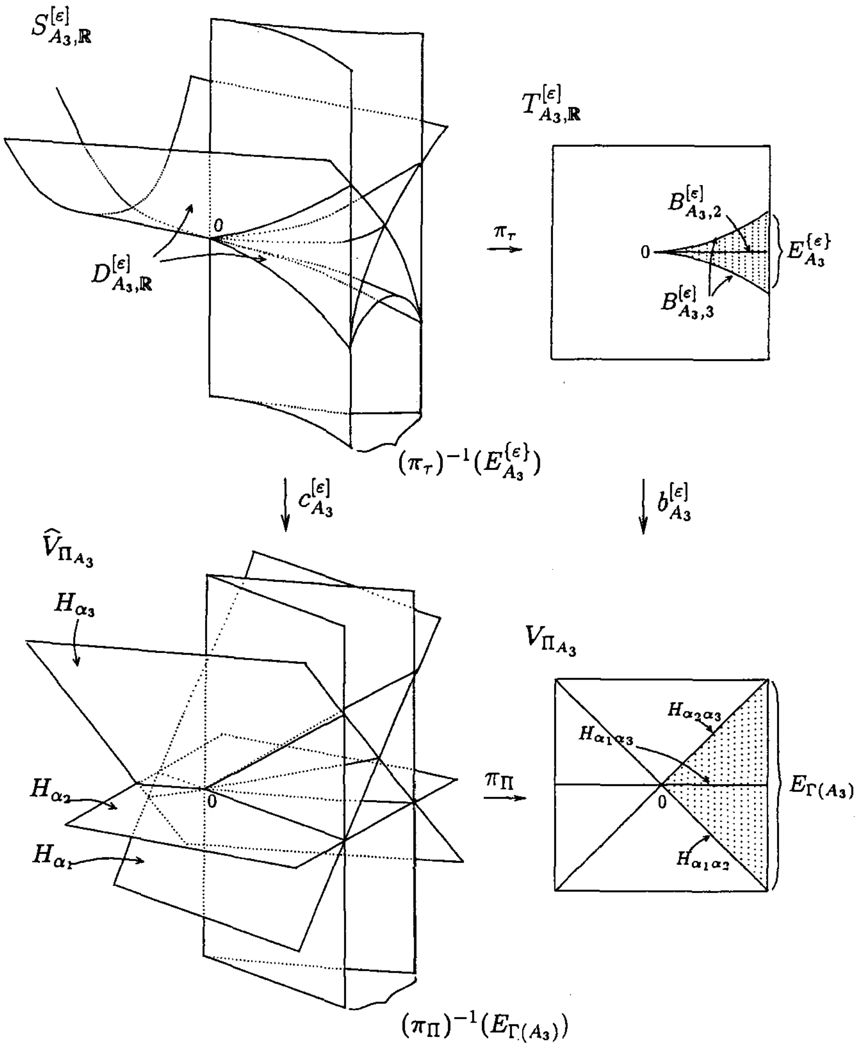
RIMS, KYOTO UNIVERSITY
E-mail address: saito@kurims.kyoto-u.ac.jp

Appendix I

Fig.16. The linearization morphisms of type A_3:
$c^{[\varepsilon]}_{A_3} : (\pi_\tau)^{-1}E^{\{\varepsilon\}}_{A_3} \simeq \widehat{T}_{\Gamma(A_3)}$ **and** $b^{[\varepsilon]}_{A_3} : E^{\{\varepsilon\}}_{A_3} \simeq E_{\Gamma(A_3)}$.

The shaded aria in the two figures in RHS are the total real region $E^{\{\varepsilon\}}_{A_3}$ in $T^{[\varepsilon]}_{A_3,\mathbb{R}}$ and the cone $E_{\Gamma(A_3)}$ in V_Π, respectively. Their inverse images $(\pi_\tau)^{-1}(E^{\{\varepsilon\}}_{A_3})$ in $S^{[\varepsilon]}_{A_3,\mathbb{R}}$ and $\widehat{T}_{\Gamma(A_3)}$ in $\widehat{V_\Pi}$ are illustrated as the vertical cylindrical domains sandwiched inside (either semialgebraic or straight) booklets in the two figures in LHS, respectively.

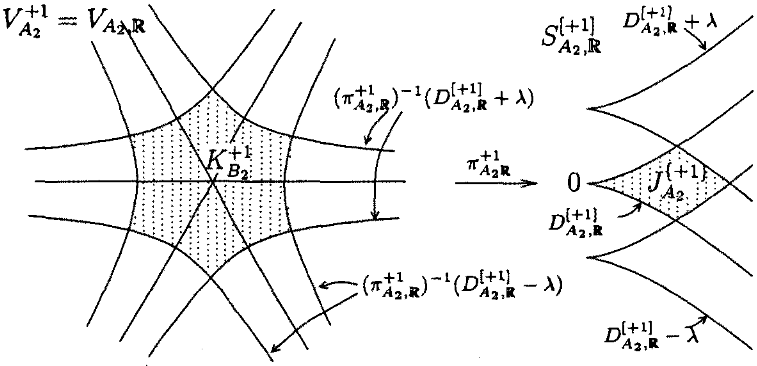


Appendix 2

1. h is odd and coset [1]:

a) $\mathrm{Inv}([1]) = \{\{1\}, \{\alpha_1, \alpha_2, \cdots, \alpha_{(h+3)/2} = \Delta, \cdots, \alpha_h\}\}$,
b) $Z_W(1) = W(1) = W(I_2(h))$,
$Z_W(\alpha_k) = W(\alpha_k) = \langle\alpha_k\rangle \ (k = 1, \cdots, h)$.

c) Fig.19. Polyhedra $J^{\{+1\}}_{A_2}$ and $K^{+1}_{A_2}$ for $\lambda = 1$

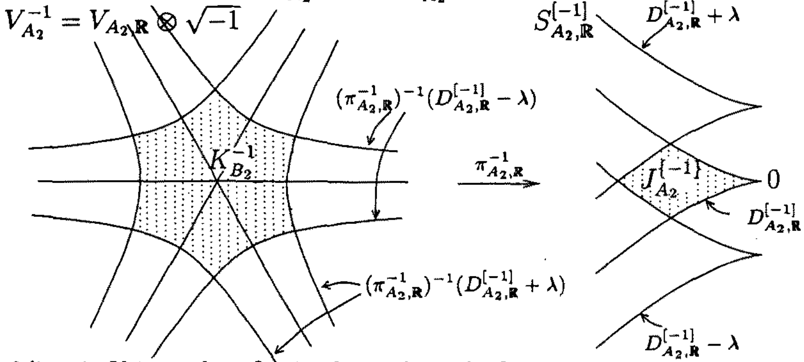


3 lines in $V^{+}_{A_2}$ are the reflection hyperplanes for W_{A_2}.

2. h is odd and coset $[\beta] = [-1]$:

a) $\mathrm{Inv}([\beta]) = \mathrm{Inv}([-1]) = \{\{-\alpha_1, -\alpha_2, \cdots, -\alpha_{(h+3)/2} = \beta, \cdots, -\alpha_h\}, \{-1\}\}$.
b) $Z_W(-1) = W(-1) = W(I_2(h))$
$Z_W(-\alpha_k) = W(-\alpha_k) = \langle\alpha_k\rangle \ (k = 1, \cdots, h)$.

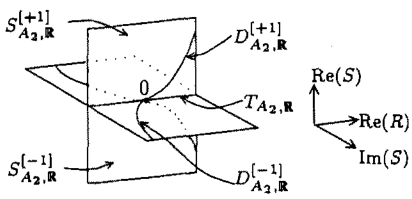
c) Fig.20. Polyhedra $J^{\{-1\}}_{A_2}$ and $K^{-1}_{A_2}$ for $\lambda = 1$



3 lines in $V^{-}_{A_2}$ are the reflection hyperplanes for W_{A_2}.

Fig. 21.

c)* Positions of $S^{[+1]}_{A_2,\mathbb{R}}$ and $S^{[-1]}_{A_2,\mathbb{R}}$ inside $S_{A_2,\mathbb{C}} \cap \{\mathrm{Im}(R) = 0\}$.



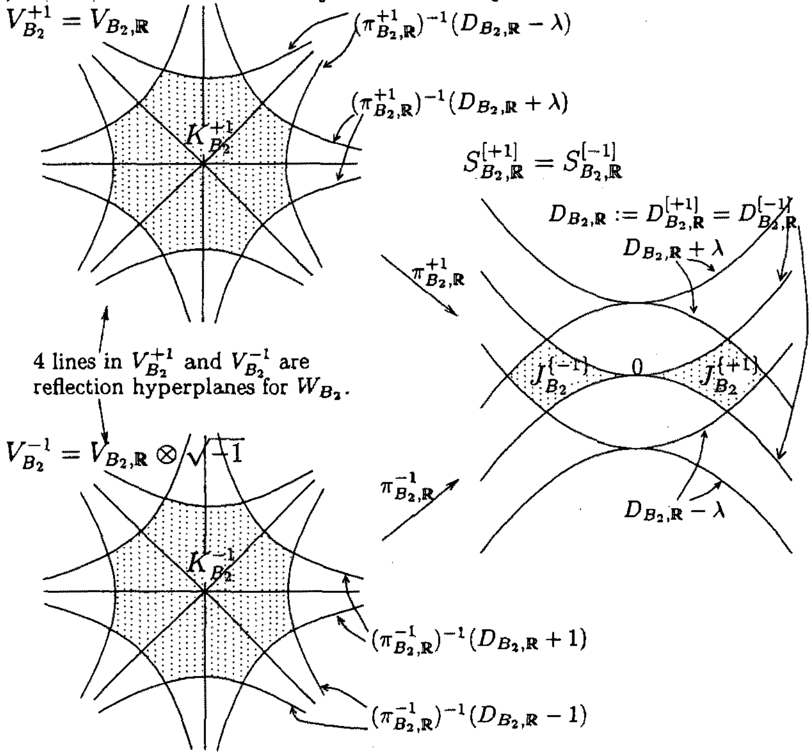
Appendix 3

3. h is even and coset $[1] = [-1]$: In this case, $-1 = \Delta \in W(I_2(h))$.

a) $\mathrm{Inv}([1]) = \mathrm{Inv}([-1]) = \{\{1\}, \{-1\}, \{\alpha_1, \alpha_3, \cdots, \alpha_{h-1}\}, \{\alpha_2, \alpha_4, \cdots, \alpha_h\}\}$,

b) $Z_W(1) = W(1) = W(I_2(h))$,
$Z_W(-1) = W(-1) = W(I_2(h))$,
$Z_W(\alpha_k) = W(\alpha_k) = \langle \alpha_k, \alpha_{k+h/2} \rangle \ (k = 1, \cdots, h)$.

c) **Fig. 22.** Polyhedra $J^{\{\pm 1\}}_{B_2}(\lambda)$ and $K^{\pm 1}_{B_2}(\lambda)$ for $\lambda = 1$.



4. h is even and coset $[\beta]$:

a) $\mathrm{Inv}([\beta]) = \{\{\beta_k := \beta(\alpha_1\alpha_2)^{k-1} \mid k = 1, \cdots, h\}$.

b) $Z_W(\beta_k) = \langle \Delta \rangle \supset W(\beta_k) = \{1\}$, $Z_W(\beta_k, C^\beta) \simeq \mathbb{Z}_2$.

c)

Fig. 23.

Twisted real forms:

$\pi^\beta_{B_2,\mathbb{R}} : U^\beta_{B_2} \to S^{[\beta]}_{B_2,\mathbb{R}}$.